Cardinal Invariants of the Continuum

George Tsoukalas

Rutgers University - New Brunswick

george.tsoukalas@rutgers.edu

May 16, 2021

Mentor: Brian Pinsky
Is there a set $K \subset \mathbb{R}$ such that

$$\omega = \text{card } \mathbb{N} < \text{card } K < \text{card } \mathbb{R} = c = 2^\omega$$

In 1963, Paul Cohen gave the (non) answer:

Theorem (Cohen 1963)
The existence of such a set $K$ is independent of ZFC.

For our purposes, suppose it’s true! Even more, that there are $K_1, K_2, \ldots, K_n$ such that

$$\text{card } \mathbb{N} < \text{card } K_1 < \text{card } K_2 < \cdots < \text{card } K_n < \text{card } \mathbb{R}?$$

corresponding to cardinalities

$$\omega < \omega_1 < \omega_2 < \cdots < \omega_n < 2^\omega$$
Unbounded and Dominating Families

It is more convenient to regard $\mathbb{R}$ as countable sequences of integers $(\omega^\omega)$, or equivalently as functions from $\omega = \mathbb{N} \to \mathbb{N} = \omega$.

### Dominating Families

A family $D \subset \omega^\omega$ is **dominating** if for each $f \in \omega^\omega$ there is a $g \in D$ such that $f \leq^* g$. $\frak{d}$ is defined to be the smallest cardinality of a dominating family.

$\leq^*$ reads as "is eventually less than." For example $f(n) = n^2 + 1000$, $g(n) = n^3$, then $f \leq^* g$.

### Unbounded Families

A family $B \subset \omega^\omega$ is **unbounded** if there is no single $f \in \omega^\omega$ such that $g \leq^* f$ for all $g \in B$. $\frak{b}$ is defined to be the smallest cardinality of an unbounded family.

An unbounded family has no upper bound. We now investigate the relationship between $\frak{b}, \frak{d}$. 

George Tsoukalas (DRP Fall 2021)
Theorem

\[ \omega < b \leq d \leq c = 2^\omega \]

Proof

We concern ourselves with the first inequality. To show \( \omega < b \), we need to show that any countable family of functions is not unbounded. Suppose \( B \) is a countable family of functions enumerated \( \{g_n : n \in \mathbb{N}\} \). Then the function

\[
f(x) = \max_{n \leq x} g_n(x)
\]

is an upper bound on \( B \). This gives \( \omega < b \).

The construction of \( f \) is natural, let’s see an example:

\[ g_1 = (1, 2, 3, 4, \ldots), \quad g_2 = (3, 1, 8, 10, \ldots), \quad g_3 = (0, 0, 2, 9, \ldots) \]

\[ f = (1, 2, 8, \ldots) \]
Next we show that $\mathfrak{b} \leq \mathfrak{d}$. To show this, we want to take an arbitrary dominating family and show that it’s unbounded too. Let $\mathcal{D}$ be a dominating family. Suppose it is actually bounded, say by $f$. This means for all $g \in \mathcal{D}$, $g \leq^* f$. By definition of a dominating family, let $g'$ be a function that has $f \leq^* g'$, this $g \leq^* f$ was utter nonsense! Thus $\mathcal{D}$ is unbounded, and $\mathfrak{b} \leq \mathfrak{d}$.

The final inequality $\mathfrak{d} \leq \text{card } \mathbb{R}$ is trivial, for $\omega^\omega$ is a dominating family.
Splitting Number

A set $X \subset \omega$ splits $Y \subset \omega$ if both $Y \cap X$, $Y - X$ are infinite. A splitting family is a family $S$ of subsets of $\omega$ such that each $Y \subset \omega$ is split by at least one $X \in S$. $s$ is the smallest cardinality of a splitting family.

Unsplittable Number

A family $\mathcal{R}$ of infinite subsets of $\omega$ is unsplittable if no single set splits all members of $\mathcal{R}$. It is $\sigma$-unsplittable if no countably many sets suffice to split all members of $\mathcal{R}$. The reaping number $r$ is the smallest cardinality of any unsplittable family. The $\sigma$-unsplitting number $r_\sigma$ is the smallest cardinality of any $\sigma$ unsplittable family.
Consider the following classical result in analysis:

**Baire Category Theorem (Variant)**

\( \mathbb{R} \) is not the union of countably many nowhere dense sets.

Is \( \mathbb{R} \) the union of \( \omega_1 \)-many nowhere dense sets? How far can we extend this result until it’s no longer true?

**Fact about Sets of Measure Zero**

The countable union of measure 0 sets has measure 0.

If this fact can be extended to \( \omega_1 \)-unions, then the Baire Category Theorem also holds for \( \omega_1 \)-unions.
A Note and a Nice Picture

One maybe realizes that \( b, d \) are quite related, one is somehow the ”dual” of the other. This is also true of \( s, r \). Galois-Tukey Theory gives a way of cutting down the work we need to do. For example, if \( b \leq s \), Galois-Tukey theory gives us too that \( d \geq r \). And to end, a nice picture illustrating many relationships between defined cardinals:
Sources:

1. Combinatorial Cardinal Characteristics of the Continuum, Andrew Blass (Department of Mathematics, University of Michigan, 2003)

2. Invariants Of Measure and Category, Tomek Bartoszynski (Department of Mathematics and Computer Science, Boise State University, 1999)
The End
Thank you!