SOLUTIONS TO HW 6

Section 3.2

- 4. For each of the following, verify that the relation is an equivalence relation. Then give the information about the equivalence class as specified.
  - (a) The relation R on $\mathbb{Z}$ is given by xRy iff $x^2 = y^2$.
    * (Reflexivity:) Let $n \in \mathbb{Z}$. Since $n^2 = n^2$, we have nRn.
    * (Symmetry:) Let $n, m \in \mathbb{Z}$. Suppopse nRm. Thus $n^2 = m^2$. Thus $m^2 = n^2$. Thus mRn.
    * (Transitivity:) Let $n, m, k \in \mathbb{Z}$. Suppose nRm and mRk. Then $n^2 = m^2 = k^2$. Thus nRk.

    $0/R = \{0\}$. $4/R = \{4, -4\}$. $-72/R = \{72, -72\}$.
  - (b) The relation R on $\mathbb{N}$ is given by mRn iff m and n have the same digit in the tens places.
    * (Reflexivity:) Let $n \in \mathbb{N}$. Since n has the same digit in its tens place as n, we have nRn.
    * (Symmetry:) Let $n, m \in \mathbb{N}$. Suppopse nRm. Thus n and m both have the same digit in the tens place. Thus mRn.
    * (Transitivity:) Let $n, m, k \in \mathbb{Z}$. Suppose nRm and mRk. Then n, m, and k all have the same digit in the tens place. Thus nRk.

    The elements of $106/R$ that are less than 50 are 1,2,3,4,5,6,7,8, and 9. The elements of $106/R$ between 150 and 300 are 200, 201, . . . , 209. An element of $106/R$ greater than 1000 is, e.g., 1100. Three elements of $635/R$ are, e.g., 32, 635, and 1131539.
  - (c) The relation V on $\mathbb{R}$ is given by xVy iff $x = y$ or $xy = 1$.
    * (Reflexivity:) Let $x \in \mathbb{R}$. Since $x = x$, we have xVx.
    * (Symmetry:) Suppopse xVy. Thus $x = y$ or $xy = 1$. Thus $y = x$ or $yx = 1$, so we have yVx.
    * (Transitivity:) Let $x, y, z \in \mathbb{V}$. Suppose xVy and yVz. Thus there are four cases. (1) If $x = y$ and $y = z$ then $x = z$, so xVz. (2) If $x = y$ and $yz = 1$ then $xz = 1$, so xVz. (3) If $xy = 1$ and $y = z$, then $xz = 1$, so xVz. (4) If $xy = yz = 1$ then $x = z = \frac{1}{y}$, so xVz.

$3/V = \{3, \frac{1}{3}\}$. $\quad -\frac{2}{3}/V = \{-\frac{2}{3}, -\frac{3}{2}\}$ $\quad 0/V = \{0\}$.

- (j) The relation R on the set $\mathcal{D}$ of all differentiable functions is given by fRg iff $f' = g'$.
  - (Reflexivity:) Let $f \in \mathcal{D}$. Since $f' = f'$, we have fRf.
  - (Symmetry:) Let $f, g \in \mathcal{D}$. Suppopse fRg. Thus $f' = g'$, so $g' = f'$, so we have gRf.
  - (Transitivity:) Let $f, g, h \in \mathcal{D}$. Suppose fRg and gRh. Then $f' = g' = h'$. Thus fRh.

  $x^2/R = \{x^2 + C : C \in \mathbb{R}\}$. $\quad (4x^3 + 10x)/R = \{4x^3 + 10x + C : C \in \mathbb{R}\}$. $\quad x^3/R = \{x^3 + C : C \in \mathbb{R}\}$. $\quad 7/R = \mathbb{R}$.

- 6(d) Calculate the equivalence classes for the relation of congruence modulo 7.

$$
\begin{aligned}
0/\equiv_7 &= \bar{0} = \{\dots, -14, -7, 0, 7, 14, 21, \dots\} \\
1/\equiv_7 &= \bar{1} = \{\dots, -13, -6, 1, 8, 15, 22, \dots\} \\
2/\equiv_7 &= \bar{2} = \{\dots, -12, -5, 2, 9, 16, 23, \dots\} \\
3/\equiv_7 &= \bar{3} = \{\dots, -11, -4, 3, 10, 17, 24, \dots\} \\
4/\equiv_7 &= \bar{4} = \{\dots, -10, -3, 4, 11, 18, 25, \dots\} \\
5/\equiv_7 &= \bar{5} = \{\dots, -9, -2, 5, 12, 19, 26, \dots\} \\
6/\equiv_7 &= \bar{6} = \{\dots, -8, -1, 6, 13, 20, 27 \dots\}
\end{aligned}
$$

- 9. Suppose that R and S are equivalence relations on a set A. Prove that $R \cap S$ is an equivalence relation on A.
  - (Reflexivity:) Let $x \in A$. We have $(x, x) \in R$ (since R is reflexive) and $(x, x) \in S$ (since S is reflexive). Thus $(x, x) \in R \cap S$.
  - (Symmetry:) Let $(x, y) \in R \cap S$. Thus $(x, y) \in R$ and $(x, y) \in S$. By the symmetry of R, $(y, x) \in R$. By the symmetry of S, $(y, x) \in S$. Thus $(y, x) \in R \cap S$.
  - (Transitivity:) Let $(x, y), (y, z) \in R \cap S$. Thus $(x, y), (y, z) \in R$ and $(x, y), (y, z) \in S$. We have $(x, z) \in R$ (resp. S) by the transitivity of R (resp. S). Thus $(x, z) \in R \cap S$.

Section 3.4

- 8. Define the relation on $\mathbb{R} \times \mathbb{R}$ by $(a, b)R(x, y)$ iff $a \leq x$ and $b \leq y$. Prove that R is a partial order on $\mathbb{R} \times \mathbb{R}$.

  - (Reflexivity:) Let $(x, y) \in \mathbb{R} \times \mathbb{R}$. Since $x \leq x$ and $y \leq y$, we have $(x, y)R(x, y)$.
  - (Anti-symmetry:) Let $(a, b)R(x, y)$ and $(x, y)R(a, b)$. Thus $a \leq x$ and $x \leq a$. Also, $b \leq y$ and $y \leq b$. Thus $a = x$ and $b = y$, so $(a, b) = (x, y)$.
  - (Transitivity:) Let $(a, b)R(c, d)$ and $(c, d)R(e, f)$. Thus $a \leq c \leq e$ and $b \leq d \leq f$. So, $(a, b)R(e, f)$.

- Prove that the partial order in problem 8 is *not* a total (linear) order. All we need to do is produce two ordered pairs (a, b) and (x, y) in $\mathbb{R} \times \mathbb{R}$ such that neither $(a, b)R(x, y)$ nor $(x, y)R(a, b)$ holds. One such counterexample is given by $(1, 2)$ and $(2, 1)$.