On Simplifications of Certain q-Multisums

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Abstract

Some examples of naturally arising multisum q-series which turn out to have representations as fermionic single sums are presented. The resulting identities are proved using transformation formulas from the theory of basic hypergeometric series.

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1 Introduction

It is certainly useful to be able to transform a naturally arising q-multisum into a single-fold sum. Many such identities are well known, e.g. the "a-generalized Andrews-Gordon theorem," (see Eq. (1.2) below), and Bressoud's generalization to even moduli [14,15], Krattenthaler and Rosengren's q-analog [20] of an identity of Gelfand, Graev, and Retakh [18], the many identities of the physicists Berkovich, McCoy, Orrick, Pearce, Schilling, and Warnaar [8,9,11–13,28], to name a few. In these papers, the multisums generally fall into a category of what physicists call "fermionic" representations, while the single sum representations are "bosonic."

A different category of q-identities, often called "fermionic reduction formulas," first appears in Andrews' 1981 paper on multiple q-Series identities [4], where it is shown how to simplify certain fermionic q-multisums via two "amalgamation lemmas" [4, pp. 19, 20; Lemmas 1, 2]. Another important result in this genre was conjectured by Melzer [21], and proved by Bressoud, Ismail, and

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Stanton using Bailey lemma techniques [16, Thms. 5.1 and 5.2]. The Melzer conjecture was subsequently reproved by Warnaar [30, Thm. 4.4] using other methods. In [29], Warnaar provides another interesting fermionic reduction formula. The results of this paper fall into the category of fermionic reduction formulas.

The terms "fermionic" and "bosonic" arise in statistical physics. For proper definitions of "fermionic" and "bosonic," one should consult an appropriate paper written by a physicist; see e.g. [10, p. 165 ff.]. However, since q-series identities arising in physics also occur in combinatorics and mathematical analysis, it seems reasonable, by analogy, to attach the term "fermionic" to any set of integer partitions with difference conditions and to its associated generating function, and the term "bosonic" to a set of integer partitions associated with congruence conditions, and to its associated generating function. This is the sense in which I use the terms herein.

Before proceeding to the identities, it would be useful to point out the combinatorial context which paved the way to their discovery.

A *partition* π of an integer n is a nonincreasing sequence $(\pi_1, \pi_2, \pi_3, \dots)$ of nonnegative integers such that $\sum_{j=1}^{\infty} \pi_j = n$. Each nonzero π_i is called a *part* of π. The number of times j appears in π is called the *multiplicity* of j in π and is denoted $m_j(\pi)$.

Recall Gordon's combinatorial generalization of the Rogers-Ramanujan identities [19]:

Gordon's Partition Theorem *Let $A_{k,i}(n)$ denote the number of partitions of the integer n into parts $\not\equiv 0, \pm i \pmod{2k+1}$. Let $B_{k,i}(n)$ denote the number of partitions π of n such that $m_1(\pi) < i$, and for any positive integer j, $m_j(\pi) + m_{j+1}(\pi) < k$. Then for $1 \leqq i \leqq k$, $A_{k,i}(n) = B_{k,i}(n)$.*

It is well known [1, p. 111, Theorem 7.8] that Andrews' generalization of the Rogers-Ramanujan identities for odd moduli [2, p. 4082, Theorem 1] is a q-series counterpart to Gordon's theorem, and as such is often called the "Andrews-Gordon theorem."

The Andrews-Gordon Theorem *For $1 \leqq i \leqq k$, $k \geqq 1$,*

$$\sum_{n_1, n_2, \dots, n_{k-1} \geqq 0} \frac{q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_i + N_{i+1} + \cdots + N_{k-1}}}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_{k-1}}} = \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}, \tag{1.1}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$, and

$$(a)_n = (a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}),$$

$$(a)_\infty = (a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots,$$
$$(a_1,a_2,\ldots,a_r;q)_n = (a_1)_n(a_2)_n\cdots(a_r)_n.$$

Furthermore, a standard refinement of the $B_{k,i}(n)$ of Gordon's partition theorem counts $\mathcal{B}_{k,i}(m,n)$, the number of partitions of n counted by $B_{k,i}(n)$ which have exactly m parts. The analogous refinement of (1.1) is [1, p. 112, Eq. (7.3.8)]

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{n}&\mathcal{B}_{k,i}(m,n)a^m q^n\\ &= \sum_{n_1,n_2,\ldots,n_{k-1}\geqq 0}\frac{a^{N_1+N_2+\cdots+N_{k-1}}q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}}\\ &= \frac{1}{(aq)_\infty}\sum_{n\geqq 0}\frac{(-1)^n a^{kn}q^{(2k+1)n(n+1)/2-in}(1-a^iq^{(2n+1)i})(aq)_n}{(q)_n}\end{aligned}\tag{1.2}$$

In a recent paper [24], I showed that certain q-series were related to dilated versions of special cases of Gordon's partition theorem. Accordingly, the following identities between multisum and single sum q-series, although not explicitly stated in [24], follow immediately from the results therein:

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{a^{r+2s}q^{2(r+s)^2+2s^2}}{(q^2;q^2)_r(q^2;q^2)_s} = (aq;q^2)_\infty\sum_{j=0}^{\infty}\frac{a^jq^{j^2}}{(aq;q^2)_j(q)_j}\tag{1.3}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{a^{r+2s+3t}q^{3(r+s+t)^2+3(s+t)^2+3t^2}}{(q^3;q^3)_r(q^3;q^3)_s(q^3;q^3)_t} = \frac{(aq)_\infty}{(aq^3;q^3)_\infty}\sum_{j=0}^{\infty}\frac{a^jq^{j^2}(a;q^3)_j}{(a)_{2j}(q)_j}.\tag{1.4}$$

Note that the series on the left hand sides of (1.3) and (1.4) are special cases of the Andrews-Gordon theorem (with $q\to q^2$, $k=i=3$; and $q\to q^3$, $k=i=4$ respectively), while the right hand sides arise in Bailey's two-variable generalizations [7, pp. 6–7] of Rogers' first mod 14 identity [22, p. 341, Ex. 2] and Dyson's first mod 27 identity [6, p. 433, Eq. (B4)] respectively.

Note that the Andrews-Gordon theorem, and indeed all of the identities presented herein, may be regarded as identities of analytic functions and are thus subject to convergence conditions. However, since the underlying motivation is combinatorial (and so the series may be regarded as generating functions), the convergence conditions will not be explicitly mentioned.

Although proved in [24] with the aid of systems of q-difference equations, it is not at all obvious *why* the right hand side of (1.3) enumerates the partitions

from the $k = i = 3$ case of Gordon's partition theorem (dilated by a factor of 2) and the right hand side of (1.4) enumerates the parititons from the $k = i = 4$ case of Gordon's partition theorem (dilated by a factor of 3), but once this fact is established, their equality with their respective left hand sides is immediate thanks to the Andrews-Gordon theorem.

The purpose of this note is to present q-hypergeometric proofs of (1.3) and (1.4) in order to gain an understanding of these identities from the standpoint of basic hypergeometric series. The identities (1.3) and (1.4) will be derived as corollaries of the more general identities

$$\begin{aligned}\sum_{j=0}^{n}\sum_{h=0}^{j}&\frac{(q^{-2n})_{2j}(yq;q^2)_j(aq/xy;q^2)_j(q^{-2j};q^2)_h(xq;q^2)_h(y;q^2)_h}{(q^2;q^2)_j(aq/x;q^2)_j(aq^2/y;q^2)_j(yq^{2-4n}/a;q^2)_j(q^2;q^2)_h(aq^2/b;q^2)_h}\\ &\times\frac{(aq^2/bx;q^2)_h}{(aq^2/x;q^2)_h(xyq^{1-2j};q^2)_h}q^{2h+2j}\\ =\frac{(aq;q^2)_{2n}(aq/xy)_{2n}}{(aq/x)_{2n}(aq^2/y;q^2)_{2n-1}(1-aq^{2n}/y)}&\sum_{j=0}^{2n}\frac{(x)_j(y)_j(aq/b;q^2)_j(q^{-2n})_jq^j}{(q)_j(aq;q^2)_j(aq/b)_j(xyq^{-2n}/a)_j}\end{aligned}\tag{1.5}$$

and

$$\begin{aligned}\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{m=0}^{k}&\frac{(q^{-3n})_{3j}(a/y^2;q^3)_j(q^{-3j};q^3)_k(yq;q^3)_k(yq^2;q^3)_k}{(q^3;q^3)_j(q^3;q^3)_k(q^3;q^3)_m(aq^2/y;q^3)_j(aq/y;q^3)_j(aq/x;q^3)_k}\\ \times&\frac{(aq/xy;q^3)_k(q^{3k};q^3)_m(xq^2;q^3)_m(y;q^3)_m(aq^2/x^2;q^3)_mq^{3j+3k+3m}}{(aq^2/x;q^3)_m(aq^3/x;q^3)_m(q^{3-9n};q^3)_j(y^3q^{3-3j}/a;q^3)_k(xyq^{2-3j}/a;q^3)_m(aq^3/y;q^3)_k}\\ =\frac{(aq)_{3n}(aq/xy)_{3n}(aq^{3n+1};q^3)_n(aq^{3n+2};q^3)_n}{(aq/x)_{3n}(aq/y)_{3n}(aq^3;q^3)_n(aq^{6n};q^3)_n}&\sum_{j=0}^{3n}\frac{(a;q^3)_j(x)_j(y)_j(q^{-3n})_jq^j}{(q)_j(a;q^2)_{2j}(xyq^{-3n}/a)_j}.\end{aligned}\tag{1.6}$$

In §2, proofs of (1.3)–(1.6) are presented. These proofs suggest additional results, presented in §3. Finally, related open questions are presented in §4.

2 q-Hypergeometric proofs of (1.3)–(1.6)

The basic hypergeometric series is defined, as in [17], by

$$ {}_r\phi_s\left[\begin{matrix}a_1,a_2,\dots,a_r\\ b_1,b_2,\dots,b_s\end{matrix};q,z\right] := \sum_{j=0}^{\infty}\frac{(a_1;q)_j(a_2;q)_j\cdots(a_r;q)_j}{(q;q)_j(b_1;q)_j(b_2;q)_j\cdots(b_s;q)_j}z^j\left[(-1)^jq^{j(j-1)/2}\right]^{1+s-r}, \quad (2.1) $$

Note that a basic hypergeometric series (2.1) is called *well-poised* if $s=r-1$ and $a_1q=a_2b_2=\cdots=a_r=b_{r-1}$ and *very-well-poised* if, in addition, $a_2=q\sqrt{a_1}$ and $a_3=-q\sqrt{a_1}$. Very-well-poised series are central to the study and turn out to be the common link between the multisum and single sum representations in this paper. It will be convenient to employ the following condensed notation for very-well-poised basic series:

$$ {}_{r+1}W_r\left(a;a_4,a_5,\dots,a_{r+1};q,z\right) := {}_{r+1}\phi_r\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},a_4,a_5,\dots,a_{r+1}\\ \sqrt{a},-\sqrt{a},aq/a_4,aq/a_5,\dots,aq/a_{r+1}\end{matrix};q,z\right]. $$

In [3], Andrews presents a very general series transformation formula whereby a very-well-poised ${}_{2k+4}\phi_{2k+3}$ is transformed into a $(k-1)$-fold multisum representation. The $k=2$ case is equivalent to Watson's q-analogue of Whipple's theorem [26]. The $k=3$ case of Andrews' transformation may be stated as

$$ {}_{10}W_9\left(a;b,c,d,e,f,g,q^{-n};q,a^3q^{n+3}/bcdefg\right) = \frac{(aq)_n(aq/fg)_n}{(aq/f)_n(aq/g)_n}\sum_{j=0}^{n}\frac{(q^{-n})_j(f)_j(g)_j(aq/de)_jq^j}{(q)_j(aq/d)_j(aq/e)_j(fgq^{-n}/a)_j} \times {}_4\phi_3\left[\begin{matrix}q^{-j},d,e,aq/bc\\ aq/b,aq/c,deq^{-j}/a\end{matrix};q,q\right], \quad (2.2) $$

where, here and throughout, n is a nonnegative integer. While (2.2) transforms a fairly general very-well-poised ${}_{10}\phi_9$ to a double sum, there are a number of transformation formulas known which transform a somewhat more specialized very-well-poised ${}_{10}\phi_9$ to a single sum. For instance, Verma and Jain [27, p. 232, Eq. (1.4)] found

$$ {}_{10}W_9\left(a;b,x,xq,y,yq,q^{1-n},q^{-n};q^2,a^3q^{2n+3}/bx^2y^2\right) = \frac{(aq)_n(aq/xy)_n}{(aq/x)_n(aq/y)_n}{}_5\phi_4\left[\begin{matrix}x,y,\sqrt{aq/b},-\sqrt{aq/b},q^{-n}\\ \sqrt{aq},-\sqrt{aq},aq/b,xyq^{-n}/a\end{matrix};q,q\right] \quad (2.3) $$

Note: the $n \to \infty$ case of (2.3) is given by Bailey (in a somewhat disguised form) as [7, p. 6, Eq. (6.3)]. With (2.2) and (2.3) in hand, it is now time to establish (1.5).

Theorem 2.1 *Identity* (1.5) *is valid.*

PROOF. The result follows from the observation that

$$ {}_{10}W_9\left(a; b, x, xq, y, yq, q^{1-2n}, q^{-2n}; q^2, a^3q^{4n+3}/bx^2y^2\right) \tag{2.4} $$

can be transformed via either (2.2) or (2.3). Transforming (2.4) via (2.2) yields

$$ \frac{(aq^2;q^2)_n(aq^{2n}/y;q^2)_n}{(aq/y;q^2)_n(aq^{1+2n};q^2)_n}\sum_{j=0}^{n}\frac{(q^{-2n};q^2)_j(yq;q^2)_j(q^{1-2n};q^2)_j(aq/xy;q^2)_j}{(q^2;q^2)_j(aq/x;q^2)_j(aq^2/y;q^2)_j(yq^{2-4n}/a;q^2)_j} \times {}_4\phi_3\left[\begin{matrix} q^{-2j}, xq, y, aq^2/bx \\ aq^2/b, aq^2/x, xyq^{1-2j}/a \end{matrix}; q^2, q^2\right], \tag{2.5} $$

while transforming (2.4) via (2.3) yields

$$ \frac{(aq)_{2n}(aq/xy)_{2n}}{(aq/x)_{2n}(aq/y)_{2n}}{}_5\phi_4\left[\begin{matrix} x, y, \sqrt{aq/b}, -\sqrt{aq/b}, q^{-2n} \\ \sqrt{aq}, -\sqrt{aq}, aq/b, xyq^{-2n}/a \end{matrix}; q, q\right]. \tag{2.6} $$

Thus (2.5)= (2.6). □

While it must be admitted that Identity (1.5) is probably not the most beautiful of identities, it nonethless gives rise to elegant corollaries, which may now be easily deduced.

Corollary 2.2 *Identity* (1.3) *is valid.*

PROOF. Let $b, x, y, n \to \infty$ in Eq. (1.5). □

Actually, (1.3) is just one of a set of three closely related identities. With (1.3) established, it is straightforward to deduce its two partners:

Corollary 2.3

$$ \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{a^{r+2s}q^{2(r+s)^2+2s^2+2r+4s}}{(q^2;q^2)_r(q^2;q^2)_s} = (aq;q^2)_\infty\sum_{j=0}^{\infty}\frac{a^jq^{j^2+2j}}{(aq;q^2)_{j+1}(q)_j} \tag{2.7} $$

$$ \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{a^{r+2s}q^{2(r+s)^2+2s^2+2s}}{(q^2;q^2)_r(q^2;q^2)_s} = (aq;q^2)_\infty\sum_{j=0}^{\infty}\frac{a^jq^{j^2+j}}{(aq;q^2)_{j+1}(q)_j} \tag{2.8} $$

PROOF. To obtain (2.7), replace a by aq^2 in (1.3). To obtain (2.8), subtract a^2q^4 times (2.7) with a replaced by aq^2 from (1.3). $\square$

Not surprisingly, other limiting cases of (1.5) reduce a particular double series to a familiar single sum.

Corollary 2.4

$$(-q)_\infty \sum_{j,k\geqq 0} \frac{q^{4j^2+6k^2+8jk-k}}{(-q;q^4)_{j+k}(q^4;q^4)_j(q^4;q^4)_k} = \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_{2j}} \tag{2.9}$$

PROOF. In (1.5), let $b, y, n \to \infty$, set $x = -\sqrt{q}$, $a = 1$, and replace q by q^2 throughout. $\square$

The right hand side of (2.9) appears twice on Slater's list [25, p. 160, Eq. (79) and p. 162, Eq. (98)], as the series expansion of the (equivalent) infinite products $(q^2;q^2)_\infty^{-1}(-q;q^2)_\infty(q^8,q^{12},q^{20};q^{20})_\infty$ and $(q)_\infty^{-1}(q^2,q^8,q^{10};q^{10})_\infty(q^6,q^{14};q^{20})_\infty$ respectively. This series expansion on the right hand side of (2.9) is originally due to L.J. Rogers [22, p. 330, 2nd eq.].

Andrews [5] pointed out the following alternate simplification of the double sum in the left hand side of (2.9):

$$\begin{aligned}
&\sum_{j,k\geqq 0} \frac{q^{4j^2+6k^2+8jk-k}}{(-q;q^4)_{j+k}(q^4;q^4)_j(q^4;q^4)_k} \\
&= \sum_{t=0}^{\infty} \frac{q^{4t^2}}{(-q^4;q^4)_t(q^4;q^4)_t} \sum_{k=0}^{t} \frac{q^{2k^2-k}(q^4;q^4)_t}{(q^4;q^4)_k(q^4;q^4)_{t-k}} \qquad \text{(by letting } t=j+k\text{)} \\
&= \sum_{t=0}^{\infty} \frac{q^{4t^2}}{(q^4;q^4)_t} \qquad \text{(by the } q\text{-binomial theorem [17, p. 8, Eq. (1.3.2)]).}
\end{aligned}$$

Notice that the last expression is the series portion of the first Rogers-Ramanujan identity (the $k = i = 2$ case of (1.1)) with $q \to q^4$.

Next, consider the $k = 4$ case of Andrews' transformation:

$$\begin{aligned}
&{}_{12}W_{11}\left(a;b,c,d,e,f,g,h,i,q^{-n};q,a^4q^{n+4}/bcdefghi\right) \\
&\qquad = \frac{(aq)_n(aq/hi)_n}{(aq/h)_n(aq/i)_n} \sum_{j=0}^{n} \frac{(q^{-n})_j(h)_j(i)_j(aq/fg)_jq^j}{(q)_j(aq/f)_j(aq/g)_j(hiq^{-n}/a)_j} \\
&\times \sum_{k=0}^{j} \frac{(q^{-j})_k(f)_k(g)_k(aq/de)_kq^k}{(q)_k(aq/d)_k(aq/e)_k(fgq^{-j}/a)_k} {}_4\phi_3\left[\begin{matrix} q^{-k},d,e,aq/bc \\ aq/b,aq/c,deq^{-j}/a \end{matrix};q,q\right]
\end{aligned} \tag{2.10}$$

and Verma and Jain's transformation [27, p. 232, Eq. (1.5)]:

$$
\begin{aligned}
&{}_{12}W_{11}\left(a; x, xq, xq^2, y, yq, yq^2, q^{2-n}, q^{1-n}, q^{-n}; q^3, a^4q^{3n+3}/x^3y^3\right) \\
&= \frac{(aq)_n(aq/xy)_n}{(aq/x)_n(aq/y)_n}{}_6\phi_5\left[\begin{matrix}\sqrt[3]{a}, \omega\sqrt[3]{a}, \omega^2\sqrt[3]{a}, x, y, q^{-n} \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, xyq^{-n}/a\end{matrix}; q, q\right], \qquad (2.11)
\end{aligned}
$$

where ω is a primitive cube root of unity.

Theorem 2.5 *Identity* (1.6) *is valid.*

PROOF. The proof is completely analogous to that of identity (1.5), with (2.10) playing the role of (2.2), and (2.11) playing the role of (2.3). This time the "very-well-poised link" is

$${}_{12}W_{11}\left(a; x, xq, xq^2, y, yq, yq^2, q^{2-3n}, q^{1-3n}, q^{-3n}; q^3, a^4q^{9n+3}/x^3y^3\right).$$

□

Corollary 2.6 *Identity* (1.4) *is valid.*

PROOF. Let $b, x, y, n \to \infty$ in Eq. (1.6). □

Just like (1.3), Eq. (1.4) is one of a set of closely related identities; the three partners of (2.14) are

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{a^{r+2s+3t}q^{3[(r+s+t)^2+(s+t)^2+t^2+3r+2s+t]}}{(q^3;q^3)_r(q^3;q^3)_s(q^3;q^3)_t} = \frac{(aq)_\infty}{(aq^3;q^3)_\infty}\sum_{j=0}^{\infty}\frac{a^jq^{j^2+3j}(aq^3;q^3)_j}{(aq)_{2j+2}(q)_j}, \qquad (2.12)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{a^{r+2s+3t}q^{3[(r+s+t)^2+(s+t)^2+t^2+2r+2s+t]}}{(q^3;q^3)_r(q^3;q^3)_s(q^3;q^3)_t} = \frac{(aq)_\infty}{(aq^3;q^3)_\infty}\sum_{j=0}^{\infty}\frac{a^jq^{j^2+2j}(aq^3;q^3)_j}{(aq)_{2j+2}(q)_j}, \qquad (2.13)$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{a^{r+2s+3t}q^{3[(r+s+t)^2+(s+t)^2+t^2+r+s+t]}}{(q^3;q^3)_r(q^3;q^3)_s(q^3;q^3)_t} = \frac{(aq)_\infty}{(aq^3;q^3)_\infty}\sum_{j=0}^{\infty}\frac{a^jq^{j^2+j}(aq^3;q^3)_j}{(aq)_{2j+1}(q)_j}. \qquad (2.14)$$

3 Additional Results

In light of the previous section, it makes sense to consider another transformation formula of Verma and Jain [27, p. 232, Eq. (1.3)]; see Bailey [7, p. 6,

Eq. (6.1)] for the $n \to \infty$ case.

$$ {}_{10}W_9\left(a; b, x, -x, y, -y, -q^{-n}, q^{-n}; q, -a^3q^{2n+3}/bx^2y^2\right) \tag{3.1} $$
$$ = \frac{(a^2q^2;q^2)_n(a^2q^2/x^2y^2;q^2)_n}{(a^2q^2/x^2;q^2)_n(a^2q^2/y^2)_n} {}_5\phi_4\left[\begin{matrix} x^2, y^2, -aq/b, -aq^2/b, q^{-2n} \\ -aq, -aq^2, a^2q^2/b^2, x^2y^2q^{-2n}/a^2 \end{matrix}; q^2, q^2\right]. $$

Here all that is needed is to make the substitutions $c = x$, $d = -x$, $e = y$, $f = -y$, $g = -q^{-n}$ in (2.2), equate its right hand side with the right hand side of (3.1), and after some routine algebra, results in the following theorem:

Theorem 3.1

$$ \sum_{j=0}^{n} \frac{(q^{-2n};q^2)_j(-y)_j(-aq/xy)_jq^j}{(q)_j(-aq/x)_j(aq/y)_j(yq^{-2n}/a)_j} {}_4\phi_3\left[\begin{matrix} q^{-j}, -x, y, aq/bx \\ aq/b, aq/x, -xyq^{-j}/a \end{matrix}; q, q\right] $$
$$ = \frac{(-aq)_{2n}(a^2q^2/x^2y^2;q^2)_n}{(a^2q^2/x^2;q^2)_n(aq/y)_{2n}} {}_5\phi_4\left[\begin{matrix} q^{-2n}, x^2, y^2, -aq/b, -aq^2/b \\ x^2y^2q^{-2n}/a^2, a^2q^2/b^2, -aq, -aq^2 \end{matrix}; q^2, q^2\right], \tag{3.2} $$

which, after suitable specialization, yields

Corollary 3.2

$$ \sum_{j,k\geqq 0} \frac{(-1)^k a^{j+2k} q^{(j+k)^2+\binom{k+1}{2}}}{(q)_j(q)_k} = (-aq)_\infty \sum_{j=0}^{\infty} \frac{a^{2j}q^{2j^2}}{(-aq)_{2j}(q^2;q^2)_j}. \tag{3.3} $$

The series on the right hand side of (3.3) with $a = 1$ is the series associated with the first Rogers-Selberg identity, an expansion of the series $(q^3, q^4, q^7; q^7)_\infty (q^2;q^2)_\infty^{-1}$, due to Rogers [22, p. 338] and recorded by Slater [25, Eq. (33)].

4 Discussion

Once I had in hand a q-hypergeometric explanation for the existence of identities like (1.3) and (1.4), it was only natural to look for additional analogous identities. I did not search exhaustively, but rather presented a couple of striking examples relating to well-known series (e.g. Rogers-Selberg). Certainly additional identities of this type exist (e.g. a multisum version of the Bailey "mod 9 identities" [6, p. 422, Eqs. (1.6)–(1.8)], [25, Eqs. (40)–(42)]), and the interested reader is encouraged to use the methods of this paper to work out additional examples.

A more ambitious project would be to look for bijective proofs of identities like (1.3) and (1.4).

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