On Series Expansions of Capparelli's Infinite Product

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Abstract

Using Lie theory, Stefano Capparelli conjectured an interesting Rogers-Ramanujan type partition identity in his 1988 Rutgers Ph.D. thesis. The first proof was given by George Andrews, using combinatorial methods. Later, Capparelli was able to provide a Lie theoretic proof.

Most combinatorial Rogers-Ramanujan type identities (e.g. the Göllnitz-Gordon identities, Gordon's combinatorial generalization of the Rogers-Ramanujan identities, etc.) have an analytic counterpart. The main purpose of this paper is to provide two new series representations for the infinite product associated with Capparelli's conjecture. Some additional related identities, including new infinite families are also presented.

1 Introduction

In 1894, L.J. Rogers was the first to discover a pair of series–product identities which are now known as the Rogers-Ramanujan identities. They may be stated compactly as follows:

Rogers-Ramanujan Identities–Analytic Form. *For $\lambda = 0$ or 1,*

$$\sum_{j=0}^{\infty} \frac{q^{j^2+\lambda j}}{(q)_j} = \frac{1}{(q^{\lambda+1}, q^{4-\lambda}; q^5)_\infty}, \tag{1.1}$$

where

$$(A)_0 := (A;q)_0 := 1,$$
$$(A)_n := (A;q)_n := (1-A)(1-Aq)\cdots(1-Aq^{n-1}),$$
$$(A)_\infty := (A;q)_\infty := \prod_{i=0}^{\infty}(1-Aq^i),$$

and

$$(A_1, A_2, \ldots, A_r; q)_s = (A_1;q)_s (A_2;q)_s \cdots (A_r,q)_s.$$

(Although the results in this paper may be considered purely from the point of view of formal power series, they also yield identities of analytic functions provided $|q| < 1$.)

A *partition* π of an integer n is a nonincreasing finite sequence of positive integers $(\pi_1, \pi_2, \ldots, \pi_s)$ such that $\sum_{i=1}^{s} \pi_i = n$. The π_i's are called the *parts* of the partition π.

MacMahon [22] and Schur [27] independently saw that the Rogers-Ramanujan identities were in fact equivalent to the following partition theoretic statement:

Rogers-Ramanujan Identities–Combinatorial Form. *Let $R_1(\lambda, n)$ denote the number of partitions $\pi = (\pi_1, \ldots, \pi_s)$ of n into parts wherein $\pi_s > \lambda$ and $\pi_i - \pi_{i+1} \geqq 2$. Let $R_2(\lambda, n)$ denote the number of partitions of n wherein all parts are congruent to $\pm(\lambda+1)$ modulo 5. Then for all integers n and for $\lambda = 0$ or 1, $R_1(\lambda, n) = R_2(\lambda, n)$.*

Over the years, many other analytic and combinatorial identities of Rogers-Ramanujan type were discovered, including the following analytic identity of Slater [28, p. 155, equations (36) and (34)] and its combinatorial counterpart due to Göllnitz [13], and rediscovered by Gordon [14]:

Slater's mod 8 Identities. *For $\lambda = 0$ or 1,*

$$\sum_{j=0}^{\infty} \frac{q^{j^2+2\lambda j}(-q;q^2)_j}{(q^2;q^2)_j} = \frac{1}{(q^{1+2\lambda}, q^4, q^{7-2\lambda}; q^8)_\infty}. \tag{1.2}$$

The Göllnitz-Gordon Partition Identities. *Let $G_1(\lambda, n)$ denote the number of partitions $\pi = (\pi_1, \ldots, \pi_s)$ of n wherein $\pi_s > 2\lambda$, $\pi_i - \pi_{i-1} \geqq 2$, and $\pi_i - \pi_{i+1} > 2$ if π_i or π_{i+1} is even. Let $G_2(\lambda, n)$ denote the number of partitions of n into parts congruent to $\pm(1+2\lambda)$ or 4 modulo 8. Then $G_1(\lambda, n) = G_2(\lambda, n)$ for all integers n and $\lambda = 0$ or 1.*

Following a program of research initiated by Lepowsky-Milne ([15], [16]), and Lepowsky-Wilson ([17],[18],[19],[20],[21]), Stefano Capparelli was able to conjecture a partition identity as a result of his studies of the standard level 3 modules associated with the Lie Algebra $A_2^{(2)}$, and included this conjecture in his Ph.D. thesis [10]:

Capparelli's Conjecture. *Let $C_1(n)$ denote the number of partitions $\pi = (\pi_1, \ldots, \pi_s)$ of n wherein $\pi_s > 1$, $\pi_i - \pi_{i+1} \geqq 2$, and if $\pi_i - \pi_{i+1} < 4$, then either π_i and π_{i+1} are are both multiples of three, or $\pi_i \equiv 1 \pmod 3$ and $\pi_{i+1} \equiv -1 \pmod 3$. Let $C_2(n)$ denote the number of partitions of n into parts congruent to ± 2 or ± 3 modulo 12. Then $C_1(n) = C_2(n)$ for all integers n.*

George Andrews, inspired by the combinatorial techniques of Wilf and Zeilberger [24], provided the first proof in [6]. Later, Lie-theoretic proofs were supplied by Tamba and Xie [29] and by Capparelli himself [11]. In [23], Meurman and Primc embed Capparelli's conjecture in an infinite family of three-color partition identities.

In [1], Alladi, Andrews, and Gordon provided refinements to Capparelli's conjecture along with a corresponding identity of generating functions. By replacing q with q^3, and setting $a = q^{-2}$, $b = q^{-4}$ and $c = 1$ in [1, p. 648–9, Lemma 2(b)], one can deduce the following analytic counterpart to Capparelli's conjecture:

$$\sum_{i,j,k \geqq 0} \frac{q^{3i^2+i+3j^2-j+\frac{3}{2}k^2+\frac{3}{2}k+3ik+3jk}(-q^3,q^3)_{i+j}}{(q^6;q^6)_i(q^6;q^6)_j(q^3;q^3)_k} = \frac{1}{(q^2,q^3,q^9,q^{10};q^{12})_\infty}. \tag{1.3}$$

The main goal of this paper is to present two additional analytic identities involving the infinite product $(q^2,q^3,q^9,q^{10};q^{12})_\infty^{-1}$, namely

$$\sum_{n=0}^{\infty}\sum_{j=0}^{2n} \frac{q^{n^2}\left(\frac{n-j+1}{3}\right)}{(q)_{2n-j}(q)_j} = \frac{1}{(q^2,q^3,q^9,q^{10};q^{12})_\infty}, \tag{1.4}$$

where $\left(\frac{n}{p}\right)$ is the Legendre symbol, and

$$1 + \sum_{\substack{n,j,r \geqq 0 \\ (n,j,r)\neq(0,0,0)}} \frac{q^{3n^2+\frac{9}{2}r^2+3j^2+6nj+6nr+6rj-\frac{5}{2}r-j}(q^3;q^3)_{2j+r-1}(1+q^{2r+2j})(1-q^{6r+6j})}{(q^3;q^3)_n(q^3;q^3)_r(q^3;q^3)_j(-1;q^3)_{j+1}(q^3;q^3)_{n+2r+2j}}$$

$$= \frac{1}{(q^2,q^3,q^9,q^{10};q^{12})_\infty}, \tag{1.5}$$

which will actually arise as a corollary to the following analytic identity, an "a-generalization of an analytic counterpart of Capparelli's conjecture":

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{a^{3n+2r+2j}q^{3n^2+\frac{9}{2}r^2+3j^2+6nj+6nr+6rj-\frac{5}{2}r-j}(a^3;q^3)_{2j+r}(1+(aq)^{2r+2j})(1-a^3q^{6j+6r})}{2(q^3;q^3)_n(q^3;q^3)_r(q^3;q^3)_j(-a^3q^3;q^3)_j(a^3,q^3)_{n+2j+2r+1}}$$
$$=\frac{1}{(a^3q^3;q^3)_\infty}\sum_{r=0}^{\infty}\frac{a^{3r}q^{3r^2}(a^3;q^3)_r(-q^3;q^3)_{r-1}(1-a^3q^{6r})\left((aq)^r+(aq)^{-r}\right)}{(q^3;q^3)_r(-a^3q^3;q^3)_r(1-a^3)}. \tag{1.6}$$

In section 2, it will be revealed how identity (1.4) arises from two of the simplest possible Bailey pairs. Section 3 will be devoted to a derivation of the Bailey pair necessary to yield identity (1.6). Once this is accomplished, the identities (1.4) and (1.5) will be embedded in infinite family of identities:

$$\sum_{n_1,\dots,n_k,j\geqq 0}\frac{q^{N_1^2+N_2^2+\cdots+N_k^2}\binom{n_k-j+1}{3}}{(q)_{2n_k-j}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_k}(q)_j}=\frac{(q^k,q^{5k},q^{6k};q^{6k})(q^{4k},q^{8k};q^{12k})}{(q)_\infty}, \tag{1.7}$$

where $N_i=n_i+n_{i+1}+\cdots+n_k$.

$$1+\sum_{\substack{n_1,\dots,n_k,r,j\geqq 0\\(n_1,\dots,n_k,r,j)\neq(0,0,\dots,0)}}\frac{q^{3(M_1^2+\cdots+M_k^2)+\frac{3}{2}r^2-\frac{5}{2}r-j}(q^3;q^3)_{2j+r-1}(1+q^{2r+2j})(1-q^{6r+6j})}{(q^3;q^3)_{n_1}\cdots(q^3;q^3)_{n_k}(q^3;q^3)_r(q^3;q^3)_j(-1;q^3)_{j+1}(q^3;q^3)_{n_k+2r+2j}}$$
$$=\frac{(-q^{3k-1},-q^{3k+1},q^{6k};q^{6k})_\infty}{(q^3;q^3)_\infty}, \tag{1.8}$$

where $M_i=n_i+n_{i+1}+\cdots+n_k+r+j$. Notice that the $k=1$ case of (1.8) is equivalent to (1.5) since

$$\frac{(-q^2,-q^4,q^6;q^6)_\infty}{(q^3;q^3)_\infty}=\frac{1}{(q^2,q^3,q^9,q^{10};q^{12})_\infty}.$$

In section 4, some related identities will be noted. In section 5, we conclude with some related open questions.

2 Implications of two simple Bailey pairs

We will require the standard machinery of Bailey's Lemma and Bailey pairs (see [8], [9], [5, Ch. 3]). Recall that two sequences of rational functions $(\alpha_n(a,q),\beta_n(a,q))$ form a *Bailey pair* if for all $n\geqq 0$,

$$\beta_n(a,q)=\sum_{r=0}^{n}\frac{\alpha_r(a,q)}{(q)_{n-r}(aq)_{n+r}}, \tag{2.1}$$

and that for any Bailey pair $(\alpha_n(a,q),\beta_n(a,q))$, the identity

$$\sum_{n=0}^{\infty}a^nq^{n^2}\beta_n(a,q)=\frac{1}{(aq)_\infty}\sum_{r=0}^{\infty}a^rq^{r^2}\alpha_r(a,q) \tag{2.2}$$

holds (Andrews [5, p. 27, equation (3.33)]).

In the literature (see e.g. Andrews [5, section 3.5]) the implications of a particular Bailey pair (often called the "unit Bailey pair") consisting of an extremely simple β_n and its corresponding α_n are considered. Here, in contrast, we consider Bailey pairs where the α_n's are of an especially simple nature.

Theorem 2.1. *Suppose*

$$\alpha_n=\begin{cases}1, & \text{if } n=0,\\ 2, & \text{if } 3|n \text{ and } n>0,\\ 0, & \text{otherwise.}\end{cases}$$

and

$$\beta_n = \sum_{r=-\infty}^{\infty} \frac{\begin{bmatrix} 2n \\ n-3r \end{bmatrix}}{(q)_{2n}},$$

where the Gaussian polynomial $\begin{bmatrix} A \\ B \end{bmatrix}$ *is defined by*

$$\begin{bmatrix} A \\ B \end{bmatrix} := \begin{cases} (q)_A (q)_B^{-1} (q)_{A-B}^{-1}, & \text{if } 0 \leqq A \leqq B, \\ 0, & \text{otherwise.} \end{cases}$$

Then (α_n, β_n) *form a Bailey pair.*

Proof. Considereing (2.1) with $a = 1$,

$$\begin{aligned} \beta_n &= \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(q)_{n+r}} \\ &= 1 + \sum_{r \geqq 1} \frac{2}{(q)_{n-3r}(q)_{n+3r}} \\ &= \sum_{r=-\infty}^{\infty} \frac{1}{(q)_{n-3r}(q)_{n+3r}} \\ &= \sum_{r=-\infty}^{\infty} \frac{\begin{bmatrix} 2n \\ n-3r \end{bmatrix}}{(q)_{2n}}. \end{aligned}$$

□

Corollary 2.2.

$$\sum_{n=0}^{\infty} \sum_{r=-\infty}^{\infty} \frac{q^{n^2} \begin{bmatrix} 2n \\ n-3r \end{bmatrix}}{(q)_{2n}} = \frac{(-q^9, -q^9, q^{18}; q^{18})_\infty}{(q)_\infty}$$

Proof. By Theorem 2.1 and (2.2) with $a = 1$,

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{r=-\infty}^{\infty} \frac{q^{n^2} \begin{bmatrix} 2n \\ n-3r \end{bmatrix}}{(q)_{2n}} &= \frac{1}{(q)_\infty} \left\{ 1 + \sum_{r=1}^{\infty} 2q^{(3r)^2} \right\} \\ &= \frac{1}{(q)_\infty} \sum_{r=-\infty}^{\infty} q^{9r^2} \\ &= \frac{(-q^9, -q^9, q^{18}; q^{18})_\infty}{(q)_\infty}, \end{aligned}$$

by Jacobi's triple product identity [12, p. 12, equation (1.6.1)]. □

Theorem 2.3. *If*

$$\alpha_n = \begin{cases} 0, & \text{if } 3|n, \\ 1, & \text{otherwise.} \end{cases}$$

and

$$\beta_n = \sum_{r=-\infty}^{\infty} \frac{\begin{bmatrix} 2n \\ n-3r+1 \end{bmatrix}}{(q)_{2n}},$$

then (α_n, β_n) *form a Bailey pair.*

Proof.

$$\begin{aligned}
\beta_n &= \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(q)_{n+r}} \\
&= \sum_{r=0}^{\lfloor n/3 \rfloor} \frac{1}{(q)_{n-3r-1}(q)_{n+3r+1}} + \sum_{r=1}^{\lfloor n/3 \rfloor} \frac{1}{(q)_{n-3r+1}(q)_{n+3r-1}} \\
&= \sum_{r=-\infty}^{\infty} \frac{1}{(q)_{n-3r+1}(q)_{n+3r-1}} \\
&= \sum_{r=-\infty}^{\infty} \frac{\left[{2n \atop n-3r+1} \right]}{(q)_{2n}}.
\end{aligned}$$

□

Corollary 2.4.

$$\sum_{n=0}^{\infty} \sum_{r=-\infty}^{\infty} \frac{q^{n^2} \left[{2n \atop n-3r+1} \right]}{(q)_{2n}} = \frac{q(-q^3, -q^{15}, q^{18}; q^{18})_\infty}{(q)_\infty}$$

Proof. By Theorem 2.3 and (2.2) with $a = 1$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{r=-\infty}^{\infty} \frac{q^{n^2} \left[{2n \atop n-3r+1} \right]}{(q)_{2n}} &= \frac{1}{(q)_\infty} \left\{ \sum_{r=0}^{\infty} q^{(3r+1)^2} + \sum_{r=1}^{\infty} q^{(3r-1)^2} \right\} \\
&= \frac{1}{(q)_\infty} \sum_{r=-\infty}^{\infty} q^{9r^2+6r+1} \\
&= \frac{q(-q^3, -q^{15}, q^{18}; q^{18})_\infty}{(q)_\infty},
\end{aligned}$$

by Jacobi's triple product identity [12, p. 12, equation (1.6.1)]. □

Theorem 2.5. *Identity (1.4) is valid.*

Proof. Essentially all we need to do is subtract the identity in Corollary 2.4 from the identity in Corollary 2.2. For the left hand side, observe that

$$\sum_{n=0}^{\infty} \sum_{r=-\infty}^{\infty} \frac{q^{n^2} \left[{2n \atop n-3r} \right]}{(q)_{2n}} - \sum_{n=0}^{\infty} \sum_{r=-\infty}^{\infty} \frac{q^{n^2} \left[{2n \atop n-3r+1} \right]}{(q)_{2n}}$$
$$= \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \frac{q^{n^2} \left(\frac{k+1}{3} \right) \left[{2n \atop n-k} \right]}{(q)_{2n}},$$

(i.e. the inner sum on k sums over the $2n$-th row of the q-Pascal triangle weighting consecutive summands in turn by the factors 1, -1, and 0,)

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{n^2} \left(\frac{n-j+1}{j} \right) \left[{2n \atop j} \right]}{(q)_{2n}} \qquad \text{(by setting } j = n-k\text{)},$$
$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{n^2} \left(\frac{n-j+1}{j} \right)}{(q)_{2n-j}(q)_j}.$$

For the right hand side, observe that

$$\begin{aligned}
& \frac{(-q^9,-q^9,q^{18};q^{18})_\infty}{(q)_\infty} - q\frac{(-q^3,-q^{15},q^{18};q^{18})_\infty}{(q)_\infty} \\
= \quad & \frac{(q,q^5,q^6;q^6)_\infty (q^4,q^8;q^{12})_\infty}{(q)_\infty} \\
& \qquad\qquad \text{(by the quintuple product identity [12, p. 134, ex. 5.6])} \\
= \quad & \frac{1}{(q^2,q^3,q^9,q^{10};q^{12})_\infty}.
\end{aligned}$$

□

Remark 2.6. Identity (1.4) can be rewritten as

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{q^{n^2+2nj+j^2}}{(q)_{2j+1}(q)_{2n}} \left(\left(\frac{j-n+1}{3}\right) + \left(\frac{j-n-1}{3}\right) q^{2j+1} + \left(\frac{j-n}{3}\right) q^{2n} \right) = \frac{1}{(q^2,q^3,q^9,q^{10};q^{12})_\infty} \tag{2.3}$$

by splitting the inner sum on j in the left hand side of (1.4) into even and odd j, interchanging the order of summation and replacing n by $n+j$. In this formulation, both sums are truly infinite over all nonnegative n and j.

3 Another Bailey pair and its implications

Recall the standard notation for basic hypergeometric series

$${}_{s+1}\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_{s+1} \\ b_1, b_2, \dots, b_s \end{matrix} ; q, z \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{s+1})_n}{(q)_n (b_1)_n (b_2)_n \dots (b_s)_n} z^n.$$

Remark 3.1. The real challenge here was to find an appropriate $\alpha_r(a,q)$ so that

- when $\alpha_r(a,q)$ is inserted into (2.1), the resulting expression is a (finite product multiplied by a) basic hypergeometric series which can be transformed appropriately, and
- when $\alpha_r(a,q)$ is inserted into (2.2) and a is set to 1, the generating function

  $$\frac{1}{(q^2,q^3,q^9,q^{10};q^{12})_\infty} = \sum_{n=0}^{\infty} C_2(n) q^n$$

  results.

Once this is achieved, the power of Bailey's lemma and Bailey chains allows us to derive a number of identities with little additional effort.

Theorem 3.2. *If*

$$\alpha_r(a,q) := \frac{(a)_r (1-aq^{2r})(-q)_{r-1}}{(q)_r (1-a)(-aq)_r} \left((aq)^{r/3} + (aq)^{-r/3} \right) \tag{3.1}$$

and

$$\beta_n(a,q) := \sum_{j=0}^{n} \sum_{r=0}^{n-j} \frac{a^{-(r+j)/3} q^{\frac12 r^2 - \frac56 r - \frac13 j} (1-aq^{2j+2r})(a)_{2j+r} (1+(aq)^{\frac23 (r+j)})}{2(q)_j (q)_r (a)_{n+j+r+1} (-aq)_j (q)_{n-j-r}}, \tag{3.2}$$

then $\left(\alpha_n(a,q), \beta_n(a,q)\right)$ *form a Bailey pair.*

Proof.

$$
\begin{aligned}
&\beta_n(a,q)\\
=&\sum_{r=0}^{n}\frac{\alpha_r(a,q)}{(q)_{n-r}(aq)_{n+r}}\\
=&\frac{1}{(q)_n(aq)_n}\sum_{r=0}^{n}\frac{(-1)^r q^{nr-\frac12 r^2+\frac12 r}}{(aq^{n+1})_r}\alpha_r(a,q)\\
=&\frac{1}{(q)_n(aq)_n}\sum_{r=0}^{n}\frac{(-1)^r q^{nr-\frac12 r^2+\frac12 r}}{(aq^{n+1})_r}\frac{(a)_r(1-aq^{2r})(-q)_{r-1}}{(q)_r(1-a)(-aq)_r}\left((aq)^{r/3}+(aq)^{-r/3}\right)\\
=&\frac{1}{(q)_n(aq)_n}\sum_{r=0}^{n}\frac{(-1)^r q^{nr-\frac12 r^2+\frac12 r}}{(aq^{n+1})_r}\frac{(a)_r(aq^2;q^2)_r(-1)_r}{2(q)_r(a;q^2)_r(-aq)_r}\left((aq)^{r/3}+(aq)^{-r/3}\right)\\
=&\frac{1}{2(q)_n(aq)_n}\lim_{\tau\to0}\left({}_6\phi_5\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},-1,\tau aq,q^{-n}\\ \sqrt{a},-\sqrt{a},-aq,\frac1\tau,aq^{n+1}\end{matrix};q,\tau^{-1}a^{\frac13}q^{n+\frac13}\right]\right.\\
&\qquad\left.+\,{}_6\phi_5\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},-1,\tau aq,q^{-n}\\ \sqrt{a},-\sqrt{a},-aq,\frac1\tau,aq^{n+1}\end{matrix};q,\tau^{-1}a^{-\frac13}q^{n-\frac13}\right]\right)\\
=&\frac{1}{2(q)_n(aq)_n}\lim_{\tau\to0}\left(\sum_{j=0}^{n}\frac{(-\frac1\tau)_j(q\sqrt{a})_j(-q\sqrt{a})_j(-1)_j(\tau aq)_j(q^{-n})_j(a)_{2j}}{(q)_j(\sqrt{a})_j(\sqrt{a})_j(-aq)_j(\frac1\tau)_j(aq^{n+1})_j}a^{j/3}q^{nj+\frac56 j-\frac12 j^2}\right.\\
&\qquad\times{}_4\phi_3\left[\begin{matrix}aq^{2j},q^{j+1}\sqrt{a},-q^{j+1}\sqrt{a},q^{j-n}\\ q^j\sqrt{a},-q^j\sqrt{a},aq^{n+j+1}\end{matrix};q,-a^{\frac13}q^{n-j+\frac13}\right]\\
&\qquad+\sum_{j=0}^{n}\frac{(-\frac1\tau)_j(q\sqrt{a})_j(-q\sqrt{a})_j(-1)_j(\tau aq)_j(q^{-n})_j(a)_{2j}}{(q)_j(\sqrt{a})_j(\sqrt{a})_j(-aq)_j(\frac1\tau)_j(aq^{n+1})_j}a^{-j/3}q^{nj+\frac16 j-\frac12 j^2}\\
&\qquad\left.\times\,{}_4\phi_3\left[\begin{matrix}aq^{2j},q^{j+1}\sqrt{a},-q^{j+1}\sqrt{a},q^{j-n}\\ q^j\sqrt{a},-q^j\sqrt{a},aq^{n+j+1}\end{matrix};q,-a^{-\frac13}q^{n-j-\frac13}\right]\right)\\
&\qquad\text{(by [12, p. 34, equation (2.4.1)])}\\
=&\frac{1}{2(q)_n(aq)_n}\sum_{j=0}^{n}\sum_{r=0}^{n-j}\frac{(aq^2;q^2)_{j+r}(q^{-n})_{j+r}(a)_{2j+r}(1+(aq)^{\frac23(j+r)})}{(q)_j(q)_r(a;q^2)_{j+r}(-aq)_j(aq^{n+1})_{j+r}}\\
&\qquad\times(-1)^{j+r}a^{-\frac13 j-\frac13 r}q^{nj+nr+\frac16 j-\frac12 j^2-\frac13 r-jr}\\
=&\sum_{j=0}^{n}\sum_{r=0}^{n-j}\frac{a^{-(r+j)/3}q^{\frac12 r^2-\frac56 r-\frac13 j}(aq^2;q^2)_{j+r}(a)_{2j+r}(1+(aq)^{\frac23(r+j)})}{2(q)_j(q)_r(aq)_{n+j+r}(a;q^2)_{j+r}(-aq)_j(q)_{n-j-r}}\\
=&\sum_{j=0}^{n}\sum_{r=0}^{n-j}\frac{a^{-(r+j)/3}q^{\frac12 r^2-\frac56 r-\frac13 j}(1-aq^{2j+2r})(a)_{2j+r}(1+(aq)^{\frac23(r+j)})}{2(q)_j(q)_r(a)_{n+j+r+1}(-aq)_j(q)_{n-j-r}}.
\end{aligned}
$$

□

Theorem 3.3. *Identity (1.6) is valid.*

Proof. Insert the Bailey pair in Theorem 3.2 into equation (2.2), and then replace a by a^3 and q by q^3 throughout. On the left hand side, interchange the order of summation bringing j out in front of n and replace n by $n+j$. Then, interchange the order of summation bringing r in front of n and replace n by $n+r$. □

Remark 3.4. Andrews [7] pointed out that a direct proof (i.e. one that is independent of Bailey's lemma) of (1.6) is possible. Start out with the left hand side of (1.6) and set $t=r+j$ so that the double sum is now on r and t. The inner sum on r is

$$
{}_2\phi_1\left[\begin{matrix}-a^{-3}q^{-3t}, q^{-3t} \\ a^{-3}q^{3-6t}\end{matrix}; q^3, q^3\right]
$$

which is summable by the q-Chu-Vandermonde formula [12, p. 236, equation (II.6)]. This form can then be converted to the right hand side using a formula due to Euler [2, p. 19, equation (2.2.5)].

Corollary 3.5. *Identity (1.5) is valid.*

Proof. Set $a = 1$ in identity (1.6), then apply Jacobi's triple product identity [5, p. 63, (7.1)] to the right hand side, and simplify the resulting product. □

Now that Bailey pairs have been established, it is a simple matter to embed the analytic Capparelli identities into infinite families of identities using the notion of the "Bailey chain" ([3], [5, p. 28 ff.]):

Theorem 3.6. *Identity (1.7) is valid.*

Proof. Insert the Bailey pairs from Theorem 2.1 and Theorem 2.3 into equation (3.34) of Andrews [5, p. 30], with $a = 1$. Subtract the second equation from the first, interchange orders of summation and change summation variables as appropriate. □

Theorem 3.7. *Identity (1.8) is valid.*

Proof. Insert $\alpha_n(a, q)$ and $\beta_n(a, q)$ into equation (3.34) of Andrews [5, p. 30], interchange orders of summation and change summation variables as appropriate. Finally, replace a by a^3 and q by q^3 throughout. □

4 Related identities

In order to obtain the a-generalization of the analytic counterpart of Capparelli's conjecture, the Bailey pair from Theorem 3.2 was inserted into equation (2.2), which is a limiting case of Bailey's lemma [5, p. 25, Thm. 3.3]. We now require a different limiting case of Bailey's lemma:
If $(\alpha_r(a, q), \beta_j(a, q))$ form a Bailey pair, then

$$
\sum_{n\geqq 0} a^n q^{n^2}(-q;q^2)_n \beta_n(a,q^2) = \frac{(-aq;q^2)_\infty}{(aq^2;q^2)_\infty} \sum_{r=0}^{\infty} \frac{a^r q^{r^2}(-q;q^2)_r}{(-aq;q^2)_r} \alpha_r(a,q^2). \tag{4.1}
$$

Now, inserting the Bailey pair from Theorem 3.2 into (4.1), we obtain the identity

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty} \frac{a^{3n+2r+2j} q^{3n^2+6r^2+3j^2+6nj+6nr+6rj-5r-2j}(-q^3;q^6)_{n+j+r}(a^3;q^3)_{2j+r}}{2(q^6;q^6)_n(q^6;q^6)_r(q^6;q^6)_j(-a^3q^6;q^6)_j(a^3,q^6)_{n+2j+2r+1}} \\
&\qquad\qquad \times(1+a^{2r+2j}q^{4r+4j})(1-a^3q^{12j+12r}) \\
&= \frac{(-a^3q^3)_\infty}{(a^3q^6;q^6)_\infty} \sum_{r=0}^{\infty} \frac{a^{3r}q^{3r^2}(-q^3;q^6)_r(a^3;q^6)_r(-q^6;q^6)_{r-1}(1-a^3q^{12r})\left((aq^2)^r+(aq^2)^{-r}\right)}{(q^6;q^6)_r(-a^3q^6;q^6)_r(1-a^3)},
\end{aligned} \tag{4.2}
$$

which, for $a = 1$, yields

$$
\begin{aligned}
1 + \sum_{\substack{n=0 \\ (n,j,r)\neq(0,0,0)}}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty} &\frac{q^{3n^2+6r^2+3j^2+6nj+6nr+6rj-5r-2j}(-q^3;q^6)_{n+j+r}(q^3;q^3)_{2j+r-1}(1+q^{4r+4j})(1-q^{12j+12r})}{(q^6;q^6)_n(q^6;q^6)_r(q^6;q^6)_j(-1;q^6)_{j+1}(q^6;q^6)_{n+2j+2r}} \\
&= (-q;q^2)_\infty.
\end{aligned} \tag{4.3}
$$

Note the extremely simple product on the right hand side of (4.3), which is the generating function for partitions into distinct odd parts.

The analogous identity relative to (1.4) is

$$\sum_{n=0}^{\infty}\sum_{j=0}^{2n} \frac{q^{n^2}(-q;q^2)_n \left(\frac{n-j+1}{3}\right)}{(q^2;q^2)_{2n-j}(q^2;q^2)_j} = \frac{(q^6;q^{12})_\infty}{(q^3,q^9;q^{12})_\infty}. \tag{4.4}$$

Of course, (4.2), (4.3), and (4.4) could easily be embedded in infinite families of identities via the Bailey chain.

5 Conclusion

While we now have in hand two series representations for the infinite product $(q^2,q^3,q^9,q^{10};q^{12})_\infty{}^{-1}$, namely the left hand sides of (1.4) and (1.5), it is not clear exactly how the partitions $C_1(n)$ are generated by them. Such an explanation would be most welcome.

Also, it should be noted that this infinite product $(q^2,q^3,q^9,q^{10};q^{12})_\infty{}^{-1}$ has appeared in the literature in at least two other combinatorial contexts beside Capparelli's conjecture: see Andrews' *Memoir* on generalized Frobenius partitions [4, p. 10, equation (5.9)], and Propp's paper on generalized Ferrers diagrams [25, p. 113, Thm. 4(a)], although in both of these cases the product contained the additional factor $(q)_\infty^{-1}$. It would be interesting to see a direct connection between Capparelli's $C_1(n)$ partitions and the combinatorial constructs of Andrews and Propp.

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