Calculus 251:C3 Reading Guide - 6/3/2020

Section 14.3 Partial Derivatives

In this section, we are going to see how the concept of the derivative extends to functions of more than one variable. When we looked at average rates of change in the previous section, we saw that at a particular point of the domain we may have different rates of change depending on the direction we travel from the point. It therefore does not make sense to talk about "the derivative" of a multivariable function at a point. What does make sense is talking about the partial derivative of a function with respect to a particular variable.

If you have a function $f(x, y, z)$, the partial derivative with respect to x at the point (a, b, c) is the rate of change of f at the point along the line $\vec{r}(t) = \langle a+t, b, c\rangle$. That is what happens when we hold y and z constant and only let x vary?

Note that there are multiple notations that get used. You may see any of

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f) = \frac{\partial}{\partial x}(f(x, y))$$

all of which mean the same thing (at least if f is a function of x, y for the last one). Other notations that are equivalent:

$$f_x(a, b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} \qquad f_y(a, b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)}$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial x^2}(f) = \frac{\partial^2 f}{\partial x^2}$$

I will usually use the subscript notation (so f_x, $f_x(a, b)$, f_{xx}, etc.) because it is a lot less writing/typing. There will be times when the Leibniz notation (the one with the ∂'s) makes it easier to see what is going on. You should become comfortable with both notations.

Find the following definitions/concepts/theorems:

- partial derivative
- partial derivative of f with respect to x
- Numerical approximation of partial derivatives
- second-order partial derivatives
- mixed partials
- Theorem: equality of mixed partials (note the condition under which this is true!)
- partial differential equation (a/k/a PDE)

The motivating examples above example 1 are intended to give you a feel for when you will need to use multivariable functions to model real-world phenomena.
Example 1 is a very basic example of partial derivatives.

Examples 2 and 3 are examples which require “The Chain Rule”, according to the text. This is not a great name for what they are doing because there are several different chain rules depending on the type of composition of functions (more in 14.5-14.6). This version of the chain rules works in problems like these where you have a function explicitly given in terms of the input variables. And it works just like the single-variable case. But do not assume that this is the end of the chain rule story...

Examples 4 and 6 are real-world examples of when we use partial derivatives and approximations.

Example 5 is a caluculation of a function of 4 variables, so you should note that the graph is a hypersurface in $\mathbb{R}^5$. Can you picture what such a graph would look like? I can’t.

Examples 7 through 10 all involve calculations of higher-order partial derivatives. Just like when you needed to find a second or third derivative, you just take derivatives multiple times. The only catch here is that you have to be careful which input variable you are differentiating with respect to. In order to find f_{xyy} in example 8, you need to differentiate with respect to x once and y twice. Note that example 10 takes advantage of the fact that the order you do these in doesn’t matter (assuming, of course, that the hypothesis of Clairaut’s Theorem is satisfied!)

Example 11 involves the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, a PDE which shows up in many problems in physics and engineering. In this case we are not solving the PDE, we are just verifying that the given function $u(x,t)$ satisfies the PDE. Solving PDE’s is beyond the scope of this course, but is a topic in Math 421. I am sure that many of the advanced physics courses also involve solving PDE’s.

Section 14.4 Differentiability, Tangent Planes, and Linear Approximation

This section talks about what it means for a multivariable function to be differentiable. You should already know that single variable functions fail to be differentiable at discontinuities (of any type), cusps, corners, and vertical tangents. As you have probably already guessed, there is a lot more going on here even with only two variables. Of particular note is that in order for $f(x,y)$ to be differentiable at a point (a,b), it is *necessary* that $f_x(a,b)$ and $f_y(a,b)$ both exist, but that condition is not *sufficient*. Figure 2c on page 818 right at the beginning of the section shows a function where both partials exist (and are actually both 0) at the origin, but the function is not differentiable there. This function is discussed in great detail in “Assumptions Matter” on pages 824-825.

The intuition for single-variable calculus is that if you zoom in far enough on the graph of a differentiable function, the graph looks like a line. We used that fact to create linearizations of differentiable functions at particular points which we could use to estimate the function at x-values near the point. Remember estimating $\sqrt{4.02}$ without a calculator in calc I? For a function of two variables, the intuition is that as you zoom in on a differentiable function the graph looks more and more like a plane. The linearization at a point on the surface $z = f(x, y)$ is going to be a plane tangent to the surface at that point.

Find the following definitions/concepts/formulas/theorems:

- tangent lines for f_x and f_y
- plane determined by f_x and f_y
- linearization of $f(x, y)$ centered at (a, b)
- Definition and formulas for "Differentiability and the Tangent Plane" - You should look at these, but you are not responsible for knowing them. They are beyond the scope of the course. Ask me in office hours if you are curious!
- Theorem: confirming differentiability
- approximation of $f(x, y)$ by $L(x, y)$
- Formulas for "Differentials and Linear Approximation"
- Linear Approximation formula $f(a + \Delta x, b + \Delta y) \approx$ something

Examples 1-3 are all straightforward, and you will hopefully not have trouble working through them. Finding tangent planes uses techniques that we developed in chapter 12. Remember the examples where you had to find the plane containing two lines? These examples are why that was so important.

Examples 4-6 all involve using partial derivatives to produce lineariazations, and then use those linearizations to estimate the function somewhere near the point at which the linearization is centered.