Lie Groups, Homogeneous Manifolds and Model Geometries

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Erlangen Program

Klein's **Erlangen Program** proposed that geometries should be characterized using the concept of symmetries

Klein described it as "Given a manifold[ness] and a group of transformations of the same; to develop the theory of invariants relating to that group"

Klein Geometries

A **Klein Geometry** is a pair (G, H) where G is a Lie Group and H is a closed Lie subgroup of G such that the coset space G/H is connected.

The space $X = G/H$ is a smooth manifold with dimension $dimX = dimG - dimH$

An example of a Klein geometry is spherical geometry S^n which group G is $O(n+1)$ and H is $O(n)$. Invariants include circles and angles.

More specifically, these are homogeneous spaces

Homogeneous Manifolds

A **homogeneous manifold** X has a group G which acts on it transitively
The elements of G are the symmetries of X
An example of a homogeneous space is Spherical space $\mathbb{S}^{n-1}$
Group G: $O(n)$, Stabilizer H: $O(n-1)$

Stabilizers

Let φ be a (left or right) action of a group G on a set M
For any $g \in G$ and $p \in M$ the element $\varphi(g)(p) \in M$ will also be denoted as "$g \star p$"
For any $p \in G$ the set $G_p = \{g \in G | g \star p = p\}$ is called the **stabilizer** of p

Lie Groups

A **manifold** is a second countable Hausdorff space locally Euclidean space

A **Lie group** is a group with a manifold structure

Examples of Lie Groups include

- GL_n: $n \times n$ invertible matrices
- SL_n $n \times n$ matrices with determinant 1

Riemannian Metric

Let M be a manifold. A **Riemannian metric** on M is a choice for each $p \in M$ of an inner product, $\langle . \, , \, . \rangle_p$ on T_pM that "varies smoothly with p in the following sense: for any pair X and Y of smooth vector fields on M, the map $p \mapsto \langle X(p) \, , \, Y(p) \rangle_p$ is a smooth function from M to $\mathbb{R}$. A manifold together with a Riemannian metric is called a **Riemannian manifold**

Geometrization Conjecture

Thurston's **Geometrization Conjecture**: Let M be a closed, orientable, prime 3-manifold. Then there is an embedding of a disjoint union of 2-tori and Klein bottles $\amalg_i T_i^2 \subset M$ such that every component of the complement admits a locally homogeneous Riemannian metric of finite volume

The Eight Model Geometries

A **model geometry** (G, X) is a manifold X together with a Lie group G for diffeomorphisms of X such that:

- X is connected and simply connected
- G acts transitively on X with compact point stabilizers
- G is not contained in any larger group of diffeomorphisms of X with compact stabilizers of points
- there exists at least one compact manifold modeled on (G, X)

The Eight Model Geometries

There are eight three-dimensional model geometries (G, X)

- If the point stabilizers are three-dimensional, X is S^3, E^3 or H^3
- If the point stabilizers are one-dimensional, X fibers over one of the two-dimensional model geometries in a way that is invariant under G. There is a G-invariant Riemannian metric on X such that the connection orthogonal to the fibers has curvature 0 or 1
    - If the curvature is zero, X is $S^2 \times E^1$ or $H^2 \times E^1$
    - If the curvature is one, it is nilgeometry which fibers over E^2 or the geometry of $\tilde{S}L(2, R)$
- If the stabilizers are zero-dimensional, the geometry is solvegeometry which fibers over the line

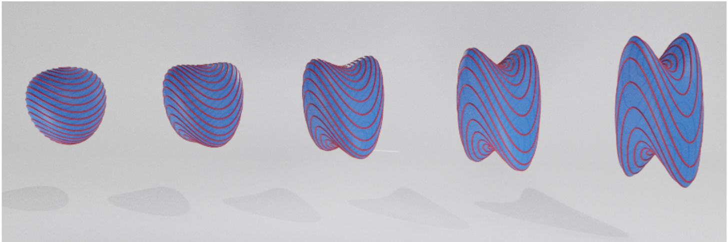
Proof of 3 of the Model Geometries

If $G^{'}$ acts with stabilizer $SO(3)$, any tangent two-plane at any point can be taken by G to any tangent two-plane at any other point, so the metric has constant sectional curvature. Same as the two-dimensional case, the geometry is either spherical, Euclidean, or hyperbolic.

Nilgeometry

Nilgeometry is the geometry of the Heisenberg Group.
The **Heisenberg Group** is the group of $3x3$ matrices of the form
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$
The Heisenberg group is **nilpotent**. It is the only three-dimensional nilpotent but non-Abelian connected and simply connected Lie group.

Nilgeometry



The spheres of radius 1, 2, 3, 4 and 5 in the Heisenberg model of Nil. The spheres have been rescaled to have approximately the same size. Image credit : © Rémi Coulon

References

"Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben. Man soll [..] die Theorie der Beziehungen, welche relativ zur Gruppe invariant sind, untersuchen." - The translation given is by M. W. Haskell and authorized by Klein; cf. New York Math. Soc. 2, 215-249 (1892/93).

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