

Midterm exam 1, Friday, Oct 4, 2024
Lev Borisov, Math 451

DO NOT OPEN THE EXAM

BEFORE THE START

ANNOUNCEMENT !

Name:

Each problem is worth 20 points, for a total of 100 points. **Calculators/notes are not allowed on this test.** Please read each question carefully, it also helps to check afterwards that you have answered each part of each question. **You must show all your work to receive credit.** Make sure that I check your name in my list or write your name down. Good luck!

1	2	3	4	5	Total

[1] Let $f : G_1 \rightarrow G_2$ be a homomorphism of finite groups. Prove that for every element $x \in G_1$ the order of its image $y = f(x)$ divides both the order of G_1 and the order of G_2 .

[1] (20 pts)

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[2] Let H_1 and H_2 be subgroups of a group G .

(a)[7pts] Prove that if some left coset of H_1 is equal to some left coset of H_2 then H_1 is equal to H_2 .

(b)[7pts] Prove that if some **left** coset of H_1 is equal to some **right** coset of H_2 then H_1 is isomorphic to H_2 .

(c)[6pts] Give an example of H_1 , H_2 and G when some left coset of H_1 is equal to some right coset of H_2 and $H_1 \neq H_2$. You need to explain why your example works.

[2] (20 pts)

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[3] Let G be a group. It is easy to see that for any $g \in G$ the map $\phi_g : G \rightarrow G$ that sends x to gxg^{-1} is an automorphism of G .

[10pts](a) Prove that the map $\rho : G \rightarrow \text{Aut}(G)$ that sends g to ϕ_g is a group homomorphism.

[10pts](b) Prove that the center $Z(G)$ of G is the kernel of ρ from part (a).

[3] (20 pts)

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[4] Let p and q be distinct prime numbers.

[10pts](a) Prove that the group $(\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/q\mathbb{Z})$ has exactly 4 subgroups, including the identity subgroup and the whole group.

[10pts](b) Find, with proof, the number of subgroups of the group $(\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$.

[4] (20 pts)

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[5] Let H_1 and H_2 be normal subgroups of a group G .

[5pts](a) Prove that the intersection subgroup $H = H_1 \cap H_2$ is normal. You don't need to check that H is a subgroup.

[15pts](b) Prove that the quotient group G/H is isomorphic to a subgroup of the direct product group $(G/H_1) \times (G/H_2)$.

[5] (20 pts)

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