Calculus 251:C3 Quiz #24 - 7/19/2021 Topic: Section 16.6

Instructions. Answer the questions in the spaces provided or on your own paper, then scan and upload to Canvas. Show and label all of your work. Responses with no work may receive no credit even if the answer is correct.

$\boxed{\text{10 pts}}$ (1) Let $\mathcal{S}$ be the part of the surface $x^2 + y^2 + z^2 = 25$ above the plane $z = 3$ oriented with outward-pointing normal. Let $\vec{F} = \langle z, z, 2\rangle$. Compute the flux of $\vec{F}$ through $\mathcal{S}$.

Hint: You can reasonably do this question using any of the three typical parametrizations. One of them has a complication in the setup, one has a complication in the integration, but any of them should work. Remember what I said about changing coordinate systems after computing the normal should you decide to do that!

Solution: The surface across which we are integrating is the part of the sphere above the plane. The intersection curve is the circle of radius 4 in the plane $z = 3$ centered at $(0, 0, 3)$. That circle lies above the disk in the xy-plane of radius 4 centered at the origin. As mentioned in the hint, we can parametrize in several different ways.

1. $G(x, y) = (x, y, \sqrt{25 - x^2 - y^2})$, $-4 \le x \le 4$, $-\sqrt{16 - x^2} \le y \le \sqrt{16 - x^2}$

$$\vec{T}_x = \left\langle 1, 0, \frac{-x}{\sqrt{25 - x^2 - y^2}} \right\rangle$$

$$\vec{T}_y = \left\langle 0, 1, \frac{-y}{\sqrt{25 - x^2 - y^2}} \right\rangle$$

$$\vec{N} = \vec{T}_x \times \vec{T}_y = \left\langle \frac{x}{\sqrt{25 - x^2 - y^2}}, \frac{y}{\sqrt{25 - x^2 - y^2}}, 1 \right\rangle$$

Check: yes, this normal points outward

$$\vec{F}(G(x, y)) = \left\langle \sqrt{25 - x^2 - y^2}, \sqrt{25 - x^2 - y^2}, 2 \right\rangle$$

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \vec{F}(G(x, y)) \cdot \vec{N}\, dy\, dx = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (2 + x + y)\, dy\, dx$$

At this point, it seems reasonable to switch the integral to polar. Remember that since we are doing this after computing the normal, we will need to multiply in the r for the Jacobian.

$$= \int_0^4 \int_0^{2\pi} (2 + r^2 \cos\theta + r^2 \sin\theta) r\, d\theta\, dr = \int_0^4 \left(2r\theta + r^2 \sin\theta - r^2 \cos\theta\right)\Big|_{\theta=0}^{\theta=2\pi} dr$$

$$= \int_0^4 4\pi r\, dr = 2\pi r^2 \Big|_{r=0}^{r=4} = 32\pi$$

2. $G(r,\theta) = (r\cos\theta, r\sin\theta, \sqrt{25-r^2})$, $0 \le r \le 4$, $0 \le \theta \le 2\pi$

$$\vec{T}_r = \left\langle \cos\theta, \sin\theta, \frac{-r}{\sqrt{25-r^2}} \right\rangle$$
$$\vec{T}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$$
$$\vec{N} = \vec{T}_r \times \vec{T}_\theta = \left\langle \frac{r^2\cos\theta}{\sqrt{25-r^2}}, \frac{r^2\sin\theta}{\sqrt{25-r^2}}, r \right\rangle$$

Check: yes, this normal points outward

$$\vec{F}(G(r,\theta)) = \left\langle \sqrt{25-r^2}, \sqrt{25-r^2}, 2 \right\rangle$$

$$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \int_0^4 \int_0^{2\pi} \vec{F}(G(r,\theta))\cdot \vec{N}\; d\theta\; dr = \int_0^4 \int_0^{2\pi} \left(r^2\cos\theta + r^2\sin\theta + 2r\right)\; d\theta\; dr$$

But this is precisely the integral we had above after the conversion to polar, so again the value is 32π.

3. $G(\theta,\phi) = (5\cos\theta\sin\phi, 5\sin\theta\sin\phi, 5\cos\phi)$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \tan^{-1}(4/3)$

The good news is that we know that $\cos(\phi_{max}) = 3/5$ and $\sin(\phi_{max}) = 4/5$. The bad news is that this parametrization is going to be more work than the other two, it seems.

$$\vec{T}_\theta = \langle -5\sin\theta\sin\phi, 5\cos\theta\sin\phi, 0\rangle$$
$$\vec{T}_\phi = \langle 5\cos\theta\cos\phi, 5\sin\theta\cos\phi, -5\sin\phi\rangle$$
$$\vec{N} = \vec{T}_\theta \times \vec{T}_\phi = \left\langle -25\cos\theta\sin^2\phi, -25\sin\theta\sin^2\phi, -25\sin\phi\cos\phi \right\rangle$$

Check: no, this normal points inward, so instead we will use

$$\vec{N} = \vec{T}_\phi \times \vec{T}_\theta = \left\langle 25\cos\theta\sin^2\phi, 25\sin\theta\sin^2\phi, 25\sin\phi\cos\phi \right\rangle$$

$$\vec{F}(G(r,\theta)) = \langle 5\cos\phi, 5\cos\phi, 2\rangle$$

$$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \int_0^{\tan^{-1}\frac{4}{3}} \int_0^{2\pi} \vec{F}(G(r,\theta))\cdot\vec{N}\; d\theta\; d\phi$$

$$= \int_0^{\tan^{-1}\frac{4}{3}} \int_0^{2\pi} 125\cos\theta\cos\phi\sin^2\phi + 125\sin\theta\cos\phi\sin^2\phi + 50\sin\phi\cos\phi\; d\theta\; d\phi$$

A bit of good news, finally: we know that the first two terms will zero out when we integrate θ from 0 to 2π first, so:

$$= \int_0^{\tan^{-1}\frac{4}{3}} 100\pi\sin\phi\cos\phi\; d\phi = 50\pi\sin^2\phi\Big|_0^{\tan^{-1}\frac{4}{3}} = 50\pi\left(\frac{4}{5}\right)^2 = 32\pi$$