5(a) • $f(x) \geqq 0 \quad \forall x$.

• $$\int_{-\infty}^{\infty} f(x)\,dx = \frac{4}{\pi}\int_0^1 \frac{dx}{1+x^2} = \frac{4}{\pi}\arctan x\Big|_0^1 = \frac{4}{\pi}\left(\frac{\pi}{4}-0\right) = 1$$

(b) $$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{4}{\pi}\int_0^1 \frac{x\,dx}{1+x^2} = \frac{2}{\pi}\int_1^2 \frac{du}{u} = \frac{2}{\pi}\ln|u|\Big|_1^2$$

$u = x^2+1$

$du = 2x\,dx$

$$= \frac{2}{\pi}(\ln 2 - \ln 1) = \boxed{\frac{2\ln 2}{\pi}}$$

$$x^2+1 \,\overline{)\, x^2+0x+0} \quad 1$$
$$x^2+0x+1$$

(c) $$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{4}{\pi}\int_0^1 \frac{x^2\,dx}{1+x^2} = \frac{4}{\pi}\int_0^1\left(1-\frac{1}{x^2+1}\right)dx$$

$$= \frac{4}{\pi}\left(1-\frac{\pi}{4}\right) = \frac{4}{\pi}-1. \quad \operatorname{Var} X = E(X^2)-(E(X))^2 = \boxed{\left(\frac{4}{\pi}-1\right)-\left(\frac{2\ln 2}{\pi}\right)^2}$$

(d) $$\int_{-\infty}^{\tilde{\mu}} f(x)\,dx = \frac{1}{2} \Rightarrow \frac{4}{\pi}\int_0^{\tilde{\mu}} \frac{dx}{1+x^2} = \frac{1}{2} \Rightarrow \frac{4}{\pi}\arctan\tilde{\mu} = \frac{1}{2}$$

$$\Rightarrow \arctan\tilde{\mu} = \frac{\pi}{8} \Rightarrow \boxed{\tilde{\mu} = \tan\left(\frac{\pi}{8}\right)}.$$

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6. (a) $E(\bar{X}) = \mu_X = \nu$. $\therefore$ $\bar{X}$ is an unbiased estimator for ν.

$\operatorname{Var}(\bar{X}) = \sigma^2_{\bar{X}} = \frac{\sigma^2_X}{n} = \frac{2\nu}{n} \to 0$ as $n \to \infty$

$\therefore$ $\bar{X}$ is a consistent estimator for ν

(b) $E(2\bar{X}) = 2E(\bar{X}) = 2\nu = \sigma^2_X$ $\therefore$ $2\bar{X}$ is an unbiased estimator for σ^2_X

(c) $$\operatorname{Var}(2\bar{X}) = E[(2\bar{X})^2] - [E(2\bar{X})]^2 = E(4\bar{X}^2) - [E(2\bar{X})]^2$$
$$= 4E(\bar{X}^2) - 4(E(\bar{X}))^2 = 4\{E(\bar{X}^2)-(E(\bar{X}))^2\} = 4\operatorname{Var}\bar{X} = 4\cdot\sigma^2_{\bar{X}}$$
$$= 4\frac{\sigma^2_X}{n} = 4\cdot\frac{2\nu}{n} = \frac{8\nu}{n} \to 0 \text{ as } n \to \infty$$

$\therefore$ $2\bar{X}$ is a consistent estimator for σ^2_X