Calculus 251:C3 Reading Guide - 6/16/2020

Section 15.1 Integration in Two Variables

In chapter 15, we will be studying multiple integration. The first step in that process is to understand the double integral. In single-variable calculus, you learned that the integral of $y = f(x)$ represents the area between the graph of the function and the x-axis. If the area was above the axis, we counted it as positive area. If the area was below the axis, we counted it as negative. Our definite integrals were written as $\int_a^b f(x)dx$, a notation that meant that we were integrating the function with respect to x on the interval $[a, b]$. Really this came down to the area bounded by the lines $x = a$, $x = b$, and $y = 0$ and the curve $y = f(x)$ with positive/negative considerations discussed above.

Now, our functions are of the form $z = f(x, y)$ and their graphs live in $\mathbb{R}^3$. When we integrate a function, we will need to look at a domain which is a subset of $\mathbb{R}^2$, which we understand to be the xy-coordinate plane. Our integrals may be written as $\int_a^b \int_c^d f(x, y)\, dy\, dx$, which means we are integrating with respect to y on $[c, d]$ and then integrating again with respect to x on $[a, b]$. What this value represents is the *volume* above the rectangle in the xy-coordinate plane and below the function $z = f(x, y)$. Again, we will consider volume below the xy-coordinate plane to be negative volume, analagously to the single-variable case. And you can think about this volume as being bounded by five planes (which ones?) and the surface $z = f(x, y)$. Most of the rest of the section is a discussion of why this all makes sense and details of how we do it.

Note: This section does introduce the notation $\iint_{\mathcal{D}} f(x, y)\, dA$, but we will only deal with integrating over rectangles. In the next section, we will see how to integrate over domains of other shapes.

Find the following definitions/concepts/formulas/theorems:

- double integral
- Riemann sum (same general idea as single-variable, but our little intervals are now little rectangles)
- limit of Riemann sums
- Double integral over a rectangle
- integrable (essentially the same as single-variable)
- regular partitions (again just adding a dimension to the definition from single-variable calculus)
- Theorem: Continuous functions are integrable
- Theorem: Linearity of the Double Integral

- Area formula (a/k/a double integral of a constant)
- iterated integral
- Theorem: Fubini's Theorem (THIS IS HUGE, arguably one of the most important theorems in all of mathematics)
- area element

The first few pages are all about how we extend our understanding of integration and Riemann sums to functions of two variables. There is a lot of notation, but it is really close enough to the single-variable case that it should be readable.

Examples 1, 2, and 3 are about how to evaulate integrals without actually integrating anything. Example 1 uses sampling to build an estimate from a Riemann sum (without taking the limit). You should have seen this in Calc I (or AP Calc in high school) as estimates using left endpoint, right endpoint, and midpoint. You probably also talked about trapezoidal approximations, but we're not going to do that here.

Example 2 is about using geometry to figure out the volume instead of integrating. The classic single-variable example is $\int_{-5}^{5} \sqrt{25-x^2}\,dx$. You *could* use trigonometric substitution to hammer out this integral, but why the heck would you do that? This is just asking us for the area of a semicircle with a radius of 5, which is $\frac{25\pi}{2}$. Life is hard enough without doing unnecessary computations.

Example 3 uses a symmetry argument to assert that an integral has to be 0 because it has the same volume above and below the xy-plane so they cancel. If you had a mean teacher for Calc I, they probably put some ridiculous question like "calculate $\int_{-47\sqrt{3}}^{47\sqrt{3}} \left(\sin^5(3x) + 43\pi x^{17}\right) dx$" on a quiz. But the silly expression in the parentheses is an odd function, so it has rotational symmetry around the origin. As long as the limits of integration are $\pm$the same constant, the integral is going to be zero.

Examples 4,5, and 6 are all very straightforward evaluations of iterated integrals. The key idea is that when you are integrating with respect to x, you treat y as though it were a constant, and vice versa. If you think that sounds like a reprise of how we compute partial derivatives, you are on the right track.

The proof of Fubini's Theorem and the Graphical Insight that follows it are pretty rough. If you want to be a math major, you should certainly spend some time trying to digest them. If you are not planning to be a math major, please don't spend your time on this proof. I also promise you that we are not going to discuss the proof in class, because it is time-consuming to do correctly and not of general interest. **The theorem itself is important** because it allows us to switch the order of integration, which will often turn a very difficult integral into a more manageable one. But you can skip the proof.

Example 7 is a volume calculation, and example 8 is a physics/oceanography example. Both are worth a read.