Bertrand's Postulate

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May 14, 2021

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Introduction

Size of its gaps between primes may not be bounded. Write $N = \prod_{\text{primes}} p_i$, for $p_i < k + 2$ and note that the sequence

$$N + 2, N + 3, N + 4, \dots, N + k, N + (k + 1)$$

contains no primes, since for $2 \leq p_i \leq k + 1$ we know that p_i has a prime factor that is smaller than $k + 2$, and this factor also divides N, and hence also $N + i$.

However, there exists **upper bounds for the gaps in the sequence of prime numbers**.

Bertrand's postulate

Bertrand's Postulate (1845)

For sufficiently large n, there is some prime number p with $n \leq p \leq 2n$.

In other words, *"the gap to the next prime cannot be larger than the number we start our search at."*

Conjectured and verified empirically for $n < 3,000,000$.

Figure: Joseph Bertrand

History of Proofs

First proved by **Chebyshev (1850)**

Ramanujan (1919): Provided a simpler proof inspired by Chebyshev

Erdős (1932): Analyzed size of the central binomial coefficient $\binom{2n}{n}$ to show that it is not "too small"

Prime Counting Function

The function $\pi(x)$ counts the number of prime numbers less than or equal to x and the prime number theorem tells us

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

Bertrand's postulate is equivalent to $\pi(x) - \pi\left(\frac{x}{2}\right) \geq 1$.

There exists "better" prime counting functions!

Better Prime Counting Functions

The **von Mangoldt function**, is defined $\Lambda(1) = 0$ and

$$\Lambda(x) = \begin{cases} \log p & \text{if } x = p^\alpha \text{ for some } \alpha \geq 1 \text{, and } p \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

The **Chebyshev's θ function**, is defined for prime p:

$$\theta(x) = \sum_{p \leq x} \log p$$

and the **Chebyshev's ψ function** is defined

$$\psi(x) = \sum_{p^\alpha \leq x} \log p = \sum_{n \leq x} \Lambda(n)$$

Relations connecting $\theta(x)$ and $\pi(x)$

Equivalent expressions to the prime number theorem are

$$\lim_{x\to\infty} \frac{\theta(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to\infty} \frac{\psi(x)}{x} = 1$$

For $x \geq 2$, we have

$$\theta(x) = \pi(x)\log x - \int_x^2 \frac{\pi(t)}{t} dt$$

and

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t\log^2 t} dt.$$

Ramanujan's Proof

Bertrand's postulate: $\theta(x) - \theta\left(\frac{x}{2}\right) > 0$, for any $x \geq 2$

Define

$$T(x) = \sum_{n \leq x} \log n = \sum_{de \leq x} \Lambda(d) = \sum_{e \leq x} \psi\left(\frac{x}{e}\right).$$

Summation Technique [Murty, Exercise 2.1.2]

$$\sum_{n \leq x} \log n = x \log x - x + O(\log x).$$

Ramanujan's Proof [cont.]

$$T(x) - 2T\left(\frac{x}{2}\right) = \sum_{n \leq x} (-1)^{n-1} \psi\left(\frac{x}{n}\right).$$

Decreasing Sequences

If $\{a_n\}$ is a decreasing sequence of real numbers tending to 0, then

$$a_0 - a_1 \leq \sum_{n=0}^{\infty} (-1)^n a_n \leq a_0 - a_1 + a_2.$$

$$T(x) - 2T\left(\frac{x}{2}\right) \leq \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right)$$

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) \geq x \log 2 + O(\log x)$$

Ramanujan's Proof [cont.]

Eventually, we find

$$\psi(x) - \psi\left(\frac{x}{2}\right) \geq \frac{x}{3}\log 2 + O(\log^2 x).$$

With $\psi(x) = \theta(x) + O(\sqrt{x}\log^2 x)$, we get

$$\theta(x) - \theta\left(\frac{x}{2}\right) \geq \frac{x}{3}\log 2 + O\left(\sqrt{x}\log^2 x\right).$$

Hence, for x sufficiently large, there is a prime between $x/2$ and x.

Generalizations

Theorem (Sylvestor)

The product of k consecutive integers greater than k is divisible by a prime greater than k.

Theorem (Erdős)

For any positive integer k, there is a natural number N such that for all $n > N$, there are at least k primes between n and $2n$.

Conjecture (Legendre)

For every $n > 1$, there is a prime between n^2 and $(n+1)^2$.

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The End
Thank you!