Calculus 251:C3 Reading Guide - 7/20/2021

Section 16.8 The Divergence Theorem and a Unified Theory

Well, this is it. You have made it to the last section of the textbook! This section is about the Divergence Theorem, the last of the Fundamental Theorems of vector calculus. It establishes the equivalence of the flux integral of a vector field through a closed surface with the triple integral of the divergence of the vector field over the enclosed region. This is the generalization of the Divergence-Flux form of Green's Theorem to $\mathbb{R}^3$. If you think about divergence in terms of sources and sinks, then you could say that divergence at a point measures how much the field represents a flow into or out of that point. Then what the Divergence Theorem is really saying is that if you add up all of the sources and sinks in a region, that tells you what the net flow across the boundary of the region has to be.

Find the following definitions/concepts/formulas/theorems:

- divergence (which we already discussed in lecture)
- Theorem: Divergence Theorem

Yes, that's really all you need to find. The rest of the section consists of examples, explanations, and extensions.

Example 1 calculates the divergence of various vector fields representing fluid flows. You should work through these to develop your intuition about what divergence really means.

Example 2 verifies the Divergence Theorem by computing the integrals on both sides of the theorem directly. The corollary after this theorem might be obvious to you, but it comes in handy sometimes. Example 3 uses the Divergence Theorem to calculate the flux out of a box. This is defintely less work than integrating across each of the six sides of the box. In example 4, they are again computing the flux out of a box, but this time they perform the calculation directly and by using the Divergence Theorem.

Theorem 9 (and its proof) are not bad. I suggest reading through the proof only because it is very straigtforward, and makes the result much more believable.

As with the first two major theorems in this chapter, the textbook proves a special case of the Divergence Theorem. They only prove the theorem for a simple convex solid, but you can write any region as the limit of some sum of very small simple convex solids as the volume of the largest solid tends to zero. So other than some technical details of that limit, this is really what the proof of the theorem looks like in the general case. If you want to be a math major/minor, you should certainly spend time on this.

The discussion of the Divergence Theorem for other regions and example 5 are somewhat illiminating. There will be times when you have a vector field whose divergence is simple, even though the field is not. Being able to apply the theorem to oddly-shaped regions gives you more options. Sometimes, we will even be able to use the theorem to calculate the flux through a surface that isn't closed, just like we were able to apply Green's Theorem to an "almost closed" curve. It is often better to do two easy calculations than one terrible one.

The subsections on Gauss's Law and Hydrodynamics are important. You should at least skim through them. And then you should look at it again sometime between the end of this course and the start of the fall semester. Again, physics is the main reason this chapter exists. If you are going to be taking physics and engineering courses, you will eventually care about these results.

The last subsection talks about unifying the integral theorems (Green's/Stokes'/Divergence). At least read through this part even if you don't fully understand it. Knowing that there is a unifying theme can be comforting, and at this point you have earned a little comfort. And then you're done!