Math 251: Multivariable Calculus, Exam #3
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1. [20 pts] Let $\mathcal{C}$ be the helix parametrized by $\vec{r}(t) = \langle 2\sin t, 2\cos t, \sqrt{5}\,t\rangle$ for $0 \le t \le 5\pi$.

   (a) Find the length of $\mathcal{C}$.

   (b) Calculate $\displaystyle\int_{\mathcal{C}} xyz\, ds$.

   (c) Let $\vec{F} = \langle 1, x^2, 0\rangle$. Calculate $\displaystyle\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$

Solution:

(a) $\vec{r}\,'(t) = \langle 2\cos t, -2\sin t, \sqrt{5}\rangle$

$ds = \|\vec{r}\,'(t)\|\,dt = \sqrt{4\cos^2 t + 4\sin^2 t + 5}\;dt = 3dt$

$\text{length}(\mathcal{C}) = \displaystyle\int_{\mathcal{C}} 1\,ds = \int_0^{5\pi} 3\,dt = 15\pi$

(b) $f(\vec{r}(t)) = (2\sin t)(2\cos t)(\sqrt{5}\,t) = 4\sqrt{5}\,t\sin t\cos t = 2\sqrt{5}\,t\sin 2t$

$$\int_{\mathcal{C}} xyz\,ds = \int_0^{5\pi} f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt = \int_0^{5\pi}(2\sqrt{5}\,t\sin 2t)(3\,dt) = 6\sqrt{5}\int_0^{5\pi} t\sin 2t\,dt$$

Integrating by parts gives $\displaystyle\int t\sin 2t\,dt = -\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t + C$, so

$$\int_{\mathcal{C}} xyz\,ds = 6\sqrt{5}\left[\frac{\sin 2t}{4} - \frac{t\cos 2t}{2}\right]\Bigg|_0^{5\pi} = 6\sqrt{5}\left[\left(0 - \frac{5\pi(1)}{2}\right) - (0-0)\right] = -15\sqrt{5}\,\pi$$

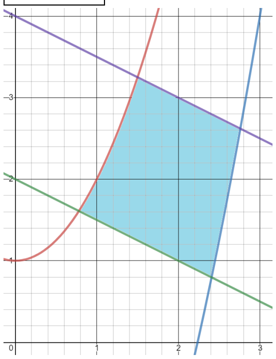
(c) $\vec{F}(\vec{r}(t)) = \langle 1, 4\sin^2 t, 0\rangle$

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_0^{5\pi}\langle 1, 4\sin^2 t, 0\rangle\cdot\langle 2\cos t, -2\sin t, \sqrt{5}\rangle dt = \int_0^{5\pi}(2\cos t - 8\sin^3 t)dt$$

$$= \int_0^{5\pi} 2\cos t\,dt + 8\int_0^{5\pi}(1-\cos^2 t)(-\sin t)dt = 0 + 8\int_1^{-1}(1-u^2)du$$

$$= \left[u - \frac{u^3}{3}\right]\Bigg|_1^{-1} = \left(-1 - \frac{-1}{3}\right) - \left(1 - \frac{1}{3}\right) = -\frac{32}{3}$$

2. 20 pts Let $\mathcal{D}$ be the part of the first quadrant shaded in the diagram.



This region is bounded on the left and right by the curves $y - x^2 = 1$ and $y - x^2 = -5$, and on the top and bottom by the lines $x + 2y = 8$ and $x + 2y = 4$.

(a) Find a rectangle $\mathcal{R}$ in the uv-plane and a map G such that $G(\mathcal{R}) = \mathcal{D}$. You may give either G or G^{-1}, but you must indicate which one your answer represents.

(b) Calculate $\text{Jac}(G)$. You may give your answer in terms of x and y or in terms of u and v

(c) Use a change of variables to calculate $\displaystyle\iint_{\mathcal{D}} (4x+1)e^{x^2+x+y}\, dx\, dy$.

Solution:
(a) Set $u = x + 2y$ and $v = x^2 - y$. This will map the rectangle $\mathcal{R} = [4, 8] \times [-1, 5]$ in the uv-plane to $\mathcal{D}$. Note that this is really an inverse map because we have u, v in terms of x, y.
$G^{-1}(x, y) = (x + 2y, x^2 - y)$

(b) $\text{Jac}(G^{-1}) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2x & -1 \end{vmatrix} = -1 - 4x$

Therefore $\text{Jac}(G) = \dfrac{1}{\text{Jac}(G^{-1})} = \dfrac{-1}{4x+1}$

(c) Since we are in the first quadrant, $\dfrac{-1}{4x+1} < 0$, so $|\text{Jac}(G)| = \dfrac{1}{4x+1}$

$$\iint_{\mathcal{D}} (4x+1)e^{x^2+x+y}\, dx\, dy = \int_{-1}^{5}\int_{4}^{8} (4x+1)e^{x^2+x+y}\frac{1}{4x+1} du\, dv$$

Next, we note that $x^2 + x + y = u + v$, so

$$\iint_{\mathcal{D}} (4x+1)e^{x^2+x+y}\, dx\, dy = \int_{-1}^{5}\int_{4}^{8} e^{u+v} du\, dv = (e^5 - e^{-1})(e^8 - e^4) = e^{13} - e^9 - e^7 + e^3$$

3. **18 pts** Let $\vec{F} = \langle e^x \sin y, e^x \cos y - \cos(z^2), 2yz\sin(z^2)\rangle$.

   (a) Calculate $\operatorname{div}(\vec{F})$.

   (b) Calculate $\operatorname{curl}(\vec{F})$.

   (c) Is $\vec{F}$ conservative? Why or why not?

   (d) If your answer to (c) is yes, find a potential for $\vec{F}$.

Solution:

(a) $\operatorname{div}(\vec{F}) = \vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right\rangle \cdot \langle e^x \sin y, e^x \cos y - \cos(z^2), 2yz\sin(z^2)\rangle$

$= \frac{\partial}{\partial x}(e^x \sin y) + \frac{\partial}{\partial y}(e^x \cos y - \cos(z^2)) + \frac{\partial}{\partial z}(2yz\sin(z^2))$

$= e^x \sin y + (-e^x \sin y) + (2y\sin(z^2) + 4yz^2\cos(z^2)) = 2y\sin(z^2) + 4yz^2\cos(z^2)$

(b) $\operatorname{curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & e^x \cos y - \cos(z^2) & 2yz\sin(z^2) \end{vmatrix}$

$= (2z\sin(z^2) - 2z\sin(z^2))\hat{\mathbf{i}} - (0 - 0)\hat{\mathbf{j}} + (e^x \cos y - e^x \cos y)\hat{\mathbf{k}} = \vec{0}$

(c) Yes. Exponential, trigonometric, and polynomial functions are all continuous on $\mathbb{R}^3$, so the domain of $\vec{F}$ is all of $\mathbb{R}^3$ which is simply connected. Since the domain is simply connected and $\operatorname{curl}(\vec{F}) = \vec{0}$, $\vec{F}$ is conservative.

(d) Let f be a potential for $\vec{F}$, i.e. $\vec{\nabla} f = \vec{F}$.
$f_x = e^x \sin y$, so $f = e^x \sin y + g(y, z)$

$e^x \cos y - \cos(z^2) = f_y = e^x \cos y + g_y$, so $g_y = -\cos(z^2)$ and $g(y, z) = -y\cos(z^2) + h(z)$
Updating: $f = e^x \sin y - y\cos(z^2) + h(z)$

$2yz\sin(z^2) = f_z = 2yz\sin(z^2) + h'(z)$, so $h'(z) = 0$ and $h(z) = C$

Updating: $f = e^x \sin y - y\cos(z^2) + C$
So $f(x, y, z) = e^x \sin y - y\cos(z^2)$ is a potential for $\vec{F}$.

4. 18 pts Let $\vec{F} = \langle 3x^2y, x^3 - 2yz, -y^2 \rangle$.
Let $\mathcal{C}_1$ be the ellipse parametrized by $\vec{r}_1 = \langle 2\cos t, 5\sin t, 3 \rangle$, $0 \le t \le 2\pi$.
Let $\mathcal{C}_2$ be the curve parametrized by $\vec{r}_2 = \left\langle 2\cos\left(\frac{\pi t}{4}\right), \frac{t^3}{25}, 2\ln(t+1) \right\rangle$, $0 \le t \le 5$.

(a) Calculate $\displaystyle\int_{\mathcal{C}_1} \vec{F} \cdot d\vec{r}_1$.

(b) Calculate $\displaystyle\int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r}_2$.

Solution:
First, we note that

$$\text{curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y & x^3 - 2yz & -y^2 \end{vmatrix} = (-2y - (-2y))\hat{\mathbf{i}} - (0 - 0)\hat{\mathbf{j}} + (3x^2 - 3x^2)\hat{\mathbf{k}} = \vec{0}$$

The domain of $\vec{F}$ is all of $\mathbb{R}^3$ which is simply connected, so $\vec{F}$ is conservative.

By inspection, we see that if $f(x,y,z) = x^3y - y^2z$, then $\vec{\nabla} f = \langle 3x^2y, x^3 - 2yz, -y^2 \rangle = \vec{F}$.
Therefore $f(x,y,z) = x^3y - y^2z$ is a potential for $\vec{F}$.

(a) $\mathcal{C}_1$ is a simple closed curve, and $\vec{F}$ is conservative. Therefore $\displaystyle\int_{\mathcal{C}_1} \vec{F} \cdot d\vec{r}_1 = 0$

(b) Since $\vec{F}$ is conservative, $\displaystyle\int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r}_2$ is path-independent.
$\vec{r}_2(0) = (2, 0, 0)$ and $\vec{r}_2(5) = (-\sqrt{2}, 5, 2\ln 6)$.

$$\int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r}_2 = f(\vec{r}_2(5)) - f(\vec{r}_2(0)) = f(-\sqrt{2}, 5, 2\ln 6) - f(2, 0, 0)$$

$$= (-10\sqrt{2} - 50\ln 6) - (0) = -10\sqrt{2} - 50\ln 6$$

5. 24 pts Let $\mathcal{S}$ be the surface $x^2 + y^2 = 16 - z$ for $z \geq 0$.

(a) Parametrize the surface with a mapping $G(r, \theta)$.

(b) Compute $\vec{T}_r, \vec{T}_\theta, \vec{N}$, orienting $\mathcal{S}$ with upward-pointing normal.

(c) Find the surface area of $\mathcal{S}$.

(d) Calculate the flux of $\vec{F} = \langle 0, 0, 3z \rangle$ across $\mathcal{S}$.

Solution:

(a) Rewriting the equation of the paraboloid in cylindrical coordinates, $z = 16 - r^2$. So the vertex of the paraboloid is at $(0, 0, 16)$. Since $z \geq 0$, we must have $r \in [0, 4]$. The parametrization is therefore:

$G(r, \theta) = (r\cos\theta, r\sin\theta, 16 - r^2)$, $r \in [0, 4]$, $\theta \in [0, 2\pi)$.

(b) $\vec{T}_r = \dfrac{\partial G}{\partial r} = \langle \cos\theta, \sin\theta, -2r \rangle$ and $\vec{T}_\theta = \dfrac{\partial G}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$$\vec{N}(r, \theta) = \vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle$$

Since $r \geq 0$, this normal points upward, and $\mathcal{S}$ is oriented correctly.

(c) $\left\| \vec{N} \right\| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = \sqrt{4r^4 + r^2} = r\sqrt{4r^2 + 1}$

$$\text{area}(\mathcal{S}) = \int_{\mathcal{S}} \left\| \vec{N} \right\| dS = \int_0^{2\pi} \int_0^4 r\sqrt{r^2 + 1}\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^4 r\sqrt{r^2 + 1} dr$$

$$= 2\pi \left(\frac{1}{8} \cdot \frac{2}{3}(4r^2 + 1)^{3/2} \right) \Bigg|_0^4 = \frac{\pi}{6}(65^{3/2} - 1)$$

(d) $\vec{F} \cdot \vec{N} = \langle 0, 0, 48 - 3r^2 \rangle \cdot \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle = 48r - 3r^3$

$$\int_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^4 \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi} \int_0^4 48r - 3r^3\, dr\, d\theta$$

$$= 2\pi \left(24r^2 - \frac{3}{4}r^4 \right) \Bigg|_0^4 = 2\pi(384 - 192) = 384\pi$$