# Differential Topology

## A Brief Journey

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# Manifolds

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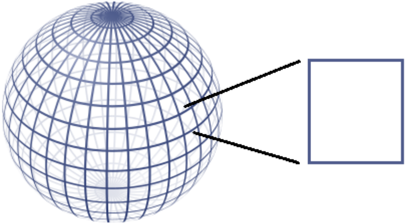

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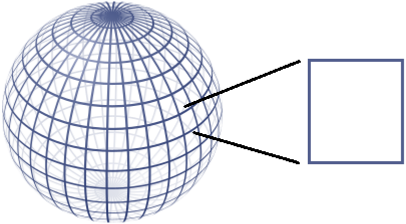

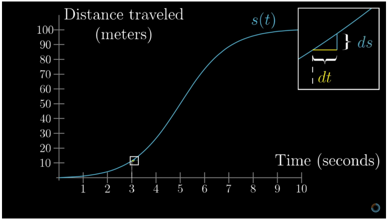

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Two manifolds are *diffeomorphic* to each other if a diffeomorphism exists between them.

# Inverse Function Theorem

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This gives us the following definitions:

- *Immersion* - $f : X \to Y$ is an immersion at $x$ if $df_x : T_x(X) \to T_y(Y)$ is injective
- *Submersion* - $f : X \to Y$ is a submersion at $x$ if $df_x : T_x(X) \to T_y(Y)$ is surjective

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## Local Immersion Theorem

If $f : X \to Y$ is an immersion at $x$, then there exists a change of variables around $x$ such that $f$ is equivalent to the *canonical immersion*, which means

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## Local Submersion Theorem

If $f : X \to Y$ is a submersion at $x$, then there exists a change of variables around $x$ such that $f$ is equivalent to the *canonical submersion*, which means

$$f(x_1, \dots, x_k, \dots, x_n) = (x_1, \cdots, x_k)$$

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*Regular Value* - If $f : X \to Y$ is a smooth map between manifolds, a point $y \in Y$ is a *regular value* of $f$ if $df_x$ is surjective at every $x$ such that $f(x) = y$.

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## Preimage Theorem

If $y \in Y$ is a regular value of $f : X \to Y$, then $f^{-1}(y)$ is a submanifold of $X$ with dimension $\dim X - \dim Y$.

# Applying to Spheres

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is defined as $f(\mathbf{x}) = |\mathbf{x}|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$. The level sets of this function are spheres.

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Recall that for a regular value $y \in Y$, we see if the derivative at each $x$ such that $f(x) = y$ is surjective. Here, we see this is true for all $y \neq 0$. By the preimage theorem, $f^{-1}(y)$ is a submanifold of dimension $n-1$, which is a sphere of radius $y$. This gives us the name $S^{n-1}$.

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$O(n)$ is the group of $n \times n$ orthogonal matrices, or all matrices $A$ such that $AA^T = I$. They preserve distance and have determinant $\pm 1$. In other words, it's the set of all rotation and reflection matrices!

# Applying to O(n)

Proof:
$M(n)$ - group of all matrices
$S(n)$ - group of all symmetric matrices Clearly, the matrix $AA^T \in S(n)$ for all matrices $A$. We define $f(A) = AA^T$. We check now if $I$ is a regular value of $f$.

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$$df_A(B) = \frac{1}{2}CAA^T + \frac{1}{2}A(CA)^T = \frac{1}{2}C(AA^T) + \frac{1}{2}(AA^T)C^T = \frac{1}{2}C + \frac{1}{2}C^T = C$$

Therefore, $I$ is a regular value, and $f^{-1}(I) = O(n)$ is a manifold.

# Thank you!

Presentation made with help from *Differential Topology* by Alan Pollack and Victor Guillemin.

A huge thanks to Soham Chanda, my DRP mentor!