Calculus 251:C3 Reading Guide - 6/2/2021

Section 12.3 The Dot Product

The main idea of this section is the introduction of another operation we can perform on vectors, namely the dot product. One very important note is that the dot product of two vectors is a *scalar*, not a vector. In slightly more formal language, the dot product is a function which maps $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$.

Find the following definitions/formulas/theorems:

- Angle between vectors
- Dot product (note that the definition assumes that we are working in $\mathbb{R}^3$, and below the defintion they define dot product in $\mathbb{R}^2$. The dot product is defined the same way for any number of dimensions. What is the dot product in $\mathbb{R}^1$? What is it in $\mathbb{R}^5$?)
- Relationship between dot product and angle (also read the proof of Theorem 1 if you need to be convinced or are curious)
- orthogonal (a/k/a perpendicular)
- Properties of the Dot Product (a set of 5 useful identities)
- conditions for an angle to be obtuse or acute
- projection of $\vec{u}$ along $\vec{v}$
- component of $\vec{u}$ along $\vec{v}$ (a/k/a scalar component of $\vec{u}$ along $\vec{v}$)
- decomposition of $\vec{u}$ with respect to $\vec{v}$
- Work done by a force

Example 1 is a pair of simple computations of dot products. Read the proof of the Theorem that follows it if you're interested. Many times in this course, it will be useful to understand the relationship between the symbolic and geometric forms of a concept, and this is one of those connections.

Examples 2 and 3 are useful examples of how we use the relationship between dot product and angle. Example 4 demonstrates how to check for orthogonality. You should certainly expect questions like these to come up in homework, quizzes, and exams, possibly as one of the steps of a larger problem.

Examples 5, 6, and 7. are about projections and decompositions. The first two are just computations and the last one is really a proof that vector decomposition works as we think it should. You should spend some time making sure that these make sense to you.

Example 8 is a simple work computation. The concept is important, though. When we talk about line integrals in chapter 16, we will often be thinking about the work exerted on a particle by the force field through which it is travelling. You can think of this example as the boring case of straight-line movement through a constant field. That description may not make sense to you if you haven't done much physics, but it will later.

Section 12.4 The Cross Product

Two important notes: First, the cross product of two vectors is a vector (remember that the dot product is a scalar). Second, while the dot product makes sense in any number of dimensions, the cross product is only defined in exactly 3 dimensions. Okay, that was actually a little white lie. The cross product makes sense in 0, 1, 3, or 7 dimensions. But in 0 or 1 dimension the cross product is always the zero vector. In 7 dimensions, there are actually 480 different possible cross products. So we are going to pretend that the cross product only exists in $\mathbb{R}^3$ where it is in fact unique. I'm sure that there are people out there who understand 7-dimensional cross products and what they might be used for, but I assure you that I am not one of those people. Please don't ask me.

One slightly less important note: recall the right hand rule from section 12.1. You are going to need it.

Find the following definitions/formulas/theorems:

- Cross product (the formula is really the definition)
- Parallel vectors (how does this relate to our prior definition of parallel vectors?)
- Basic properties of the cross product (there are 6)
- Property 3 is often called the anticommutative property. Why?
- cross products of standard basis vectors
- What does it mean for vectors to "span" a shape?
- Area of parallelogram spanned by $\vec{v}, \vec{w}$
- Area of triangle spanned by $\vec{v}, \vec{w}$
- determinant of a 2×2 matrix (in the margin comments, not in the main text)
- determinant of a 3×3 matrix
- torque (a physics concept useful for visualizing why cross products are important)
- parallelepiped
- scalar triple product (a/k/a box product)
- Volume of parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$

Example 1 is a demonstration of how to compute $\vec{v} \times \vec{w}$. Note that the top row is always $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. The second row is always $\vec{v}$. If you switch the second and third rows, you will end up with $-\vec{v} \times \vec{w}$, which is the same as $\vec{w} \times \vec{v}$. You might even want to try that on this example to confirm what I'm telling you.

Examples 2, 3, and 4 use the cross product to solve geometry problems. Make sure you understand these.

Example 5 is a simpe torque computation. I don't think that this plug-and-chug example adds much to the previous discussion of torque.

Example 6 is a straightforward volume computation. Doing it is easy. Try to convince yourself why this calculation works. That's much more meaningful.