Calculus 251:C3 Reading Guide - 5/27/2020

Section 12.3 Dot Product and the Angle Between Two Vectors

The main idea of this section is the introduction of another operation we can perform on vectors, namely the dot product. One very important note is that the dot product of two vectors is a *scalar*, not a vector. In slightly more formal language, the dot product is a function which maps $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$.

Find the following definitions/formulas/theorems:

- Dot product (note that the definition assumes that we are working in $\mathbb{R}^3$, and below the defintion they define dot product in $\mathbb{R}^2$. The dot product is defined the same way for any number of dimensions. What is the dot product in $\mathbb{R}^1$? What is it in $\mathbb{R}^5$?)
- Properties of the Dot Product (a set of 5 useful identities)
- Relationship between dot product and angle (also read the proof if you need to be convinced or are curious)
- orthogonal (a/k/a perpendicular)
- conditions for an angle to be obtuse or acute
- projection of $\vec{u}$ along $\vec{v}$
- component of $\vec{u}$ along $\vec{v}$ (a/k/a scalar component of $\vec{u}$ along $\vec{v}$)
- decomposition of $\vec{u}$ with respect to $\vec{v}$

Example 1 is a verification of the Distributive Law by example, but <u>not a proof</u>. Note that if you replaced the numbers with general component forms $\vec{u} = \langle a_1, b_1, c_1 \rangle$, $\vec{v} = \langle a_2, b_2, c_2 \rangle$, $\vec{w} = \langle a_3, b_3, c_3 \rangle$, you could perform the same calculations. That would constitute a proof of the Distributive Law.

Examples 2, 3, and 4 are useful examples of how we use the relationship between dot product and angle. You should certainly expect questions like these to come up in homework, quizzes, and exams, possibly as one of the steps of a larger problem.

Examples 5 and 6 are nice applications of the dot product in geometry. You can skip these for now if you have trouble with them, but you should look at them again after the lecture.

Examples 7-10 are about projections and decompositions. The first two are just computations and the last two are applications. You should spend some time making sure that these make sense to you.

The "Graphical Insight" at the end of the section is worth a look. The basic idea is that vector decomposition isn't really completely new. We were already thinking about vectors in component form. When we are working in $\mathbb{R}^2$, the component form of $\vec{v}$ that we have been working with is really just the decompostion of $\vec{v}$ with respect to $\hat{\mathbf{i}}$ (or $\hat{\mathbf{j}}$, I suppose). Things are a little more complicated in $\mathbb{R}^3$, but think about what would happen if you took $\vec{v} = \langle a, b, c\rangle$ and found its decomposition with respect to $\hat{\mathbf{i}}$. You would get $\langle a, 0, 0\rangle + \langle 0, b, c\rangle$, a vector lying on the x-axis and a vector in the yz-plane. You could then decompose the second of those with respect to $\hat{\mathbf{j}}$ and you would have the component form you are used to. Another way of thinking about these decompositions: you are really just changing coordinate systems. Instead of using the axes you started with, you are using some other set of orthogonal axes and rewriting your vector in terms of those other axes. This is a very important concept in linear algebra.

Section 12.4 The Cross Product

Two important notes: First, the cross product of two vectors is a vector (remember that the dot product is a scalar). Second, while the dot product makes sense in any number of dimensions, the cross product is only defined in exactly 3 dimensions. Okay, that was actually a little white lie. The cross product makes sense in 0, 1, 3, or 7 dimensions. But in 0 or 1 dimension the cross product is always the zero vector. In 7 dimensions, there are actually 480 different possible cross products. So we are going to pretend that the cross product only exists in $\mathbb{R}^3$ where it is in fact unique. I'm sure that there are people out there who understand 7-dimensional cross products and what they might be used for, but I assure you that I am not one of those people. Please don't ask me.

One slightly less important note: recall the right hand rule from section 12.2. You are going to need it.

Find the following definitions/formulas/theorems:

- torque (a physics concept useful for visualizing why cross products are important)
- determinant of a 2×2 matrix
- determinant of a 3×3 matrix
- Cross product (the formula is really the definition)
- Theorem: geometric description of the cross product
- anticommutative
- Basic properties of the cross product (there are 5)
- cross products of standard basis vectors
- What does it mean for vectors to "span" a shape?
- Area of parallelogram spanned by $\vec{v}, \vec{w}$
- Area of triangle spanned by $\vec{v}, \vec{w}$
- parallelepiped
- scalar triple product

- Volume of parallelepiped spanned by $\vec{v}, \vec{w}$

Example 1 is just a demonstration of how we compute 3×3 determinants. Note that you can calculate determinants for larger (square) matrices with a similar technique. The plus and minus signs just alternate, and you end up with determinants that are one dimension smaller than you started with. So to calculate a 5×5 determinant, you would need to do this process many times. This is why we generally use computers for anything bigger than 3 dimensions. You should be able to do 3×3's by hand.

Example 2 is a demonstration of how to compute $\vec{v} \times \vec{w}$. Note that the top row is always $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. The second row is always $\vec{v}$. If you switch the second and third rows, you will end up with $-\vec{v} \times \vec{w}$, which is the same as $\vec{w} \times \vec{v}$. You might even want to try that on this example to confirm what I'm telling you.

Exercise 3 is supposed to help your intuition regarding what cross products mean geometrically. There is one comment that I don't like: "Finally, property (iii) tells us that $\vec{u}$ points in the positive z-direction." In my mind, saying that something points in the positive z-direction means that it is in the direction of the positive z-axis, which is not the case in this example. I think that I would say that the right hand rule tells us the z-component of $\vec{u}$ must be positive. In either case, figuring out that the cross product is "up-ish" instead of "down-ish" lets us put the correct signs on the components.

Exercise 4 is a method for computing cross products that looks something like FOIL from high school algebra. Personally, I don't think that this method is appreciably easier than calculating $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 0 & 1 \\ 0 & 3 & 5 \end{vmatrix}$. On a positive note, this method does reinforce the cross products of the standard basis vectors. You should use whatever method makes you happier (and more likely to get a right answer). They are mathematically equivalent. If you want to be a math major, you should probably *prove* that they are equivalent. If you attempt such a proof, I would be happy to look it over and offer you feedback.

Examples 5 and 6 are nice physics examples. If you find yourself asking, "What are cross products useful for?" these examples are part of your answer.

Examples 7 and 8 are straightforward area and volume calculations. Make sure you can do these. You will get to practice one of these in recitation.

You should only read the Proofs of Cross-Product Properties if you are either curious about the proofs or skeptical about whether the properties are true. Otherwise, you can skip this part.