Calculus 251:C3 Reading Guide - 6/8/2021

Section 13.3 Arc Length in Space

Again, this section is a generalization of something you saw in calculus 2 (or AP Calculus BC). When you considered the arc length of a parameterized curve in $\mathbb{R}^2$, you saw the formula $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$. Much of the time, this formula produced an integrand which you did not have the techniques to integrate by hand, and that will be true here as well. The examples which you could integrate were usually specifically designed to have a "nice" integrand. As usual, this section will only discuss the situation in $\mathbb{R}^3$, but the techniques presented here generalize to dimensions beyond 3 as well.

Find the following definitions/concepts/formulas/theorems:

- Arc length in $\mathbb{R}^3$ (two different forms - one useful, one compact)
- arc length parameter
- arc length parametrization
- speed (for a vector-valued function)
- unit tangent vector

One important point that I do not see explicitly mentioned in the textbook is that arc length is independent of parametrization. We have seen that you can parameterize curves in multiple ways (starting with multiple ways to parameterize lines). If you start at a particular point on a curve and travel to a different point on the curve, you will get the same arc length regardless of the parametrization you chose for the curve. It makes sense that this has to be true geometrically. The reason that it works algebraically is that if you have two different parametrizations, your limits of integration will be different in a way that offsets the change in the integrand.

Example 1 is a standard calculation and you should work through it.

The next couple of paragraphs and Example 2 are about arc length parametrization. The way to visualize what is happening here is to picture yourself standing on the curve at some starting point at $t = 0$. You walk along the curve **at the constant speed of 1 (distance unit)/(time unit)**. The arc length parametrization is the unique parametrization which expresses your position on the curve as a function of time. As the book states, we can usually not evaluate the arc length integral, therefore we can usually not produce an explicit arc length parametrization. There are certain "nice" cases in which we can, and example 2 shows one of them.

Example 3 is important if you are going to do lots of physics problems about motion in space. If you really want to get deeper into what is going on here, you would need to talk about the Frenet-Serret frame (a/k/a TNB frame) which is covered in Math 291 but is beyond the scope of this course. Some of the groundwork for Frenet-Serret is in section 13.4 (see below). If you are really interested, take a look at the Wikipedia page as a starting point.

Section 13.4 Curvature and Normal Vectors of a Curve

This section is omitted in the department syllabus, and we are in fact going to omit it because none of the subsequent material depends on it. If you are going to be a mechanical engineer or a physicist, you will need this material eventually. If either of those applies to you or you are just curious, you are welcome to peruse this section either now or in late July when this course is over and you are missing multivariable calculus deeply. If you want some idea of what a lecture on this topic would look like, I will refer you to Dr. G.'s website.