Calculus 251:C3 Reading Guide - 6/23/2021

Section 15.1 Double and Iterated Integrals over Rectangles

In chapter 15, we will be studying multiple integration. The first step in that process is to understand the double integral. In single-variable calculus, you learned that the integral of $y = f(x)$ represents the area between the graph of the function and the x-axis. If the area was above the axis, we counted it as positive area. If the area was below the axis, we counted it as negative. Our definite integrals were written as $\int_a^b f(x)dx$, a notation that meant that we were integrating the function with respect to x on the interval $[a, b]$. Really this came down to the area bounded by the lines $x = a$, $x = b$, and $y = 0$ and the curve $y = f(x)$ with positive/negative considerations discussed above.

Now, our functions are of the form $z = f(x, y)$ and their graphs live in $\mathbb{R}^3$. When we integrate a function, we will need to look at a domain which is a subset of $\mathbb{R}^2$, which we understand to be the xy-coordinate plane. Our integrals may be written as $\int_a^b \int_c^d f(x, y)\ dy\ dx$, which means we are integrating with respect to y on $[c, d]$ and then integrating again with respect to x on $[a, b]$. What this value represents is the *volume* above the rectangle in the xy-coordinate plane and below the function $z = f(x, y)$. Again, we will consider volume below the xy-coordinate plane to be negative volume, analagously to the single-variable case. And you can think about this volume as being bounded by five planes (which ones?) and the surface $z = f(x, y)$. Most of the rest of the section is a discussion of why this all makes sense and details of how we do it.

Note: This section does introduce the notation $\iint_{\mathcal{R}} f(x, y)\ dA$ with $\mathcal{R}$ denoting a region in the domain, but we will only deal with integrating over rectangles. In the next section, we will see how to integrate over domains of other shapes.

Find the following definitions/concepts/formulas/theorems:

- partition (you should have seen this when you defined the integral in calc 1)
- Riemann sum (same general idea as single-variable, but our little intervals are now little rectangles)
- limit of Riemann sums
- norm (of a partition)
- integrable (essentially the same as single-variable)
- double integral
- area element (dA)
- Continuous functions are integrable
- iterated integral

- Theorem: Fubini's Theorem (THIS IS HUGE, arguably one of the most important theorems in all of mathematics)
- Double integral over a rectangle

The first few pages are all about how we extend our understanding of integration and Riemann sums to functions of two variables. There is a lot of notation, but it is really close enough to the single-variable case that it should be readable.

The fact that there are only two exaples in this section makes me very sad. We will do many more in lecture in recitation. The two examples are very basic, and you should not have any trouble working through them. Notice that integration of a multivariable expression with respect to one of its variables treats the other one as a constant. This is for all of the same reasons as why we take partial derivatives the way that we do, and it should not come as a great shock to you.

Section 15.2 Double Integrals over More General Regions

The bad news is this is a very long section. The good news is that it there are more examples than in the previous section. The reason there are so many examples is because the domains over which we are integrating can be many different shapes in RR^2. The main idea is that you want to set up an iterated integral as we did for rectangles, but the limits of integration for the inner integral will usually be in terms of the other variable. Honestly, you jet need to look at lots of examples to see how that works in practice.

I don't like the way the book presents this material. I am going to use what I think is clearer terminology in the lecture. You should muddle through this section as best you can, paying particular attention to the examples.

Find the following definitions/concepts/formulas/theorems:

- Fubini's Theorem (a stronger form which applies to non-rectangular regions)
- vertical/horizontal cross-sections (I will introduce the terms horizontally/vertically simple in lecture)
- Properties of Double Integrals (none of which should surprise you)

The first couple of pages deal with how we generalize domains and what properties they need to have. Not critically important to understanding what follows, but you should take a quick look to get a sense of why we can use the techniques from the previous section.

Example 1 is a basic example of integration over a triangular domain. They calculate this integral in both of the possible orders to provide evidence that Fubini's Theorem works. Notice that when the inner integral is dy, that we have an x in the upper limit of integration. When the inner integral is dx, we have a y in thr lower limit of integration. In general, you can only have variables in your limits of integration which are NOT the variable you are currently integrating with respect to but ARE variables that you will be integrating with respect to later in the iterated integral. This is true for triple integrals as well.

Example 2 is important because the order of integration matters. The choice is forced because if you try to integrate with respect to x first, the integrand is unmanageable. When we integrate with respect to y first, something good happens.

Example 3 just asks us to switch the order of integration. The technique here is that you draw the region of integration, rewrite the boundary curves so that they are functions of the other variable (recall inverse functions from high-school algebra), then use those new functions as the limits of integration. For practice, evaluate both of these integrals to make sure the book isn't lying to you about the value.

Example 4 is interesting mostly for the setup of the integral. The evaluation is annoying, and the fraction arithmetic is worse. Look at it, but don't worry about it too much.