Cardinal Invariants of the Continuum

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Introduction

Is there a set $K \subset \mathbb{R}$ such that

$$\omega = \text{card } \mathbb{N} < \text{card } K < \text{card } \mathbb{R} = \mathfrak{c} = 2^\omega$$

In 1963, Paul Cohen gave the (non) answer:

Theorem (Cohen 1963)

The existence of such a set K is independent of ZFC.

For our purposes, suppose it's true! Even more, that there are $K_1, K_2, \ldots, K_n$ such that

$$\text{card } \mathbb{N} < \text{card } K_1 < \text{card } K_2 < \cdots < \text{card } K_n < \text{card } \mathbb{R}?$$

corresponding to cardinalities

$$\omega < \omega_1 < \omega_2 < \cdots < \omega_n < 2^\omega$$

Unbounded and Dominating Families

It is more convenient to regard $\mathbb{R}$ as countable sequences of integers (ω^ω), or equivalently as functions from $\omega = \mathbb{N} \to \mathbb{N} = \omega$.

Dominating Families

A family $\mathcal{D} \subset \omega^\omega$ is **dominating** if for each $f \in \omega^\omega$ there is a $g \in \mathcal{D}$ such that $f \leq^* g$. $\mathfrak{d}$ is defined to be the smallest cardinality of a dominating family.

$\leq^*$ reads as "is eventually less than." For example $f(n) = n^2 + 1000$, $g(n) = n^3$, then $f \leq^* g$.

Unbounded Families

A family $\mathcal{B} \subset \omega^\omega$ is **unbounded** if there is no single $f \in \omega^\omega$ such that $g \leq^* f$ for all $g \in \mathcal{B}$. $\mathfrak{b}$ is defined to be the smallest cardinality of an unbounded family.

An unbounded family has no upper bound. We now investigate the relationship between $\mathfrak{b}, \mathfrak{d}$.

Unbounded and Dominating Families 2

Theorem

$$\omega < \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c} = 2^\omega$$

Proof

We concern ourselves with the first inequality. To show $\omega < \mathfrak{b}$, we need to show that any countable family of functions is not unbounded. Suppose $\mathcal{B}$ is a countable family of functions enumerated $\{g_n : n \in \mathbb{N}\}$. Then the function

$$f(x) = \max_{n \leq x} g_n(x)$$

is an upper bound on $\mathcal{B}$. This gives $\omega < \mathfrak{b}$.

The construction of f is natural, let's see an example:

$$g_1 = (1, 2, 3, 4, \dots), g_2 = (3, 1, 8, 10, \dots), g_3 = (0, 0, 2, 9, \dots)$$

$$f = (1, 2, 8, \dots)$$

Unbounded and Dominating Families 3

Proof (ctd.)

Next we show that $\mathfrak{b} \leq \mathfrak{d}$. To show this, we want to take an arbitrary dominating family and show that it's unbounded too. Let $\mathcal{D}$ be a dominating family. Suppose it is actually bounded, say by f. This means for all $g \in \mathcal{D}$, $g \leq^* f$. By definition of a dominating family, let g' be a function that has $f \leq^* g'$, this $g \leq^* f$ was utter nonsense! Thus $\mathcal{D}$ is unbounded, and $\mathfrak{b} \leq \mathfrak{d}$.

The final inequality $\mathfrak{d} \leq \text{card } \mathbb{R}$ is trivial, for ω^ω is a dominating family.

More Cardinals!

Splitting Number

A set $X \subset \omega$ **splits** $Y \subset \omega$ if both $Y \cap X$, $Y - X$ are infinite. A **splitting family** is a family S of subsets of ω such that each $Y \subset \omega$ is split by at least one $X \in S$. $\mathfrak{s}$ is the smallest cardinality of a splitting family.

Unsplittable Number

A family $\mathcal{R}$ of infinite subsets of ω is **unsplittable** if no single set splits all members of $\mathcal{R}$. It is σ-**unsplittable** if no countably many sets suffice to split all members of $\mathcal{R}$. The reaping number $\mathfrak{r}$ is the smallest cardinality of any unsplittable family. The σ-unsplitting number $\mathfrak{r}_\sigma$ is the smallest cardinality of any σ unsplittable family.

Classic Results in Analysis and Cardinals

Consider the following classical result in analysis:

Baire Category Theorem (Variant)

$\mathbb{R}$ is not the union of countably many nowhere dense sets.

Is $\mathbb{R}$ the union of ω_1-many nowhere dense sets? How far can we extend this result until it's no longer true?

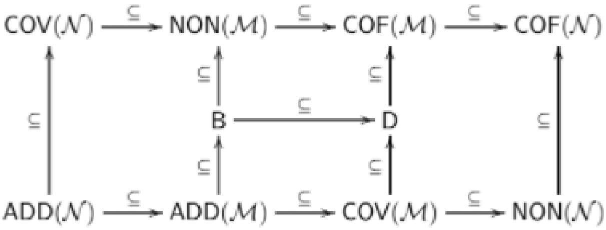
Fact about Sets of Measure Zero

The countable union of measure 0 sets has measure 0.

If this fact can be extended to ω_1-unions, then the Baire Category Theorem also holds for ω_1-unions.

A Note and a Nice Picture

One maybe realizes that $\mathfrak{b}, \mathfrak{d}$ are quite related, one is somehow the "dual" of the other. This is also true of $\mathfrak{s}, \mathfrak{r}$. Galois-Tukey Theory gives a way of cutting down the work we need to do. For example, if $\mathfrak{b} \leq \mathfrak{s}$, Galois-Tukey theory gives us too that $\mathfrak{d} \geq \mathfrak{r}$. And to end, a nice picture illustrating many relationships between defined cardinals:



Sources:

- 1. Combinatorial Cardinal Characteristics of the Continuum, Andrew Blass (Department of Mathematics, University of Michigan, 2003)
- 2. Invariants Of Measure and Catergory, Tomek Bartoszynski (Department of Mathematics and Computer Science, Boise State University, 1999)

The End

Thank you!