Math 251: Multivariable Calculus, Exam #1
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1. The planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are described by the following equations.
$\mathcal{P}_1$: $x - 2y + 4z = 2$
$\mathcal{P}_2$: $x + y - 2z = 5$

6 pts (a) Find the angle between $\mathcal{P}_1$ and $\mathcal{P}_2$.

9 pts (b) The planes $\mathcal{P}_1$ and $\mathcal{P}_2$ intersect in the line $\mathcal{L}$. Find a parametrization of $\mathcal{L}$

Solution:
(a) The angle between the planes is the same as the angle between their normal vectors. Using the coefficients in the equations of the planes gives us $\vec{n}_1 = \langle 1, -2, 4\rangle$ and $\vec{n}_2 = \langle 1, 1, -2\rangle$. Now we can use the formula for dot product: $\vec{n}_1 \cdot \vec{n}_2 = \|\vec{n}_1\| \, \|\vec{n}_2\| \cos\theta$

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \, \|\vec{n}_2\|}\right) = \cos^{-1}\left(\frac{-9}{\sqrt{21}\sqrt{6}}\right) = \cos^{-1}\left(\frac{-3}{\sqrt{14}}\right)$$

(b) $\mathcal{L}$ is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$, so its direction vector is $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle 0, 6, 3\rangle$.
Because the z-component of $\vec{v}$ is not 0, $\mathcal{L}$ intersects the xy-coordinate plane.
So we can set $z = 0$ and solve the equations of the planes for x, y.
$x - 2y = 2$ and $x + y = 5$ gives us $x = 4, y = 1$. So we use $\vec{x}_0 = \langle 4, 1, 0\rangle$
Therefore a parametrization of $\mathcal{L}$ is $\vec{r}(t) = \vec{x}_0 + t\vec{v} = \langle 4, 1, 0\rangle + t\langle 0, 6, 3\rangle$.

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2. A particle travels on a path which satisfies the equation $\dfrac{d\vec{r}}{dt} = \left\langle e^{t-2}, 3\pi\cos\left(\frac{\pi}{4}t\right), t^2\right\rangle$ for all $t \geq 0$.

8 pts (a) Find the general solution $\vec{r}(t)$ of the equation above which gives the position of the particle.

5 pts (b) Find the particular solution $\vec{r}(t)$ when $\vec{r}(2) = \langle 4, 10, 3\rangle$.

Solution:
(a) We may find the antiderivatives of a vector-valued function componentwise, so

$$\vec{r}(t) = \int \left\langle e^{t-2}, 3\pi\cos\left(\frac{\pi}{4}t\right), t^2\right\rangle dt = \left\langle e^{t-2}, 12\sin\left(\frac{\pi}{4}t\right), \frac{t^3}{3}\right\rangle + \vec{c}$$

(b) Using the condition $\vec{r}(2) = \langle 4, 10, 3\rangle$:

$$\langle 4, 10, 3\rangle = \left\langle e^{2-2}, 12\sin\left(\frac{\pi}{4}(2)\right), \frac{2^3}{3}\right\rangle + \langle c_1, c_2, c_3\rangle$$

$$\langle 4, 10, 3\rangle = \left\langle 1, 12, \frac{8}{3}\right\rangle + \langle c_1, c_2, c_3\rangle$$

$$\langle 3, -2, \frac{1}{3}\rangle = \langle c_1, c_2, c_3\rangle$$

This gives us the particular solution:

$$\vec{r}(t) = \left\langle 3 + e^{t-2}, -2 + 12\sin\left(\frac{\pi}{4}t\right), \frac{1}{3} + \frac{t^3}{3}\right\rangle$$

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3. Calculate each limit or show that the limit does not exist.

6 pts (a) $\lim\limits_{(x,y)\to(2,0)} \dfrac{x^2\sin(3y)}{y}$

6 pts (b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+xy+y^2}$

Solution:
(a)

$$\lim_{(x,y)\to(2,0)} \frac{x^2\sin(3y)}{y} = \left(\lim_{(x,y)\to(2,0)} x^2\right)\left(\lim_{(x,y)\to(2,0)} \frac{\sin(3y)}{y}\right) = \left(\lim_{x\to 2} x^2\right)\left(\lim_{y\to 0)} \frac{\sin(3y)}{y}\right) = 4\cdot 3 = 12$$

(b) We consider the paths approaching the origin along the x-axis and along the line $y = x$:
Along x-axis: $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+xy+y^2} = \lim\limits_{x\to 0} \dfrac{0}{x^2} = 0$
Along $y = x$: $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+xy+y^2} = \lim\limits_{x\to 0} \dfrac{x^2}{3x^2} = \dfrac{1}{3}$
Because we have different limits along two paths to $(0,0)$, the limit does not exist.

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4. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lines in $\mathbb{R}^3$ representing the position of two particles at time t with the following parametrizations:
$\mathcal{L}_1$: $\vec{r}_1(t) = \langle 2+3t, -4+\lambda t, -4\rangle$
$\mathcal{L}_2$: $\vec{r}_2(t) = \langle 18-t, 4-4t, -12+2t\rangle$

3 pts (a) For what value of λ do the lines intersect?

3 pts (b) What is the point of intersection?

2 pts (c) Do the particles collide?

7 pts (d) Find an equation of the plane containing both lines.

Solution: (a) In order for the lines to intersect, we must have $\vec{r}_1(t) = \vec{r}_2(s)$ for some $s, t \in \mathbb{R}$. This gives us the system of equations

(1) $2 + 3t = 18 - s$

(2) $-4 + \lambda t = 4 - 4s$

(3) $-4 = -12 + 2s$

Now, from (3) we get $s = 4$. Substitute into (1) to get $t = 4$.
Substitute both into (2) to get $\lambda = -2$.
(b) Plug 4 into either equation to get $(14, -12, -4)$ as the point of intersection.
(c) Yes, because both particles reach the intersection point at time $t = 4$.
(d) To find the normal vector of the plane, we take the cross product of two non-parallel vectors in the plane. The direction vectors of our lines are $\langle 3, -2, 0\rangle$ and $\langle -1, -4, 2\rangle$. Therefore $\vec{n} = \langle 3, -2, 0\rangle \times \langle -1, -4, 2\rangle = \langle -4, -6, -14\rangle$. We also need the position vector of a point on the plane. You could use the intersection point (or any other point on either line), but I choose to use $\vec{r}_1(0) = \langle 2, -4, -4\rangle$.
So the plane has equation $-4(x - 2) - 6(y + 4) - 14(z + 4) = 0$

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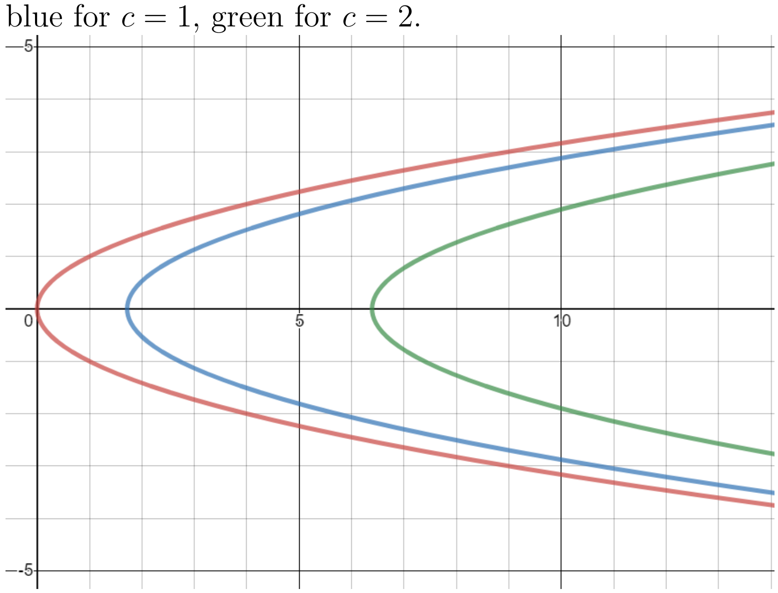
5. Consider the function $f(x, y) = \ln(x - y^2 + 1)$

5 pts (a) Sketch any 3 level curves of the function. Label each curve with the appropriate function value.

3 pts (b) Give a complete and concise English description of the set of all level curves of $f(x, y)$.

Solution: (a) This is a graph of three level curves for $f(x, y) = c$. The red curve for $c = 0$, blue for $c = 1$, green for $c = 2$.



(b) Each level curve $f(x, y) = c$ is a parabola with axis of symmetry on the x-axis and vertex at $(e^c - 1, 0)$. As c increases, the level curves become farther apart (exponentially, in fact). As

c decreases, the level curves get more closely packed with the vertices approaching the point $(-1, 0)$.

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6. Let $\vec{v} = \langle 2, -4, 8\rangle$ and $\vec{w} = \langle 1, a, b\rangle$.

5 pts (a) For what values of a and b are $\vec{v}$ and $\vec{w}$ parallel?

8 pts (b) For what values of a and b are $\vec{v}$ and $\vec{w}$ perpendicular?

Solution: (a) You could have used $\vec{v}\times\vec{w} = \vec{0}$, but that is more work than using the definition. Since both vectors are nonzero, $\vec{v} \parallel \vec{w}$ iff $\vec{v} = \lambda\vec{w}$ for some scalar λ. The first component tells us that $\lambda = 2$, so we must have $a = -2$ and $b = 4$
(b) $\vec{v} \perp \vec{w}$ iff $\vec{v}\cdot\vec{w} = 0$, so $\vec{v}\cdot\vec{w} = 2 - 4a + 8b$ must be 0.
Therefore, any pair (a, b) satisfying $a = 2b + \frac{1}{2}$ will make $\vec{w}$ perpendicular to $\vec{v}$.

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7. Let $\vec{r}(t) = (3\cos t)\hat{\mathbf{i}} + (3\sin t)\hat{\mathbf{j}} + \sqrt{7}t\hat{\mathbf{k}}$.

8 pts (a) Find the tangent vector to $\vec{r}(t)$ at $t = 0$.

6 pts (b) Find the arc length of $\vec{r}(t)$ from $t = 0$ to $t = \pi$.

Solution: (a) $\vec{r}\,'(t) = \langle -3\sin t, 3\cos t, \sqrt{7}\rangle$, so $\vec{r}\,'(0) = \langle 0, 3, \sqrt{7}\rangle$ is the tangent vector.
You could have given me the tangent *line* $\vec{L}(s) = \langle 3, 0, 0\rangle + s\langle 0, 3, \sqrt{7}\rangle$, but that was not required.
(b) The arc length calculation is usually not so easy to integrate explicitly, but this one was:

$$s = \int_0^\pi \|\vec{r}\,'(t)\|\,dt = \int_0^\pi \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (\sqrt{7})^2 dt} = \int_0^\pi 4dt = 4\pi$$

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8. Let $\beta = \dfrac{1 + \sqrt[3]{8.03}}{\sqrt{15.99}}$

10 pts Use an appropriate function $f(x, y)$ and linear approximation to estimate the value of β. Your answer should be a single fraction in lowest terms.

Solution: We should use the function $f(x, y) = \dfrac{1 + \sqrt[3]{x}}{\sqrt{y}}$ and linearize it at the point $(a, b) = (8, 16)$.

$f(8,16) = \dfrac{3}{4}$

$f_x(x,y) = \dfrac{1}{3x^{2/3}y^{1/2}}$, so $f_x(8,16) = \dfrac{1}{48}$

$f_y(x,y) = \dfrac{1+x^{1/3}}{2y^{3/2}}$, so $f_y(8,16) = \dfrac{-3}{128}$

So $L(x,y) = \dfrac{3}{4} + \dfrac{1}{48}(x-8) + \dfrac{-3}{128}(y-16)$

And $L(8,16) = \dfrac{3}{4} + \dfrac{1}{48}\left(\dfrac{3}{100}\right) + \dfrac{-3}{128}\left(\dfrac{-1}{100}\right) = \dfrac{3}{4} + \dfrac{1}{1600} + \dfrac{3}{12800} = \dfrac{9611}{12800}$
