Calculus 251:C3 Reading Guide - 7/12/2021

Section 16.4 Green's Theorem in the Plane

This section and sections §16.7 and §16.8 are about several extensions of the Fundamental Theorem of Calculus to contexts involving multiple integrals. In single-variable calculus, we did not want to compute every integral as the limit of Riemann sums, so instead we used the FTC to evaluate integrals using antiderivatives. Similarly, while we have developed a number of techniques for computing multiple integrals, line integrals, and surface integrals, we always want to have options which will simplify our computations.

This section focuses on Green's Theorem, which only applies to functions in the plane. It establishes the equivalence of the double integral of some functions over a region in the plane with the line integral along the (closed) boundary of the region.

Find the following definitions/concepts/formulas/theorems:

- circulation density (this is just the $\hat{\mathbf{k}}$-component of the curl)
- divergence a/k/a flux density (defined here only for $\mathbb{R}^2$)
- incompressible
- simple closed curve
- boundary orientation conventions
- Theorem: Circulation-Curl Form of Green's Theorem
- Flux-Divergence Form of Green's Theorem
- additivity of circulation
- Green's Theorem for more general regions (discussed, but not stated as a theorem)

You should certainly read the two pages of material before the first example and the two pages between examples 1 and 2. The more ways you think about what is going on in vector calculus, the deeper your understanding of the topic will be. The general discussion of curl and divergence, while restricted to the plane, can be summed up as follows: curl is how "spinny" the field is at any point, and divergence is how "stretchy" the field is at any point. Examples 1 and 2 ask you to analyze the curl and divergence of a variety of vector fields. These will definitely help your intuition about what these operations are all about.

Example 3 verifies both forms of Green's Theorem for a particular vector field and closed curve. The first couple of homework problems will look like this one.

Both forms of the theorem are important, if only because they foreshadow things to come at the end of the course. Stokes' Theorem in §16.7 is a generalization of the Circulation-Curl Form to $\mathbb{R}^3$ (with the boundary and surface still piecewise-smooth but not restricted to the plane). The Flux-Divergence form of the theorem generalizes to the Divergence Theorem in $\mathbb{R}^3$ (with closed surfaces and triple integrals replacing closed curves and double integrals).

Better to think about why they work in two dimensions first.

Examples 4 and 5 are the reason that we bother with this theorem. The general idea is that we get to compute one double integral instead of having to compute four line integrals. There are certainly other cases where it is easier to use Green's Theorem than to compute the circulation directly, but these two are just doing less work for the same result. There might also be times where a double-integral would be disgusting, but we could perform an easier calculation to find the circulation. We may discuss this situation when we talk about Stokes' Theorem later.

The proof of Green's Theorem isn't bad, because they chose to prove a special case where $\mathcal{R}$ is both horizontally and vertically simple. This means that you can write the boundary of $\mathcal{R}$ as two functions of either x or y by breaking the curve at the two corresponding points of tangency.

The more general form of the theorem (which they allude to but do not state in the final paragraph of the section) allows us to use it on domains with holes, as long as those holes are bounded by simple closed curves. The general idea is that you slice the domain into smaller domains, each of which is bounded by a simple closed curve. Green's Theorem then applies to each of the pieces. Then when you reassemble the pieces, you see that for each of the cuts you made you integrated across that curve once in each direction. Since changing direction changes the sign of your line integral, those two parts of the integral have to add up to zero. After you put everything back together, the only pieces of line integrals that didn't cancel out are the curves that formed the boundary of the domain you started with. It's a neat trick.