Random Walks and Electric Networks

Rajeev Atla[1] Quentin Dubroff[2]

[1]Mentee

[2]Mentor

Fall 2021 DRP Presentation

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References

- Doyle and Snell - *Random walks and electric networks* [1]

Introduction: Definitions

- **Random walk** - a random process that consists of random steps on some mathematical space e.g. $\mathbb{Z}$

Introduction: Definitions

- **Random walk** - a random process that consists of random steps on some mathematical space e.g. $\mathbb{Z}$
- **Electric network** - an interconnection of electric components
    - We only care about resistors

A Random Walk

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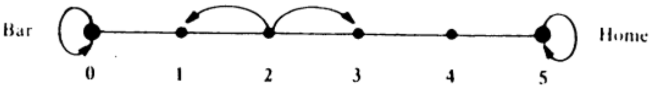

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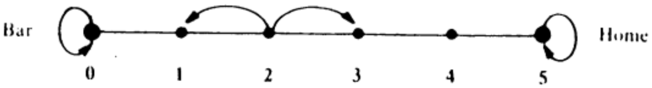

- **Escape probability** - the probability of never going to the bar (origin) i.e. going home before going to the bar

Recurrence Relation

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1. $p(0) = 0$
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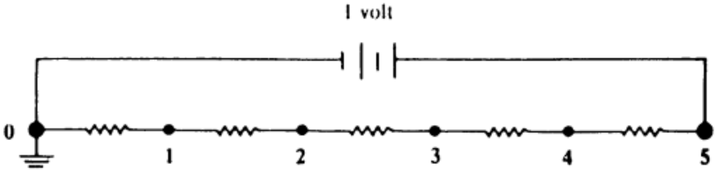
- Uniqueness theorem can be proved

Similar Problem ?!



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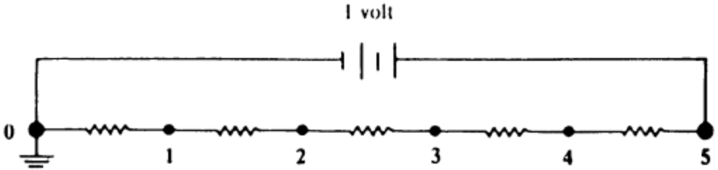

- Want to find voltage at each node $v(x)$

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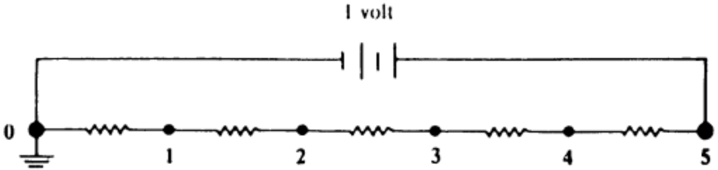

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$$v(x) = \frac{1}{2}v(x-1) + \frac{1}{2}v(x+1)$$

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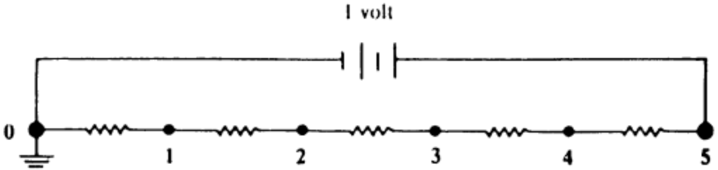

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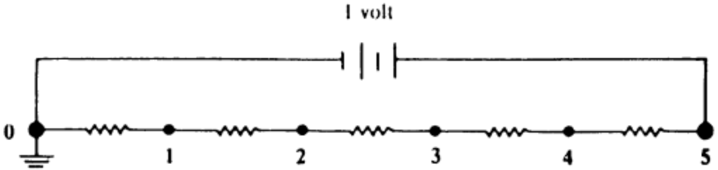

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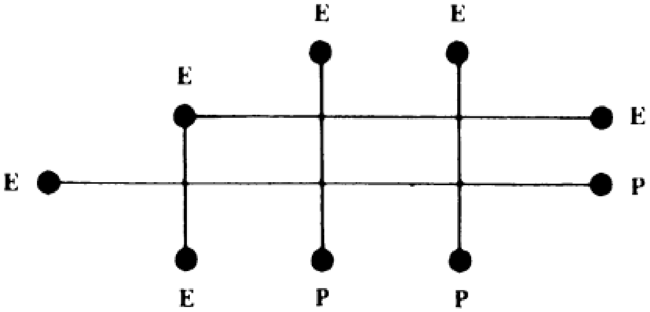
- Uniqueness theorem
- Escape probability is inverse of effective resistance

More Dimensions



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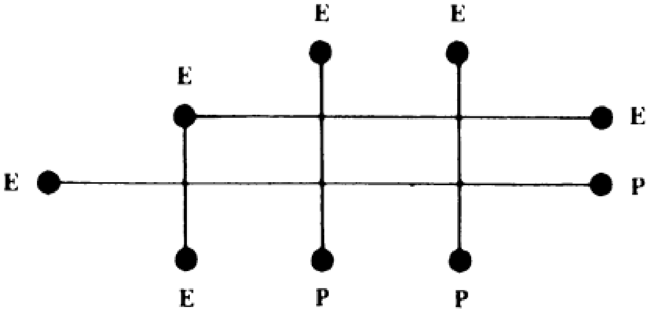


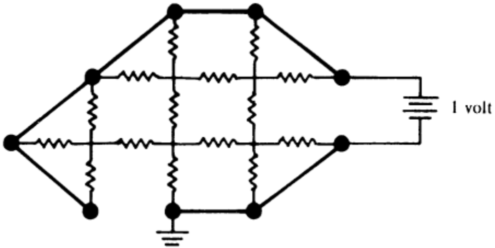
- E is escape, P is police
- What is the probability of escape before a policeman is encountered?

More Recurrences

- Averaging principle from Law of Total Probability

$$f(a,b) = \frac{1}{4}f(a+1,b)+\frac{1}{4}f(a-1,b)+\frac{1}{4}f(a,b+1)+\frac{1}{4}f(a,b-1)$$

Electric Analogue



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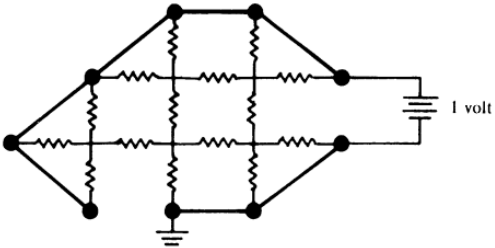
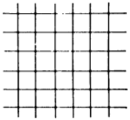


- Using Ohm's and Kirchoff's Laws, the voltage at each point $v(a,b)$ is

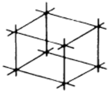
$$v(a,b) = \frac{1}{4}v(a+1,b) + \frac{1}{4}v(a-1,b) + \frac{1}{4}v(a,b+1) + \frac{1}{4}v(a,b-1)$$

Lattices and Infinite Random Walks

1-dimensional lattice



2-dimensional lattice



3-dimensional lattice

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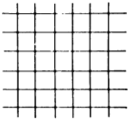

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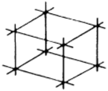

3-dimensional lattice

- $\mathbb{Z}^d$ is an infinite d-dimensional lattice

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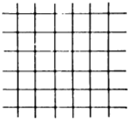
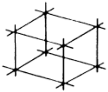


2-dimensional lattice



3-dimensional lattice

- $\mathbb{Z}^d$ is an infinite d-dimensional lattice
- Let $\mathbf{0}$ be the origin or where the random walk started
- We want to find escape probability i.e. never returning to $\mathbf{0}$
- **Recurrent** - escape probability is 0
- **Transient** - escape probability is nonzero

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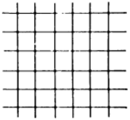
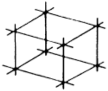


2-dimensional lattice



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- **Type problem** - determining whether or not a random walk is transient or recurrent

Electric Analogue (1D)

Resistance to infinity infinite!

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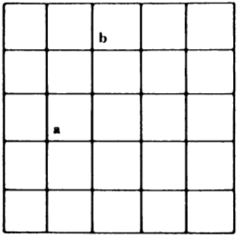
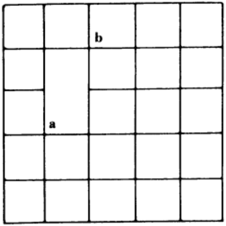
Resistance to infinity infinite!

- 1D is trivial

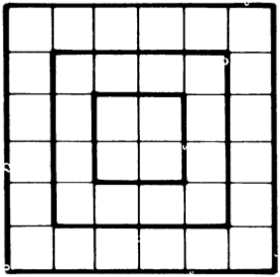
Rayleigh's Monotonicity Law





- Right is lower resistance than left

2D Random Walk



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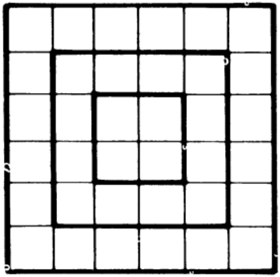
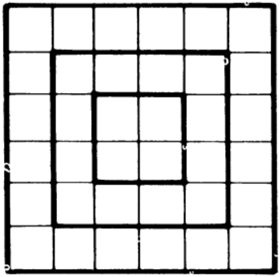


- Add up all the branches in parallel

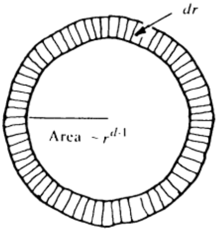
2D Random Walk



- Add up all the branches in parallel

$$\frac{1}{4} + \frac{1}{12} + \frac{1}{20} \cdots = \sum \frac{1}{8n+4} \sim \sum \frac{1}{n} = \infty$$

Higher Dimensions



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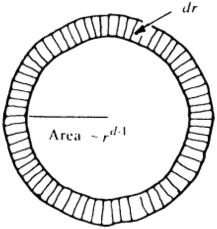

$$R_{\text{eff}} \sim \int_a^\infty \frac{dr}{r^{d-1}}$$

- Integral diverges only for $d = 1, 2$

Why This is Useful

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- Simplifies calculations greatly!

References

Doyle et al.: Random walks and electric networks

Peter G. Doyle and J. Laurie Snell. *Random walks and electric networks*. MAA Press, 2006. URL: https://math.dartmouth.edu/~doyle/docs/walks/walks.pdf.