Difficulty guide for worksheet:	
C-level or B-level exam problem:	1, 2, 3, 4, 5, 6, 7, 8
A-level exam problem or challenge for extra study:	none
beyond the scope and/or removed from syllabus:	none

1. For each pair of vectors, calculate both the dot product $\boldsymbol{u} \cdot \boldsymbol{v}$ and the cross product $\boldsymbol{u} \times \boldsymbol{v}$.

(a) $\boldsymbol{u} = \langle 1, 2, 1\rangle$ and $\boldsymbol{v} = \langle -3, 2, 4\rangle$

(b) $\boldsymbol{u} = \boldsymbol{j}$ and $\boldsymbol{v} = \boldsymbol{k}$

(c) $\boldsymbol{u} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{j}$ and $\boldsymbol{v} = -\boldsymbol{i} + \boldsymbol{j}$

2. Find the sine and cosine of the angle between each pair of vectors. Then determine whether the angle between the two vectors is acute, right, or obtuse.

(a) $\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

(b) $\langle 2, 3, -1\rangle$ and $\langle -4, -6, 2\rangle$

(c) $\boldsymbol{i} + \boldsymbol{k}$ and $\boldsymbol{i} - \boldsymbol{j}$

3. Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal with $\|\boldsymbol{u}\| = 2$ and $\|\boldsymbol{v}\| = 5$. Calculate $\|\boldsymbol{u} + \boldsymbol{v}\|$.

4. Suppose the angle between the unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is 120 degrees. Calculate the following.

(a) $\boldsymbol{u} \cdot \boldsymbol{v}$

(b) $\|\boldsymbol{u} - 2\boldsymbol{v}\|$

5. For each pair of vectors, find the projection of $\boldsymbol{v}$ along $\boldsymbol{u}$.

(a) $\boldsymbol{v} = \langle 3, -2, 1\rangle$ along $\boldsymbol{u} = \boldsymbol{j}$

(b) $\boldsymbol{v} = 2\boldsymbol{i} - \boldsymbol{j} + 6\boldsymbol{k}$ along $\boldsymbol{u} = \boldsymbol{i} + \boldsymbol{k}$

(c) $\boldsymbol{v} = 5\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$ along $\boldsymbol{u} = \langle 1, 1, -1\rangle$

6. Let $\boldsymbol{u} = \lambda\boldsymbol{i} - 2\lambda\boldsymbol{j} + \mu\boldsymbol{k}$ and $\boldsymbol{v} = 5\boldsymbol{i} - \mu\boldsymbol{j} + \lambda\boldsymbol{k}$, where λ and μ are unknown constants.

(a) Find all pairs (λ, μ) such that $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, or determine that no such pair exists.

(b) Find all pairs (λ, μ) such that $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, or determine that no such pair exists.

7. Find the area of the triangle spanned by the vectors $\boldsymbol{u} = 2\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{v} = \boldsymbol{i} + 4\boldsymbol{j}$.

8. Calculate the following determinants. Fully simplify your answer.

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 2 & -3 \\ 4 & -2 & 1 \end{vmatrix}, \quad \begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix}, \quad \begin{vmatrix} \sin(\theta)\cos(\varphi) & \rho\cos(\theta)\cos(\varphi) & -\rho\sin(\theta)\sin(\varphi) \\ \sin(\theta)\sin(\varphi) & \rho\cos(\theta)\sin(\varphi) & \rho\sin(\theta)\cos(\varphi) \\ \cos(\theta) & -\rho\sin(\theta) & 0 \end{vmatrix}$$

Solutions

(1) (a) $\vec{u} \cdot \vec{v} = 5$

$\vec{u} \times \vec{v} = 6\hat{\imath} - 7\hat{\jmath} + 8\hat{k}$

(b) $\vec{u} \cdot \vec{v} = 0$

$\vec{u} \times \vec{v} = \hat{\imath}$

(c) $\vec{u} \cdot \vec{v} = -5$

$\vec{u} \times \vec{v} = -\hat{\imath} - \hat{\jmath} - \hat{k}$

(2) (a) $\cos(\theta) = \dfrac{-8}{\sqrt{180}}$ obtuse

$\sin(\theta) = \dfrac{\sqrt{116}}{\sqrt{180}}$

(b) $\cos(\theta) = -1$ obtuse $(180°)$

$\sin(\theta) = 0$

(c) $\cos(\theta) = \dfrac{1}{2}$ acute

$\sin(\theta) = \dfrac{\sqrt{3}}{2}$

(3) $\|\vec{u} + \vec{v}\| = \sqrt{29}$

④ (a) $\vec{u}\cdot\vec{v} = \cos(120°) = -\frac{1}{2}$

(b) $\|\vec{u}-2\vec{v}\|^2 = \|\vec{u}\|^2 - 4\,\vec{u}\cdot\vec{v} + 4\,\|\vec{v}\|^2$

$= 1 - 4\left(-\frac{1}{2}\right) + 4\cdot 1 = 7$

$\Longrightarrow \|\vec{u}-2\vec{v}\| = \sqrt{7}$

⑤ (a) $-2\hat{j}$

(b) $4\hat{i} + 4\hat{k}$

(c) $4\hat{i} + 4\hat{j} - 4\hat{k}$

⑥ (a) $0 = \vec{u}\cdot\vec{v} = 5\lambda + 2\mu\lambda + \mu\lambda = \lambda(5+3\mu)$

$\lambda = 0$ <u>or</u> $\mu = -5/3$

(b) $\vec{0} = \vec{u}\times\vec{v} =$

$= (-2\lambda^2+\mu^2)\hat{i} - (\lambda^2-5\mu)\hat{j} + (-\lambda\mu+10\lambda)\hat{k}$

$$\left.\begin{array}{r} -2\lambda^2+\mu^2=0 \\ \lambda^2-5\mu=0 \\ -\lambda\mu+10\lambda=0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \lambda=\mu=0 \\ \underline{\text{OR}} \\ \mu=10,\ \lambda=\sqrt{50} \\ \underline{\text{OR}} \\ \mu=10,\ \lambda=-\sqrt{50} \end{array}\right.$$

7. $A = \frac{1}{2} \| \vec{u} \times \vec{v} \| = \frac{9}{2}$

8. (a) 8

   (b) adf

   (c) $\rho^2 \sin(\varphi)$