Calculus 251:C3 Reading Guide - 7/19/2021

Section 16.7 Stokes' Theorem

This section focuses on Stokes' Theorem, a generalization of the Circulation-Curl form of Green's Theorem to $\mathbb{R}^3$. It establishes the equivalence of the flux integral of the curl of a vector field through a surface with the vector line integral of the field along the boundary of the surface. A very important specific case is that for a *closed* surface, the flux integral of the curl will always be 0 (as long as $\vec{F}$ is reasonably well-behaved).

Find the following definitions/concepts/formulas/theorems:

- del operator (which we already discussed in lecture)
- curl in $\mathbb{R}^3$ (which we already discussed in lecture)
- piecewise smooth (for a surface)
- Theorem: Stokes' Theorem
- boundary orientation (for the boundary of a surface, not a plane region as in §17.1)
- identity about the curl of a gradient field (how does this relate to our discussion of conservative vector fields?)
- Theorem related to closed-loop property (how does this relate to our discussion of conservative vector fields?)

The start of the section introduces the del operator ∇ and the curl operation on a vector field, both of which we have discussed in lecture. Example 1 is a curl calculation which you should already know how to do, but review it to make sure.

Examples 2 and 3 verifiy Stokes' Theorem for a combination of $\vec{F}$ and $\mathcal{S}$ by computing the integrals on both sides of the theorem directly. Example 3 illustrates one of the applications of Stokes' Theorem: if you are trying to compute the flux integral of a curl through a surface, you can use any surface which has the same boundary curve. In order to be able to use this trick for some arbitrary vector field $\vec{F}$, you would need to know that $\vec{F}$ can be written as the curl of some vector field, i.e. there is a field $\vec{G}$ such that $\vec{F} = \vec{\nabla} \times \vec{G}$. While this is beyond the scope of the course, it is cool enough that we will talk about it in lecture.

Examples 4 and 5 show how to use Stokes' Theorem to find a circulation. Note that since we are finding the circulation, we can use any surface that has that boundary without worrying about finding a vector potential. That is because we are taking the curl of the given field to find the flux of the curl, rather than trying to find the flux of the field. Think about it a little

Examples 6 and 7 are further examples of verifying the theorem and calculating a circulation.

The discussion of the paddle-wheel interpretation of curl is a slightly more formal discussion than the statements we made in class about curl being a measure of how "spinny" a field is, but it is the same general idea.

Examples 8, 9, and 10 are further examples of how to apply the theorem to a variety of situations. None of these examples are terribly difficult, and you should try to work through them.

The proof outline (of a special type of surface) is not a bad read, but you can skip it if you are not interested. The paragraph about Stokes' Theorem for surfaces with holes should remind you af our discussion of Green's Theorem for regions that are not simply connected. The rest of the section is a discussion which relates curl and Stokes' Theorem back to the section on conservative vector fields.