Lie Groups, Homogeneous Manifolds and Model Geometries

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Klein’s **Erlangen Program** proposed that geometries should be characterized using the concept of symmetries. Klein described it as ”Given a manifold[ness] and a group of transformations of the same; to develop the theory of invariants relating to that group”
A **Klein Geometry** is a pair \((G, H)\) where \(G\) is a Lie Group and \(H\) is a closed Lie subgroup of \(G\) such that the coset space \(G/H\) is connected.

The space \(X = G/H\) is a smooth manifold with dimension 
\[ \dim X = \dim G - \dim H \]

An example of a Klein geometry is spherical geometry \(S^n\) which group \(G\) is \(O(n+1)\) and \(H\) is \(O(n)\). Invariants include circles and angles.

More specifically, these are homogeneous spaces
A **homogeneous manifold** $X$ has a group $G$ which acts on it transitively.
The elements of $G$ are the symmetries of $X$.
An example of a homogeneous space is Spherical space $\mathbb{S}^{n-1}$.
Group $G$: $O(n)$, Stabilizer $H$: $O(n-1)$
Let \( \varphi \) be a (left or right) action of a group \( G \) on a set \( M \).
For any \( g \in G \) and \( p \in M \) the element \( \varphi(g)(p) \in M \) will also be denoted as \( g \ast p \).
For any \( p \in G \) the set \( G_p = \{ g \in G | g \ast p = p \} \) is called the stabilizer of \( p \).
A **manifold** is a second countable Hausdorff space locally Euclidean space.

A **Lie group** is a group with a manifold structure.

Examples of Lie Groups include:

- $GL_n$: $n \times n$ invertible matrices
- $SL_n$: $n \times n$ matrices with determinant 1
Let $M$ be a manifold. A **Riemannian metric** on $M$ is a choice for each $p \in M$ of an inner product, $\langle \cdot, \cdot \rangle_p$ on $T_p M$ that ”varies smoothly with $p$ in the following sense: for any pair $X$ and $Y$ of smooth vector fields on $M$, the map $p \mapsto \langle X(p), Y(p) \rangle_p$ is a smooth function from $M$ to $\mathbb{R}$. A manifold together with a Riemannian metric is called a **Riemannian manifold**.
Thurston’s **Geometrization Conjecture**: Let $M$ be a closed, orientable, prime 3-manifold. Then there is an embedding of a disjoint union of 2-tori and Klein bottles $\biguplus_i T^2_i \subset M$ such that every component of the complement admits a locally homogeneous Riemannian metric of finite volume.
A model geometry \((G, X)\) is a manifold \(X\) together with a Lie group \(G\) for diffeomorphisms of \(X\) such that:

- \(X\) is connected and simply connected
- \(G\) acts transitively on \(X\) with compact point stabilizers
- \(G\) is not contained in any larger group of diffeomorphisms of \(X\) with compact stabilizers of points
- there exists at least one compact manifold modeled on \((G, X)\)
There are eight three-dimensional model geometries \((G, X)\)

- If the point stabilizers are three-dimensional, \(X\) is \(S^3, E^3\) or \(H^3\)
- If the point stabilizers are one-dimensional, \(X\) fibers over one of the two-dimensional model geometries in a way that is invariant under \(G\). There is a \(G\)-invariant Riemannian metric on \(X\) such that the connection orthogonal to the fibers has curvature 0 or 1
  - If the curvature is zero, \(X\) is \(S^2 \times E^1\) or \(H^2 \times E^1\)
  - If the curvature is one, it is nilgeometry which fibers over \(E^2\) or the geometry of \(\tilde{SL}(2, R)\)
- If the stabilizers are zero-dimensional, the geometry is solvegeometry which fibers over the line
If $G'$ acts with stabilizer $SO(3)$, any tangent two-plane at any point can be taken by $G$ to any tangent two-plane at any other point, so the metric has constant sectional curvature. Same as the two-dimensional case, the geometry is either spherical, Euclidean, or hyperbolic.
Nilgeometry is the geometry of the Heisenberg Group. The Heisenberg Group is the group of 3x3 matrices of the form
\[
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix}
\]
The Heisenberg group is nilpotent. It is the only three-dimensional nilpotent but non-Abelian connected and simply connected Lie group.
The spheres of radius 1, 2, 3, 4 and 5 in the Heisenberg model of Nil. The spheres have been rescaled to have approximately the same size. Image credit: © Rémi Coulon
