Complex Numbers & The Fundamental Theorem of Algebra

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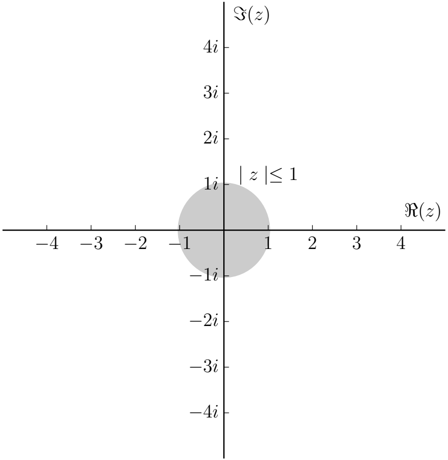
Complex Numbers

Complex Number: $z = x + iy$ where x and y are real numbers.

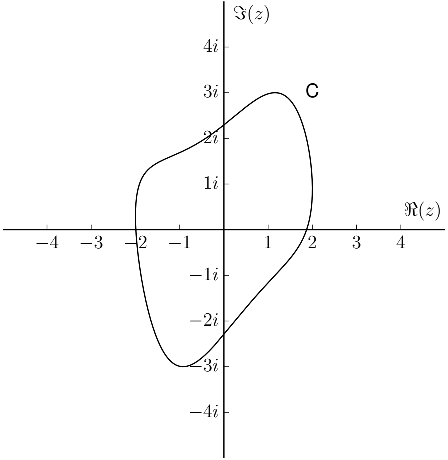
- How we represent them visually:

- $\mid z \mid = \sqrt{x^2 + y^2}$



Complex Numbers, cont.

- Contour Integral: $\oint_C f(z)dz$
- We say a function $f(z)$ of a complex variable is ***analytic*** at a point z_0 if it has a derivative everywhere in some neighborhood of z_0.
- An ***entire*** function is analytic at each point in the complex plane.



The Fundamental Theorem of Algebra

Statement:

Any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n \qquad (a_n \neq 0)$$

of degree n ($n \geq 1$) has at least one zero. That is, there exists at least one point z_0 such that

$$P(z_0) = 0.$$

"All proofs...involve some mathematical analysis, or at least the topological concept of continuity of real or complex functions. Some also use differentiable or even analytic functions. This fact has led to the remark that the Fundamental Theorem of Algebra is neither fundamental, nor a theorem of algebra." -Wikipedia

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Outline

- Step 1: Find an upper bound for $\oint\limits_C f(z)dz$.
- Step 2: Find an upper bound for $|\ f'(z)\ |$.
- Step 3: Prove Liouville's Theorem (any function that is entire and bounded in the complex plane is constant).
- Step 4: Prove the Fundamental Theorem of Algebra.
- Interpretation/Generalization

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Step 1

Want: Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If M is a nonnegative constant such that

$$(1) \mid f(z) \mid \leq M$$

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$$(1)\ |\ f(z)\ | \leq M$$

for all points z on C at which $f(z)$ is defined, then

$$(2)\ \left|\ \oint_C f(z)dz\ \right| \leq ML.$$

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Step 1, cont.

Proof: Assume (1). Let $z = z(t)$ $(a \leq t \leq b)$.

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Then, $\left| \oint_C f(z)dz \right| = \left| \int_a^b f[z(t)]z'(t)dt \right|$

Step 1, cont.

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$$\text{Then, } \left| \oint_C f(z)dz \right| = \left| \int_a^b f[z(t)]z'(t)dt \right|$$

$$\leq \int_a^b \left| f[z(t)]z'(t)dt \right| \leq M \int_a^b \left| z'(t) \right| dt = ML$$

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Note: M always exists.

Step 2

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Want: Suppose that a function f is analytic inside and on a positively oriented circle C_R centered at z_0 and with radius R. If M_R denotes the maximum value of $f(z)$ on C_R, then

$$\left| f'(z_0) \right| \leq \frac{M_R}{R}.$$

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Cauchy's Integral Formula - Extension

Statement:

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)^{n+1}} \qquad (n = 0, 1, 2, ...)$$

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Proof: An extension of Cauchy's Integral Formula states that $f'(z_0) = \frac{1}{2\pi i} \oint\limits_C \frac{f(z)dz}{(z-z_0)^2}$.

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Since C_R is a circle, $z - z_0 = R$, and, by the previous theorem, we have that

$$\begin{aligned} |\ f'(z_0)\ | &= \left| \frac{1}{2\pi i} \oint\limits_{C_R} \frac{f(z)dz}{(z-z_0)^2} \right| \leq \frac{1}{2\pi} \frac{M_R}{R^2} 2\pi R \\ &= \frac{M_R}{R}. \end{aligned}$$

Step 3

We have: $| f'(z_0) | \leq \frac{M_R}{R}$.

Now, assume f is bounded in the complex plane. Then, there exists an M (independent of R) such that $f(z) \leq M$ for all z. Since $M_R \leq M$, $| f'(z_0) | \leq \frac{M}{R}$. If we also assume f is entire (analytic everywhere), then R can be arbitrarily large. Thus,

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$$| f'(z_0) | = 0$$

and f is constant.

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Liouville's Theorem

Statement:

If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Step 4: The Fundamental Theorem

Take a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \qquad (a_n \neq 0)$$

of degree $n \geq 1$. Suppose, for the sake of contradiction, that $P(z) \neq 0$ anywhere. Now let's look at the quotient $\frac{1}{P(z)}$.

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The Fundamental Theorem, cont.

Since $P(z)$ is nonzero everywhere, $\frac{1}{P(z)}$ is analytic everywhere.

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(Boundedness: There is an R such that $P(z) \leq \frac{2}{|a_n|R^n}$ whenever $\mid z \mid \geq R$)

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Interpretation/Generalization

- It turns out that a polynomial $P(z)$ of degree n has n roots, not just one.
- Thus, $P(z)$ can be factored into n factors.
- This means that the roots of $P(z)$ determine $P(z)$.

Interpretation/Generalization

Weierstrass Factorization Theorem

Given any sequence $\{a_n\}$ of complex numbers with $|a_n| \to \infty$ as $n \to \infty$, there exists an entire function f that vanishes at $z = a_n$ and nowhere else.

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