Calculus 251:C3 Reading Guide - 7/7/2020

Section 17.1 Green's Theorem

This chapter is about several extensions of the Fundamental Theorem of Calculus to contexts involving multiple integrals. We did not want to compute every single-variable integral as the limit of Riemann sums, so instead we used the FTC to evaluate integrals using antiderivatives. Similarly, while we have developed a number of techniques for computing multiple integrals, line integrals, and surface integrals, we always want to have options which will simplify our computations.

This section focuses on Green's Theorem, which only applies to functions in the plane. It establishes the equivalence of the double integral of some functions over a region in the plane with the line integral along the (closed) boundary of the region.

Find the following definitions/concepts/formulas/theorems:

- simple closed curve
- boundary orientation
- Theorem: Green's Theorem
- formula for area enclosed by a closed curve
- Theorem: Circulation Form of Green's Theorem
- additivity of circulation
- Green's Theorem for more general regions
- Flux form of Green's Theorem

You should certainly read the paragraphs before the theorem. The proof of Green's Theorem isn't bad, because they chose to prove a special case where the boundary of $\mathcal{D}$ can be written as two functions of x, i.e. that the domain is vertically simple.

Example 1 is similar to the first few homework problems, in which you are asked to verify Green's Theorem by computing the integrals on both sides of the theorem directly. Example 2 uses Green's Theorem to compute one double integral instead of three line integrals. This is the setting in which we usually apply the theorem: to save ourselves extra work!

Example 3 uses Green's Theorem to derive the area formula for an ellipse. Definitely worth your time.

The two subsections on circulation aren't extremely important in their own right, but there are two reasons two work your way through them anyway. First, as always, the more different ways you think about a concept the better your intuition will be for problems involving that concept. Second, Stokes' Theorem in §17.2 is a generalization of Green's Theorem to $\mathbb{R}^3$ (with closed surfaces and triple integrals replacing closed curves and double integrals), and Stokes' Theorem is stated in terms of curl. Better to think about why it works in two dimensions first.

The more general form of the theorem allows us to use it on domains with holes, as long as those holes are bounded by simple closed curves. The general idea is that you slice the domain into smaller domains, each of which is bounded by a simple closed curve. Green's Theorem then applies to each of the pieces. Then when you reassemble the pieces, you see that for each of the cuts you made you integrated across that curve once in each direction. Since changing direction changes the sign of your line integral, those two parts of the integral have to add up to zero. After you put everything back together, the only pieces of line integrals that didn't cancel out are the curves that formed the boundary of the domain you started with. It's a neat trick.

Example 4 is extremely important if you are ever going to need complex analysis for anything in your life. Please work through it.

The Flux form of the theorem is important. Please spend some time with it. When we generalize this form of the theorem to $\mathbb{R}^3$, we get the Divergence Theorem which is the subject of §17.3. The example and the conceptual insight after it are both pretty straightforward. If you have made it this far into the section, finish strong and power through them.