Calculus 251:C3 Reading Guide - 7/7/2021

Section 16.2 Vector Fields and Line Integrals; Work, Circulation, and Flux

This section introduces the concept of a vector field. We have actually already seen one type of vector field (although we didn't call it that), the gradient. Recall that the gradient of a function is the vector of its partial derivatives, so if you evaluate the gradient at a particular point you get back a vector representing the rate of change at that point. The main idea is that a vector field $\vec{F}$ is a function $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$, that is a function which has the same vector space as its domain and codomain. You can visualize a vector field as a collection of vectors with one attached to each point in the plane or in space, perhaps representing the velocity of a fluid at that position.

The other main topic of this section is (vector) line integrals. It might be appropriate to call these "curve integrals" or "path integrals" instead. But the same was true of contour lines, which might more accurately be called contour curves. There are two types, scalar line integrals (which were covered in §16.1) and vector line integrals, the difference being the type of function you are integrating. The general intuition is that you are moving along some path in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and you are adding up the function values along that path. In the case of a scalar line integral, you were adding up the function values (so very much like an integral from Calc II, except that you are summing along a path instead of just an interval). In the case of a vector field, you are summing the tangential components along the path. The way to think about this is that at each point (P) on the path you are only really interested in the projection of the value of $\vec{F}(P)$ onto the direction vector, because that is the part of the vector field in the direction you are moving.

Note on notation: $\oint_{\mathcal{C}}$ really means the same thing as $\int_{\mathcal{C}}$. The circle on the first one is just an indication that $\mathcal{C}$ is a closed curve, but doesn't affect how you compute it.

Find (or figure out) the following definitions/concepts/formulas/theorems:

- vector field
- domain/codmain (for a vector field - basically the same as for any other function)
- component function
- differentiable (for a vector field - basically the same as for any other function)
- radial vector field (What should the definitions of "constant vector field" and "unit vector field" be?)
- gradient field
- orientation/oriented curve (and forward/backward a/k/a positive/negative in this context)
- tangential component
- line integral definition/formula/process for computing

- formula for line integrals with respect to dx, dy, dz
- Work done by a force (concept/definition/formulas
- flow along curve, circulation
- simple curve
- closed curve
- flux across a curve (somewhat interesting in its own right, but flux across a surface is the central concept in §16.6, so this is your chance to think about it in two dimensions)

Example 1 just serves to back up my statement above that the gradient of a multivariable function is a vector field. Nothing new here.

Example 2 is a basic computation of a (vector) line integral. You should be sure to understand the process here. Example 3 is a similar computation, but starting with the scalar differential form of the integral. Note that we just use the parametrization of the curve to rewrite the field as a vector-valued function of the parameter.

Examples 4 and 5 and the discussion before them are all about work calculations, but they also further illustrate the general technique of computing line integrals. Work is the prototypical example of "When are we ever going to use vector line integrals?" in much the same way as mass of a wire is the prototypical example for scalar line integrals.

Examples 6 and 7 just do all of the same things in the context of flows. Again, the principal value here may just be additional examples of the computation process.

Example 8 is the only one which is fundamentally different. Notice that the flux across the curve uses a normal vector, so we are integrating the normal component instead of the tangential component. Think about it this way: if the field is a fluid flow, the flow integral calculates flow *along* the curve while the flux integral calculates flow *across* the curve. You should try to follow through this example, because the concept of flux will return when we study Green's Theorem (§16.4) and surface integrals (§16.6).