Calculus 251:C3 Reading Guide - 6/9/2020

Section 14.6 Multivariable Calculus Chain Rules

The first thing that crossed your mind when you saw the title of this section was probably, "Wait. Chain rules? Why is there more than one?" Think back to the product rules for vector-valued functions. We had defined three different products (scalar, dot, and cross), so we had three different product rules. Similarly, there are different ways to compose functions of several variables. In single-variable calculus, we only had functions from $\mathbb{R} \to \mathbb{R}$, so there was really only one type of composition. In multivariable calculus, we have already seen the chain rule for paths. That was a type of composition where $f : \mathbb{R}^3 \to \mathbb{R}$ and $\vec{r} : \mathbb{R} \to \mathbb{R}^3$, so we had $f(\vec{r}) : \mathbb{R} \to \mathbb{R}$. In this section, we will look more closely at that chain rule, and then generalize it to other types of compositions.

Find the following definitions/concepts/formulas/theorems:

- Theorem: Chain rule for paths (you have seen this before)
- independent variables
- Theorem: General Version of the Chain Rule (note there is also a gradient version on the next page)
- primary derivatives (*not* standard terminology)
- implicit differentiation (recall that you used this in calc one for relations like $xy^2 - y^3 + x^2 = 3$)

The proof of the chain rule for paths uses the limit definition of differentiability. If you plan to be a math major (or are just curious), you should probably spend some time trying to understand it.

Example 1 is another example of using the chain rule for paths. This one should be okay.

Example 2 is a motivating example for the general chain rule. We have a function $f : \mathbb{R}^3 \to \mathbb{R}$, and we have parametrizations for x, y, z which have two parameters instead of the one we are used to.

Examples 3 and 4 are standard examples where f is a multivariable function each of whose inputs is a function of several independent parameters.

Example 5 is very important because it deals with switching between polar and rectangular coordinates in $\mathbb{R}^2$. In the not-too-distant future, we will be looking at spherical and cylindrcal coordinates in $\mathbb{R}^3$. You should perhaps spend a moment here thinking about what spherical and cylindrical coordinates would look like, and how the technique introduced in this example would generalize to $\mathbb{R}^3$.

Example 6 and the "Assumptions matter" subsection are about implicit differentiation. We often want to analyze surfaces where either z is not a function of x, y (see figure 5 on p. 846

- fails a vertical line test). We may also have a relation where it is difficult or impossible to produce an explicit equation for the function even if one variable is actually a function of the others. Think a bit about figure 6 and the discussion next to it. Why does it make sense that $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ don't exist? What would it mean if they did exist?