**Calculus 251:C3 Reading Guide - 6/1/2020**

## Section 13.1 Vector-Valued Functions

You can think of vector-valued functions as functions which take a real number as input and return a vector as output. We have already seen some vector-valued functions, specifically the parametrizations of lines. This section will extend our view of vector-valued functions to include general curves in $\mathbb{R}^3$. There are many useful pictures of curves and surfaces in $\mathbb{R}^3$, both in the body of the section and in the exercises. I won't be able to draw any of these, so enjoy them in the textbook!

Find the following definitions/concepts:

- vector-valued function
- parameter
- coordinate functions (a/k/a components)
- vector parametrization
- space curve
- plane curve
- What is the difference between the path and the curve traced by a vector-valued function?

Example 1 explains the answer to the question above this sentence.

Example 2 is a nice visual way to think about how we would describe a helix. Note that in the parametrization, the first two components would describe a circle if the third component were constant. But the third component is linear in $t$ (this time, it is just $t$), which produces motion in the $z$-direction. When you see a parametrization that looks like this, you should recognize that you have a helix. What would happen to the graph if we switched the first and third components?

Example 3 is important. This will be our general technique for describing the intersection of two surfaces in $\mathbb{R}^3$. We will do an extension of this problem in class. If you would like to see more examples, including the ones we will do in class, please read this pdf from Dr. G's website.

Example 4 is relatively simple. Think about some of the changes you could make to the circle in this question. Could you tilt it 45°? Could you stretch it into an ellipse?

## Section 13.2 Calculus of Vector-Valued Functions

This section can be summarized in the following two sentences. When you want the {limit, derivative, integral} of a vector-valued function, do everything component-wise. Pay attention to the order of the functions when you use the three different product rules, because it matters for one of them.

Find the following definitions/concepts/formulas/theorems:

- limit of a vector-valued function
- Theorem: vector-valued limits
- continuous (as it applies to a vector-valued function)
- formula for derivative of a vector-valued function
- Theorem: derivatives vector-valued function
- differentiation rules
- Theorem: derivatives of dot product and cross product
- tangent vector (a/k/a velocity vector)
- tangent line (as it applies to vector-valued functions)
- Theorem: two antiderivatives of a vector-valued function differ by a constant
- Fundamental Theorem of Calculus: vector-valued function edition

Most of the examples in this section show how to use the various formulas, so they are worth a read. The proofs (including example 4) are definitely worth reading because they will help convince you that these formulas all make sense. You can skim examples 5 and 6 which are about plotting tangent vectors, but the visuals may help develop your intuition about what is happening as we move along a parameterized curve.

**Section 13.3 Arc Length and Speed**

Again, this section is a generalization of something you saw in calculus 2 (or AP Calculus BC). When you considered the arc length of a parameterized curve in $\mathbb{R}^2$, you saw the formula $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$. Much of the time, this formula produced an integrand which you did not have the techniques to integrate by hand, and that will be true here as well. The examples which you could integrate were usually specifically designed to have a "nice" integrand. As usual, this section will only discuss the situation in $\mathbb{R}^3$, but the techniques presented here generalize to dimensions beyond 3 as well.

Find the following definitions/concepts/formulas/theorems:

- Theorem: Arc length in $\mathbb{R}^3$
- arc length function
- speed (for a vector-valued function)
- arc length parametrization

One important point that I do not see in the textbook is that arc length is independent of parametrization. We have seen that you can parameterize curves in multiple ways (starting with multiple ways to parameterize lines). If you start at a particular point on a curve and travel to a different point on the curve, you will get the same arc length regardless of the parametrization you chose for the curve. It makes sense that this has to be true geometrically. The reason that it works algebraically is that if you have two different parametrizations, your limits of integration will be different in a way that offsets the change in the integrand.

Examples 1, 2, and 3 are standard calculations and you should work through them.

Everything from just below example 3 through the end of example 4 is about arc length parametrization. The way to visualize what is happening here is to picture yourself standing on the curve at some starting point at $t = 0$. You walk along the curve **at the constant speed of 1 (distance unit)/(time unit)**. The arc length parametrization is the unique parametrization which expresses your position on the curve as a function of time. As the book states, we can usually not evaluate the arc length integral, therefore we can usually not produce an explicit arc length parametrization. There are certain "nice" cases in which we can, and example 4 shows one of them.

**Section 13.4 Curvature**

This section is listed as optional in the department syllabus, and we are in fact going to omit it because none of the subsequent material depends on it. If you are curious, you are welcome to peruse this section either now or in late July when this course is over and you are missing multivariable calculus deeply. If you want some idea of what a lecture on this topic would look like, I will again refer you to Dr. G.'s website.