Difficulty guide for worksheet:	
C-level or B-level exam problem:	1a, 1b, 1c, 1d, 1e, 1g
A-level exam problem or challenge for extra study:	1f
beyond the scope and/or removed from syllabus:	2

1. For each vector field $\boldsymbol{F}$ and oriented surface $\mathcal{S}$, calculate the flux integral $\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S}$.

(a) $\boldsymbol{F} = \langle e^z,\ z,\ x\rangle$; $\mathcal{S}$ is parametrized by $G(u,v) = (uv,\ u+v,\ u)$ on the domain $[0,1]\times[0,1]$

(b) $\boldsymbol{F} = \langle x,\ x,\ y\rangle$; $\mathcal{S}$ is the triangle with vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$, and oriented upward

(c) $\boldsymbol{F} = \langle z^2,\ x,\ -3z\rangle$; $\mathcal{S}$ is the portion of the graph of $z = 4-y^2$ cut by the planes $x=0$, $x=1$, and $z=0$, and oriented by a normal vector pointing away from the x-axis

(d) $\boldsymbol{F} = \langle -x,\ -y,\ z^2\rangle$; $\mathcal{S}$ is the portion of the cone $z=\sqrt{x^2+y^2}$ between the planes $z=1$ and $z=2$, and oriented by a normal vector pointing away from the z-axis

(e) $\boldsymbol{F} = \langle xz,\ yz,\ 1\rangle$; $\mathcal{S}$ is the upper cap of the sphere $x^2+y^2+z^2=25$ cut by the plane $z=3$, and oriented inward

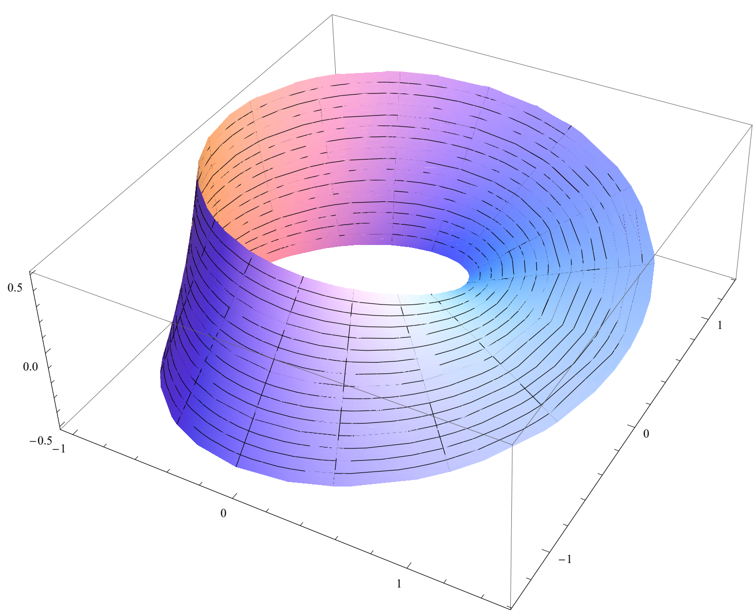
(f) $\boldsymbol{F} = \langle 2xy,\ 2yz,\ 2xz\rangle$; $\mathcal{S}$ is the boundary of the cube $[0,a]\times[0,b]\times[0,c]$ with $a,b,c>0$, and oriented outward

(g) $\boldsymbol{F} = \langle 4x,\ 4y,\ 2\rangle$; $\mathcal{S}$ is the bottom portion of the paraboloid $z=x^2+y^2$ cut by the plane $z=3$, and oriented downward

2. Let $\mathcal{S}$ be the surface with parametrization

$$G(u,v) = \big((1+v\cos(u/2))\cos(u),\ (1+v\cos(u/2))\sin(u),\ v\sin(u/2)\big)$$

with $0\le u\le 2\pi$ and $-\frac{1}{2}\le v\le\frac{1}{2}$. A graph of $\mathcal{S}$ is shown below.



This surface is known as a Möbius strip and is an example of a non-orientable surface. This roughly means that it is not possible to choose an “inward side” and an “outward side” for this surface. Indeed, this surface seems to have only one side!

(a) The intersection of $\mathcal{S}$ with the xy-plane is the unit circle $G(u,0) = (\cos(u),\ \sin(u)\ 0)$. Show that the normal vector along this circle is

$$\boldsymbol{N}(u,0) = \langle \cos(u)\sin(u/2),\ \sin(u)\sin(u/2),\ -\cos(u/2)\rangle$$

(b) Show that $\boldsymbol{N}(u,0)$ is a unit vector that varies continuously for $0 < u < 2\pi$ but $\boldsymbol{N}(2\pi,0) = -\boldsymbol{N}(0,0)$.

Remark: The result of part (b) shows that $\mathcal{S}$ is not orientable. That is, it is not possible to choose a nonzero normal vector at each point on $\mathcal{S}$ that varies continuously on all of $\mathcal{S}$. If this were possible, the unit normal vector $\boldsymbol{N}(u,0)$ would return to itself rather than to its negative when carried around the unit circle.

Since $\mathcal{S}$ is not orientable, the flux integral $\displaystyle\iint_{\mathcal{S}} \boldsymbol{F}\cdot d\boldsymbol{S}$ cannot be defined. Now although we cannot integrate vector fields over $\mathcal{S}$, we can integrate scalar fields over $\mathcal{S}$ since the scalar surface integral $\displaystyle\iint_{\mathcal{S}} f\ dS$ is defined irrespective of orientation.

(c) Verify that

$$\|\boldsymbol{N}(u,v)\|^2 = 1 + \frac{3}{4}v^2 + 2v\cos(u/2) + \frac{1}{2}v^2\cos(u)$$

(d) Calculate the surface area of $\mathcal{S}$ to four decimal places. (You will need a computer algebra system for this.)