Math 251: Multivariable Calculus, Exam #2
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1. $\boxed{\text{12 pts}}$ Find the point or points on the surface $z^2 = xy + 4$ closest to the origin.
Solution:

We want the point on the surface closest to the origin, so we are minimizing $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ subject to the constraint $z^2 = xy+4$. We can equivalently minimize $f(x, y, z) = (d(x, y, z))^2 = x^2 + y^2 + z^2$. We will use the technique of Lagrange multipliers to do so.

Let $g(x, y, z) = z^2 - xy - 4$. We compute the gradients of f and g:

$$\nabla f = \langle 2x, 2y, 2z\rangle$$

$$\nabla g = \langle -y, -x, 2z\rangle$$

Now setting $\nabla f = \lambda\nabla g$ and using the constraint gives us four equations:

$$2x = -\lambda y$$

$$2y = -\lambda x$$

$$2z = 2\lambda z$$

$$z^2 = xy + 4$$

From the third equation, we see that either $z = 0$ or $\lambda = 1$

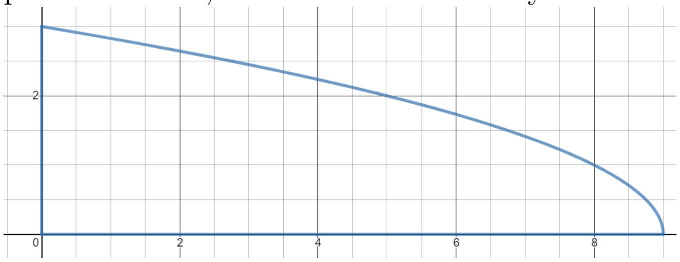
Case 1: $z = 0$
From the first two equations, we get $2x^2 = -\lambda xy = 2y^2$, so $x^2 = y^2$.
From the fourth equation, we get $0 = xy + 4$. Combining these gives $(x, y) = (-2, 2)$ or $(x, y) = (2, -2)$. Therefore we have the points $(-2, 2, 0)$ and $(2, -2, 0)$ which are both at distance $2\sqrt{2}$ from the origin.

Case 2: $\lambda = 1$
Now the first two equations give us $2x = -y$ and $2y = -x$, which are only satisfied by $(x, y) = (0, 0)$. Then the constraint equation becomes $z^2 = 4$, so $z = \pm 2$. This gives us the points $(0, 0, -2)$ and $(0, 0, 2)$ which are both at distance 2 from the origin.

Therefore $(0, 0, -2)$ and $(0, 0, 2)$ are the points on the surface closest to the origin.

2. $\boxed{\text{16 pts}}$ Calculate $\displaystyle\int_0^3 \int_0^{9-y^2} \frac{ye^{2x}}{9-x}\,dx\,dy$ **Solution:**

We don't know how to integrate $\displaystyle\int \frac{e^{2x}}{9-x}dx$ (in fact this integral does not have a closed form), so we will need to switch the order of integration. The domain of integration is the region pictured below, which is both vertically and horizontally simple.



We can write the equation of the parabola as $y = \sqrt{9-x}$, which allows us to proceed.

$$\int_0^3 \int_0^{9-y^2} \frac{ye^{2x}}{9-x}\,dx\,dy = \int_0^9 \int_0^{\sqrt{9-x}} \frac{ye^{2x}}{9-x}\,dy\,dx = \int_0^9 \frac{e^{2x}}{9-x} \int_0^{\sqrt{9-x}} y\,dy\,dx$$

$$= \int_0^9 \frac{e^{2x}}{9-x}\left(\frac{y^2}{2}\bigg|_0^{\sqrt{9-x}}\right)dx = \int_0^9 \frac{e^{2x}}{9-x}\frac{9-x}{2}\,dx = \int_0^9 \frac{e^{2x}}{2}\,dx = \frac{e^{2x}}{4}\bigg|_0^9 = \frac{e^{18}-1}{4}$$

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3. $\boxed{\text{10 pts}}$ Let x, y, and z be related implicitly by the equation $xy^2 - 2xz + 5z^2 = 11$.

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

Solution:

Because we are computing partial derivatives of z with respect to the other variables, we are treating z as an implicit function of independent variables x and y.

First we differentiate with respect to x:

$$y^2 - 2z - 2xz_x + 10zz_x = 0$$
$$z_x(10z - 2x) = 2z - y^2$$
$$z_x = \frac{2z - y^2}{10z - 2x}$$

Then we differentiate with respect to y:

$$2xy - 2xz_y + 10zz_y = 0$$
$$z_y(-2x + 10z) = -2xy$$
$$z_y = \frac{-2xy}{-2x + 10z} = \frac{xy}{x - 5z}$$

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4. $\boxed{\text{12 pts}}$ Let $f(x,y) = x^3 + y^3 - 3xy + 15$. Find all critical points and critical values of f.

Classify each critical point as a maximum, minimum, or saddle point.

Solution:

$$f_x = 3x^2 - 3y \qquad \text{and} \qquad f_y = 3y^2 - 3x$$

In order to satisfy $f_x = f_y = 0$, we must have $y = x^2$ and $x = y^2$. The only such points are $(x,y) = (0,0)$ or $(x,y) = (1,1)$, so those are the only critical points. The associated critical values are $f(0,0) = 15$ and $f(1,1) = 14$.

$$f_{xx} = 6x \quad f_{xy} = -3 \quad f_{yy} = 6y$$

So the discriminant is

$$D = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix} = 36xy - 9$$

At $(0,0)$, we have $D < 0$, so $(0,0,15)$ is a saddle point.
At $(1,1)$, we have $D > 0$ and $f_{xx} > 0$, so $(1,1,14)$ is a minumum.

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5. $\boxed{\text{16 pts}}$ Let $\mathcal{R}$ be the rectangle $\{(x,y) : -2 \le x \le 2, 0 \le y \le 2\}$ and let $f(x,y) = \dfrac{y}{4+x^2}$.
Calculate $\displaystyle\iint_{\mathcal{R}} f(x,y)\, dA$.

Solution:
We are integrating over a rectangle, so the limits of integration are the obvious ones:

$$\iint_{\mathcal{R}} f(x,y)\, dA = \int_{-2}^{2}\int_{0}^{2} \frac{y}{4+x^2}\, dy\, dx = \left(\int_0^2 y\, dy\right)\left(\int_{-2}^{2} \frac{1}{4+x^2}\, dx\right)$$

$$= \left(\frac{y^2}{2}\bigg|_0^2\right)\left(\frac{1}{4}\int_{-2}^{2} \frac{1}{1+\left(\frac{x}{2}\right)^2}\, dx\right) = (2)\left(\frac{1}{4}\right)\left(2\tan^{-1}\left(\frac{x}{2}\right)\bigg|_{-2}^{2}\right)$$

$$= \left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi}{2}$$

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6. $\boxed{\text{16 pts}}$ The temperature in a room is modeled by $T(x,y,z) = xz^2 - 4y$. An insect flies through the room along the path $\vec{r}(t) = \left\langle e^{t-1}, \frac{4}{\pi}\sin\left(\frac{\pi}{2}t\right), 4-t \right\rangle$.

   (a) Calculate $\nabla T(x,y,z)$.

   (b) Find a parametrization of the line tangent to the insect's path at $t = 2$.

   (c) What is the rate of change in temperature for the insect at $t = 2$? (i.e. what is the change in temperature along the insect's path at that moment?)

Solution:
(a) $\nabla T(x,y,z) = \langle z^2, -4, 2xz \rangle$

(b) We need the tangent vector and the point of tangency.
$\vec{r}(2) = \langle e, 0, 2 \rangle$
$\vec{r}\,'(t) = \left\langle e^{t-1}, 2\cos\left(\dfrac{\pi}{2}t\right), -1 \right\rangle$
$\vec{r}\,'(2) = \langle e, -2, -1 \rangle$
A parametrization of the line tangent to the insect's path is $\vec{L}(s) = \langle e, 0, 2 \rangle + s\langle e, -2, -1 \rangle$

(c) By the Chain Rule for Paths,

$$\frac{d}{dt}T(\vec{r}(2)) = \nabla T(\vec{r}(2)) \cdot \vec{r}\,'(2) = \nabla T(e,0,2) \cdot \langle e, -2, -1 \rangle$$
$$= \langle 4, -4, 4e \rangle \cdot \langle e, -2, -1 \rangle = 4e + 8 - 4e = 8$$

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7. $\boxed{\text{18 pts}}$ Let $\mathcal{S}$ be the sphere of radius 2 centered at the origin.
Let $\mathcal{P}$ be the paraboloid $x^2 + y^2 = 3z$.
Let $\mathcal{W}$ be the region inside $\mathcal{S}$ and above $\mathcal{P}$ (so $\mathcal{W}$ includes part of the positive z-axis).

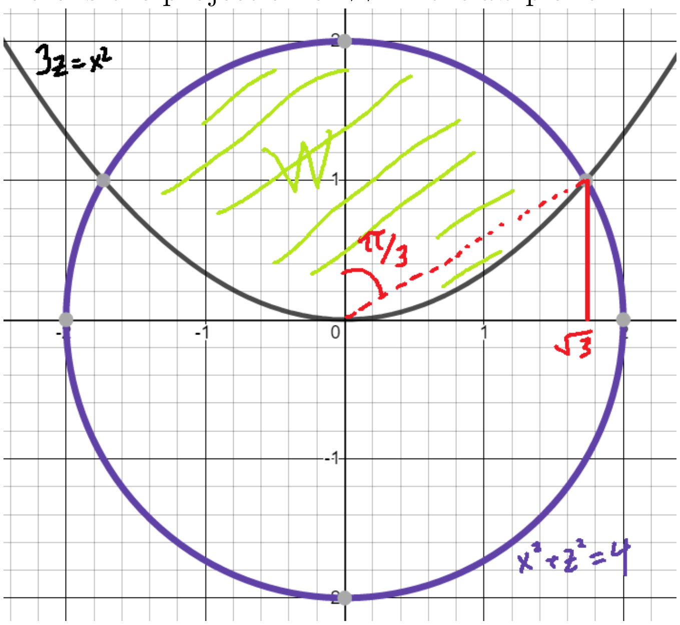
   (a) Write $\iiint_{\mathcal{W}} z\, dV$ as an iterated integral in rectangular coordinates.

   (b) Write $\iiint_{\mathcal{W}} z\, dV$ as an iterated integral in cylindrical coordinates.

   (c) Write $\iiint_{\mathcal{W}} z\, dV$ as an iterated integral in spherical coordinates (Warning: this one is significanty more difficult than the first two).

   (d) Choose any one of these integrals and use it to calculate $\iiint_{\mathcal{W}} z\, dV$.

Solution:
Here is the projection of $\mathcal{W}$ in the xz-plane.



The projection of $\mathcal{W}$ in the xy-plane is the circle of radius $\sqrt{3}$ centered at the origin, which has the equation $x^2+y^2=3$. The sphere has equation $x^2+y^2+z^2=4$ in rectangular coordinates.

(a) If we integrate in the order $dz\ dy\ dx$, we first have to move in the z direction. If we start in the xy-plane and travel upwards, we first intersect the paraboloid and then the sphere. So we need to integrate w.r.t. z on the interval $[(x^2+y^2)/3, \sqrt{4-x^2-y^2}]$. Then we need to integrate over the circle in the xy-plane, so the integral we need is:

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{(x^2+y^2)/3}^{\sqrt{4-x^2-y^2}} z\ dz\ dy\ dx$$

(b) This is the same basic idea as (a). We can integrate in the order $dz\ dr\ d\theta$. The equations for the sphere and the paraboloid are $r^2+z^2=4$ and $r^2=3z$. This gives us the interval $[r^2/3, \sqrt{4-r^2}]$ for z. In cylindrical coordinates, the circle is easier. We do need to remember that $dV = r\ dz\ dr\ d\theta$:

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_{r^2/3}^{\sqrt{4-r^2}} zr\ dz\ dr\ d\theta$$

(c) I did warn you this one is hard. If you look at the projection in the xz-plane, you can see that when $\phi \in [0, \pi/3]$ that ρ goes from 0 to the sphere. But when $\phi \in [\pi/3, \pi/2]$, ρ goes from 0 to the paraboloid. Our region is not radially simple, so we will end up with two integrals. The equation of the sphere is just $\rho = 2$, so that part is easy. The equation of the paraboloid is $\rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) = 3\rho\cos\phi$, or $\rho = 3\cot\phi\csc\phi$. Also remembering that our integrand is $z = \rho\cos\phi$, we get:

$$\int_0^{2\pi} \int_0^{\pi/3} \int_0^{2} (\rho\cos\phi)(\rho^2\sin\phi)\ d\rho\ d\phi\ d\theta + \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^{3\cot\phi\csc\phi} (\rho\cos\phi)(\rho^2\sin\phi)\ d\rho\ d\phi\ d\theta$$

$$\int_0^{2\pi} \int_0^{\pi/3} \int_0^{2} (\frac{1}{2}\rho^3\sin(2\phi))\ d\rho\ d\phi\ d\theta + \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^{3\cot\phi\csc\phi} (\frac{1}{2}\rho^3\sin(2\phi))\ d\rho\ d\phi\ d\theta$$

I assure you that all three of these integrals do have the same value. If you tried computing anything other than the one in cylindrical coordinates during a timed exam, you are a braver soul than I am.

$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_{r^2/3}^{\sqrt{4-r^2}} zr\;dz\;dr\;d\theta = 2\pi\int_0^{\sqrt{3}} r\int_{r^2/3}^{\sqrt{4-r^2}} z\;dz\;dr = 2\pi\int_0^{\sqrt{3}} r\left(\frac{z^2}{2}\bigg|_{r^2/3}^{\sqrt{4-r^2}}\right)dr$$

$$= \pi\int_0^{\sqrt{3}} r\left[(4-r^2)-\left(\frac{r^4}{9}\right)\right]dr = \pi\int_0^{\sqrt{3}}\left[4r-r^3-\left(\frac{r^5}{9}\right)\right]dr$$

$$= \pi\left(2r^2-\frac{r^4}{4}-\frac{r^6}{54}\right)\bigg|_0^{\sqrt{3}} = \pi\left(6-\frac{9}{4}-\frac{27}{54}\right) = \pi\left(\frac{24}{4}-\frac{9}{4}-\frac{2}{4}\right) = \frac{13\pi}{4}$$
