**Calculus 251:C3 Reading Guide - 6/10/2021**

## Section 14.3 Partial Derivatives

In this section, we are going to see how the concept of the derivative extends to functions of more than one variable. When we looked at the graphs of functions of two variables in section 14.1, we saw that at a particular point of the domain we may have different rates of change depending on the direction we travel from the point. It therefore does not make sense to talk about "the derivative" of a multivariable function at a point. What does make sense is talking about the partial derivative of a function with respect to a particular variable.

If you have a function $f(x, y)$, the partial derivative with respect to $x$ at the point $(a, b)$ in the domain of the function is the rate of change of $f$ at that point along the line $\vec{r}(t) = \langle a+t, b\rangle$. That is what happens when we hold $y$ constant and only let $x$ vary.

Note that there are multiple notations that get used. You may see any of

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f) = \frac{\partial}{\partial x}(f(x, y))$$

all of which mean the same thing (at least if $f$ is a function of $x, y$ for the last one). Other notations that are equivalent:

$$f_x(a, b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} \qquad f_y(a, b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)}$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial x^2}(f) = \frac{\partial^2 f}{\partial x^2}$$

I will usually use the subscript notation (so $f_x$, $f_x(a, b)$, $f_{xx}$, etc.) because it is a lot less writing/typing. There will be times when the Leibniz notation (the one with the $\partial$'s) makes it easier to see what is going on. You should become comfortable with both notations.

Find the following definitions/concepts/theorems:

- partial derivative
- partial derivative of $f$ with respect to $x$ (or $y$)
- Numerical approximation of partial derivatives
- second-order (and higher) partial derivatives
- mixed partials (a/k/a mixed derivatives)
- Theorem: The Mixed Derivative Theorem a/k/a Clairault's Theroem (note the conditions under which this is true!)
- differentiability
- smooth surface

- Differentiability implies continuity

The motivating examples above example 1 are intended to give you a feel for when you will need to use multivariable functions to model real-world phenomena.
Examples 1, 2, 5, and 7 are very basic examples of partial derivatives. Example 3 requires the quotient rule, but is otherwise no more scary than the others.

Examples 4 and 6 require versions of the chain rule. Much more on this in section 14.4.

The section on Partial Derivatives and Continuity and example 8 are a cautionary tale. A function can have continuous partial derivatives at a point, but still not be a differentiable function at that point. In order for the function to be differentiable, both (or all if there are more than two independent variables) partials must be continuous in some open region around the point. Most of the functions we deal with in this course will be differentiable everywhere (or almost everywhere). You will certainly need to think about this if and when you take a real analysis class.

Examples 9 and 10 involve second-order partials. Neither one is terribly complicated, and you should make sure you understand them. In particular, notice that applying Theorem 2 makes example 10 much easier. If you intend to be a math major, take a look at the proof of Theorem 2 in Appendix 9 in the back of the book.

You should give the section on Differentiability a quick read, but don't get hung up on it. The only big takeaway is that if a function is well-enough behaved to be differentiable a point (so its partials exist in some open region around the point), then it must be continuous there.

**Section 14.4 The Chain Rule**

In this section, there are a bunch of theorems each of which defines one or more chain rules. At this point, you should be asking "Wait. Chain rules? Why is there more than one?" Think back to the product rules for vector-valued functions. We had defined three different products (scalar, dot, and cross), so we had three different product rules. Similarly, there are different ways to compose functions of several variables. In single-variable calculus, we only had functions from $\mathbb{R} \to \mathbb{R}$, so there was really only one type of composition. In multivariable calculus, we have already seen the chain rule for paths. That was a type of composition where $f : \mathbb{R}^3 \to \mathbb{R}$ and $\vec{r} : \mathbb{R} \to \mathbb{R}^3$, so we had $f(\vec{r}) : \mathbb{R} \to \mathbb{R}$. In this section, we will look more closely at that chain rule, and then generalize it to other types of compositions.

Find the following definitions/concepts/formulas/theorems:

- Chain rule for paths (go back and look at this on p. 768 - it's number 7 in the box)
- Theorems: all of the different chain rules - see if you can figure out what they all have in common!
- dependency diagrams (these are all in the margins, but they are very useful)
- implicit differentiation (recall that you used this in calc one for relations like $xy^2 - y^3 + x^2 = 3$)

Please work through as many of these examples as you can. In class, we are going to do a bunch of examples, and I am going to show you other ways to think about these problems. In particular, I am going to show you a much easier (at least for me) way of doing implicit differentiation. Don't worry too much about the theory, unless you are really curious.