SOME MORE IDENTITIES OF THE ROGERS-RAMANUJAN TYPE

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Abstract. In this we paper we prove several new identities of the Rogers-Ramanujan-Slater type. These identities were found as the result of computer searches. The proofs involve a variety of techniques, including series-series identities, Bailey pairs, a theorem of Watson on basic hypergeometric series and the method of q-difference equations.

1. introduction

The most famous of the "q-series=product" identities are the Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})}. \tag{1.2}$$

These identities have a curious history ([15], p. 28). They were first proved by L.J. Rogers in 1894 ([20]) in a paper that was completely ignored. They were rediscovered (without proof) by Ramanujan sometime before 1913. In 1917, Ramanujan rediscovered Rogers's paper. Also in 1917, these identities were rediscovered and proved independently by Issai Schur ([22]). They were also discovered independently by R. Baxter (see [4] for details). An account of the many proofs of the Rogers-Ramanujan identities can be found in [3].

There are numerous identities that are similar to the Rogers-Ramanujan identities. These include identities by Jackson ([16]), Rogers ([20] and [21]) and Bailey ([8] and [9]). Of special note is Slater's 1952 paper [27], which contains a list of 130 such identities, many of them new (see the paper by the third author [24], for an annotated version of Slater's list). There are also other identities of Rogers-Ramanujan type in the literature.

In the present paper we prove several new identities, mostly found through computer searches. The proof of these identities lets us present examples of the various methods used to prove identities of the Rogers-Ramanujan type. These methods include using series-series identities, q-difference equations, Bailey pairs, and some miscellaneous methods.

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One search involved computing to high precision, for $q = 0.1$, series of the form

$$S := \sum_{n=0}^{\infty} \frac{q^{(an^2+bn)/2}(-1)^{cn}(d,e;q)_n}{(f,g,q;q)_n}, \tag{1.3}$$

where

$$d,e,f,g \in \{0,-1,q,-q,-q^2,q^2\}, \qquad c \in \{0,1\},$$

and where a and b are integers such that $1 \le a \le 10$ and $-a \le b \le 10$. This choice for the form of S was motivated by the fact that many series on the Slater list have this form.

Suppose, for example, that for particular values of a, b, c, d, e, f and g, such a series is identically (i.e. for all q) equal to an infinite product, say modulo 20. In other words,

$$S = \prod_{j=1}^{20} (q^j;q^{20})_\infty^{s_j},$$

for integers s_j. Upon taking logarithms of both sides we have that

$$\log S - \sum_{j=1}^{20} s_j \log(q^j;q^{20})_\infty = 0. \tag{1.4}$$

If the LLL-algorithm is applied to the set

$$\{\log S, \log(q;q^{20})_\infty, \log(q^2;q^{20})_\infty, \dots, \log(q^{20};q^{20})_\infty\},$$

it will hopefully output the integers s_j, thus giving a series-product identity which must then be proved. (Since computations are not exact, the output may be a set of s_j's that just make the left side of (1.4) just very small, and not zero.)

In practice we select a modulus, say 20 as above, and set $q = 0.1$. Then we compute the set of values

$$A = \{\log(q;q^{20})_\infty, \log(q^2;q^{20})_\infty, \dots, \log(q^{20};q^{20})_\infty\}$$

to sufficiently high precision. For our purposes, using *pari-gp*, 200 decimal places gave sufficient precision, and it was enough to compute the first 50 terms in each of the infinite products.

Next, we let the parameters a, b, c, d, e, f and g loop through their allowed values, and compute S for each particular set of choices (for precision purposes, computing the first 30 terms in the series was sufficient in each case). We append $\log S$ to the list A and then apply the LLL-algorithm, in the form of Pari's *lindep* command, to the new list A.

For practical reasons, we had the program output the set of s_j's, and the choices for the parameters a, b, c, d, e, f and g, only when $\max_j |s_j| < 100$. Thus each line of output, listing the choices for the parameters and the s_j's, either represented a known identity or a potential new identity.

We looked for identities where the products had moduli 20, 24, 28, 32, 36, 40 and 44. We also tried searches where the series had the forms

$$S' := \sum_{n=0}^{\infty} \frac{q^{(an^2+bn)/2}(-1)^{cn}(d,e;q^2)_n}{(f,g,q^2;q^2)_n},$$

$$S'' := \sum_{n=0}^{\infty} \frac{q^{(an^2+bn)/2}(-1)^{cn}(d;q^2)_n}{(e;q^2)_{n+1}(q;q)_{n+1}},$$

the latter form being motivated by identity (56) on Slater's list.

It is possible that other choices for the form of the series, or extending the choices for the various parameters, may turn up new identities.

2. New Identities

2.1. **Infinite Series Transformations.** We first recall Heine's q-Gauss sum.

$$\sum_{n=0}^{\infty} \frac{(a,b;q)_n}{(c,q;q)_n}\left(\frac{c}{ab}\right)^n = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}$$

Let $b \to \infty$ to get

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n(n-1)/2}(-c/a)^n}{(c;q)_n(q;q)_n} = \frac{(c/a;q)_\infty}{(c;q)_\infty}. \tag{2.1}$$

Theorem 2.1.

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q;q)_{2n+1}} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}.$$

Proof. In (2.1), replace q by q^2 and set $a=-q^2$ and $c=q^3$.

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q;q)_{2n+1}} = \frac{1}{1-q}\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q^3;q^2)_n(q^2;q^2)_n}$$
$$= \frac{1}{1-q}\frac{(-q;q^2)_\infty}{(q^3;q^2)_\infty} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}.$$

□

Shortly after proving the above result, we discovered while reading a pre-print version of [6] that it had previously been stated by Ramanujan (see [6, **Entry 1.7.13**]).

We next recall a result of Ramanujan.

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n(n-1)/2}\gamma^n}{(b;q)_n(q;q)_n} = \frac{(-\gamma;q)_\infty}{(b;q)_\infty}\sum_{n=0}^{\infty} \frac{(-a\gamma/b;q)_n q^{n(n-1)/2}(-b)^n}{(-\gamma;q)_n(q;q)_n}, \tag{2.2}$$

This identity, in a more symmetric form, is found in Ramanujan's lost notebook [19] and a proof can be found in the recent book by Andrews and Berndt [5]. It also follows from the second iteration of Heine's transformation for a ${}_2\phi_1$ series [14, equations III.1 and III.2, page 359]. The form at (2.2) is better suited to our present requirements than Ramanujan's more symmetric form.

Theorem 2.2.

$$\sum_{n=0}^{\infty} \frac{(-1;q)_{2n} q^{n^2+n}}{(q^2;q^2)_n(q^2;q^4)_n} = \frac{(-q^3;q^6)_\infty^2(q^6;q^6)_\infty(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Proof. We use the following identity ((25) from Slater's list, with q replaced by $-q$):

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(q^4;q^4)_n} = \frac{(-q^3;q^6)_\infty^2(q^6;q^6)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{2.3}$$

In (2.2), replace q by q^2 and set $a=-1$, $b=q$ and $\gamma=q^2$. Then

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-1;q)_{2n}q^{n^2+n}}{(q^2;q^2)_n(q^2;q^4)_n}\\
&=\sum_{n=0}^{\infty}\frac{(-1;q^2)_n(-q;q^2)_nq^{n^2+n}}{(q^2;q^2)_n(q;q^2)_n(-q;q^2)_n}\\
&=\sum_{n=0}^{\infty}\frac{(-1;q^2)_nq^{n^2-n}(q^2)^n}{(q;q^2)_n(q^2;q^2)_n}\\
&=\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-q)^nq^{n^2-n}(q;q^2)_n}{(-q^2;q^2)_n(q^2;q^2)_n}\\
&=\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2}(q;q^2)_n}{(-q^2;q^2)_n(q^2;q^2)_n}\\
&=\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2}(q;q^2)_n}{(q^4;q^4)_n}\\
&=\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\frac{(-q^3;q^6)_\infty^2(q^6;q^6)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty}\quad\text{(by (2.3)).}
\end{aligned}$$

The result now follows. □

Theorem 2.3.

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{n^2+2n}}{(q^4;q^4)_n}=\frac{(q,q^5,q^6;q^6)_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Proof. In (2.2), replace q by q^2 and set $a=-q$, $b=-q^2$ and $\gamma=q^3$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{n^2+2n}}{(q^4;q^4)_n}&=\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{n^2-n}(q^3)^n}{(-q^2;q^2)_n(q^2;q^2)_n}\\
&=\frac{(-q^3;q^2)_\infty}{(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-q^2;q^2)_n}{(-q^3;q^2)_n(q^2;q^2)_n}\\
&=\frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-q^2;q^2)_n}{(-q;q^2)_{n+1}(q^2;q^2)_n}\\
&=\frac{(q,q^5,q^6;q^6)_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}.
\end{aligned}$$

The last equality follows from the identity below ((28) from Slater's list, with q replaced by $-q$):

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-q^2;q^2)_n}{(q;q)_{2n+1}}=\frac{(-q,-q^5,q^6;q^6)_\infty(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}.\tag{2.4}$$

□

The result above was also stated by Ramanujan and alternative proofs can be found in [6, **Entry 4.2.11**] and [28].

2.2. **Watson's Transformation.** Before proving the next identity, we introduce some notation.

An ${}_r\phi_s$ *basic hypergeometric series* is defined by

$$ {}_r\phi_s\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, x\right) := \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \dots (a_r;q)_n}{(q;q)_n (b_1;q)_n \dots (b_s;q)_n} \left((-1)^n q^{n(n-1)/2}\right)^{s+1-r} x^n, $$

for $|q| < 1$.

Watson's identity is the following.

$$ {}_8\phi_7\left(\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ q^{-n} \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq^{n+1} \end{matrix}; q, \frac{a^2 q^{n+2}}{bcde}\right) = \frac{(aq)_n (aq/de)_n}{(aq/d)_n (aq/e)_n} {}_4\phi_3\left(\begin{matrix} aq/bc, d, e, q^{-n} \\ aq/b, aq/c, deq^{-n}/a \end{matrix}; q, q\right), \tag{2.5} $$

where n is a non-negative integer. Watson [29] used his transformation in his proof of the Rogers-Ramanujan identities (1.1).

Theorem 2.4. *Let a, b and $q \in \mathbb{C}$ satisfy $|q| < \min\{|b|, 1\}$. Then*

$$ \sum_{r=0}^{\infty} \frac{(1 + aq^r)(a^2;q)_r (b;q)_r \,(-a/b)^r\, q^{r(r+1)/2}}{(a^2q/b;q)_r (q;q)_r} = \frac{(-a;q)_\infty (a^2q;q^2)_\infty (aq/b;q)_\infty}{(a^2q/b;q)_\infty}. \tag{2.6} $$

Proof. We will use Bailey's identity [7]:

$$ \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(aq/b;q)_n (q;q)_n} \left(-\frac{q}{b}\right)^n = \frac{(aq;q^2)_\infty (-q;q)_\infty (aq^2/b^2;q^2)_\infty}{(aq/b;q)_\infty (-q/b;q)_\infty}. \tag{2.7} $$

First, let $n, b \to \infty$ in (2.5) to get

$$ \sum_{r\ge 0} \frac{(1 - aq^{2r})(a)_r (c)_r (d)_r (e)_r \left(a^2/cde\right)^r q^{r(r-1)+2r}}{(1-a)(aq/c)_r (aq/d)_r (aq/e)_r (q)_r} = \frac{(aq)_\infty (aq/de)_\infty}{(aq/d)_\infty (aq/e)_\infty} \sum_{r\ge 0} \frac{(d)_r (e)_r (aq/de)^r}{(aq/c)_r (q)_r}. \tag{2.8} $$

Next, replace a by $-a$, set $c = -b$, $d = a$ and $e = b$, so that (2.8) becomes

$$ \sum_{r\ge 0} \frac{(1 + aq^{2r})(-a)_r (-b)_r (a)_r (b)_r \left(-a/b^2\right)^r q^{r(r-1)+2r}}{(1+a)(aq/b)_r (-q)_r (-aq/b)_r (q)_r} = \frac{(-aq)_\infty (-q/b)_\infty}{(-q)_\infty (-aq/b)_\infty} \sum_{r\ge 0} \frac{(a)_r (b)_r (-q/b)^r}{(aq/b)_r (q)_r}, $$

and the result now follows from (2.7), after replacing b^2 by b and q^2 by q. ∎

2.3. **Bailey Pairs.** A pair of sequences (α_n, β_n) that satisfy $\alpha_0 = 1$ and

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \tag{2.9}$$

is termed a *Bailey pair relative to a*. Baily [8, 9] showed that, for such a pair,

$$\sum_{n=0}^{\infty}(y,z;q)_n \left(\frac{aq}{yz}\right)^n \beta_n = \frac{(aq/y, aq/z;q)_\infty}{(aq, aq/yz;q)_\infty} \sum_{n=0}^{\infty} \frac{(y,z;q)_n}{(aq/y, aq/z;q)_n} \left(\frac{aq}{yz}\right)^n \alpha_n. \tag{2.10}$$

We note two special cases which will be needed later. Firstly, upon letting y, $z \to \infty$ we get that

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq;q)_\infty} \sum_{n=0}^{\infty} a^n q^{n^2} \alpha_n. \tag{2.11}$$

Secondly, upon setting $y = q^{1/2}$ and letting $z \to \infty$ we get that

$$\sum_{n=0}^{\infty}(q^{1/2};q)_n(-1)^n a^n q^{n^2/2}\beta_n = \frac{(aq^{1/2};q)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{1/2};q)_n}{(aq^{1/2};q)_n}\, a^n(-1)^n q^{n^2/2}\alpha_n. \tag{2.12}$$

Lemma 2.5. *The pair (α_n, β_n) is a Bailey pair relative to 1, where*

$$\alpha_n = \begin{cases} 1, & n = 0, \\ 2(-1)^n q^{n^2/2}, & n \geq 1, \end{cases}$$

$$\beta_n = \frac{(\sqrt{q};q)_n}{(-\sqrt{q}, -q, q;q)_n}.$$

Proof. Set $a = 1$, $c = -\sqrt{q}$, $d = -1$ in Slater's equation (4.1) from [26, page 468]:

$$\sum_{r=0}^{n} \frac{(1-aq^{2r})(a,c,d;q)_r q^{(r^2+r)/2}}{(a;q)_{n+r+1}(q;q)_{n-r}(aq/c, aq/d, q;q)_r} \left(\frac{-a}{cd}\right)^r = \frac{(aq/cd;q)_n}{(aq/c, aq/d, q;q)_n}.$$

The result follows from (2.9), after a little simplification. □

Theorem 2.6.

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n} = \frac{(q^3;q^6)_\infty^2(q^6;q^6)_\infty}{(q^2;q^2)_\infty}. \tag{2.13}$$

Proof. Substitute the Bailey pair from Lemma 2.5 into (2.11), with $a = 1$, and replace q with q^2. The result follows after using using the Jacobi triple product identity

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-q/z, -qz, q^2;q^2)_\infty \tag{2.14}$$

to sum the resulting right side. □

Remark: This is a companion identity to number (27) on Slater's list, with q replaced by $-q$:

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n} = \frac{(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty}{(q^2;q^2)_\infty}.$$

2.4. **An identity of Bailey.** Before coming to the next identity we recall a result of Bailey([10], p. 220):

$$(-z^2q,-z^{-2}q^3,q^4;q^4)_\infty + z(-z^2q^3,-z^{-2}q,q^4;q^4)_\infty = (-z,-z^{-1}q,q;q)_\infty. \tag{2.15}$$

We also recall Slater's Bailey pair **G3** (relative to 1) from [26].

$$\alpha_n = \begin{cases} 1, & n=0,\\ q^{3r^2}(q^{3r/2}+q^{-3r/2}), & n=2r,\ r\ge 1,\\ -q^{3r^2}(q^{3r/2}+q^{9r/2+3/2}), & n=2r+1, \end{cases} \tag{2.16}$$
$$\beta_n = \frac{q^n}{(q^2;q^2)_n(-q^{1/2};q)_n}.$$

We note that Slater used **G3** to derive two other series-product identities, (16) and (32) in [27], so we may regard the identity in Theorem 2.7 as one she missed. [1]

Theorem 2.7. *Let $|q|<1$. Then*

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+2n}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n} = \frac{(-q;q^5)_\infty(-q^4;q^5)_\infty(q^5;q^5)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{2.17}$$

Remark: This identity is clearly a companion to Identity (21) on Slater's list:

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n} = \frac{(-q^3;q^5)_\infty(-q^2;q^5)_\infty(q^5;q^5)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{2.18}$$

Proof of Theorem 2.7. We insert the Bailey pair (2.16) into (2.12), set $a=1$ and replace q by q^2 to get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+2n}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n}\\
&\quad= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left(1+\sum_{r=1}^{\infty}q^{10r^2}(q^{3r}+q^{-3r})+\sum_{r=0}^{\infty}q^{10r^2+4r+1}(q^{3r}+q^{9r+3})\right)\\
&\quad= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{r=-\infty}^{\infty}q^{10r^2+3r}+q^4\sum_{r=-\infty}^{\infty}q^{10r^2+13r}\right)\\
&\quad= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left((-q^7,-q^{13},q^{20};q^{20})_\infty+q^4(-q^{-3},-q^{23},q^{20};q^{20})_\infty\right)\\
&\quad= \frac{(-q^3;q^5)_\infty(-q^2;q^5)_\infty(q^5;q^5)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty}.
\end{aligned}$$

The next-to-last equation follows from (2.14), and the last equation follows from (2.15), with q replaced by q^5 and $z=q^4$. □

[1] In an earlier version of this paper we proved Theorem 2.7 by the method of q-difference equations. However, that proof was much longer and less transparent than the present proof.

2.5. **Miscellaneous Methods.** Before coming to the next identity, we recall two other necessary results. The first of these is an identity of Blecksmith, Brillhart and Gerst [11] (a proof is also given in [13]):

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{5n^2} = 2q\frac{(q^4,q^6,q^{10},q^{14},q^{16},q^{20};q^{20})_\infty}{(q^3,q^7,q^8,q^{12},q^{13},q^{17};q^{20})_\infty}. \tag{2.19}$$

If we replace q by $-q$ and apply the Jacobi triple product identity to the left side, (2.19) may be re-written as

$$(q^5,q^5,q^{10};q^{10})_\infty - (q,q,q^2;q^2)_\infty = 2q\frac{(q^4,q^6,q^{10};q^{10})_\infty}{(-q^3,-q^7;q^{10})_\infty(q^8,q^{12};q^{20})_\infty}. \tag{2.20}$$

The second is the following identity, due to Rogers [21, p. 330 (4), line 3, corrected] recently generalized by the third author [25, p. 404, Eq. (3)]:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1;q)_n}{(q;q)_n(q;q^2)_n} = \frac{(q^5,q^5,q^{10};q^{10})_\infty}{(q;q)_\infty(q;q^2)_\infty}. \tag{2.21}$$

We are now able to prove another identity discovered during the present investigations.

Theorem 2.8. *Let $|q|<1$. Then*

$$\sum_{n=0}^{\infty} \frac{q^{(n^2+3n)/2}(-q;q)_n}{(q;q^2)_{n+1}(q;q)_{n+1}} = \frac{(q^{10};q^{10})_\infty}{(q;q)_\infty(q;q^2)_\infty(-q^3,-q^4,-q^6,-q^7;q^{10})_\infty}. \tag{2.22}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{(n^2+3n)/2}(-q;q)_n}{(q;q^2)_{n+1}(q;q)_{n+1}} &= \frac{1}{2q}\sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}(-1;q)_{n+1}}{(q;q^2)_{n+1}(q;q)_{n+1}} \\
&= \frac{1}{2q}\left(\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1;q)_n}{(q;q^2)_n(q;q)_n} - 1\right) \\
&= \frac{1}{2q}\left(\frac{(q^5,q^5,q^{10};q^{10})_\infty}{(q;q)_\infty(q;q^2)_\infty} - 1\right) \text{ (by (2.21))} \\
&= \frac{1}{2q(q;q)_\infty(q;q^2)_\infty}\left((q^5,q^5,q^{10};q^{10})_\infty - (q;q^2)_\infty(q;q^2)_\infty(q^2;q^2)_\infty\right) \\
&= \frac{(q^4,q^6,q^{10};q^{10})_\infty}{(q;q)_\infty(q;q^2)_\infty(-q^3,-q^7;q^{10})_\infty(q^8,q^{12};q^{20})_\infty} \text{ (by (2.20))} \\
&= \frac{(q^{10};q^{10})_\infty}{(q;q)_\infty(q;q^2)_\infty(-q^3,-q^4,-q^6,-q^7;q^{10})_\infty}.
\end{aligned}$$

□

3. An Application of the Two-Variable Generalization of a Rogers-Ramanujan Type Series

In [2], Andrews showed that a certain two-variable generalization $f(t,q)$ of a Rogers-Ramanujan type series $\Sigma(q)$ served as a generating function in t of a sequence of polynomials $P_n(q)$ for which $\lim_{n\to\infty} P_n(q) = \Sigma(q)$.

For example, if

$$\Sigma := \Sigma(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}, \tag{3.1}$$

the series associated with the first Rogers-Ramanujan identity, and the two variable generalization $f(t,q)$ of Σ is given by

$$f(t,q) := \sum_{n=0}^{\infty} \frac{t^{2n} q^{n^2}}{(t;q)_{n+1}}, \tag{3.2}$$

then it is also the case that

$$f(t,q) = \sum_{n=0}^{\infty} P_n(q) t^n \tag{3.3}$$

where

$$P_0(q) = P_1(q) = 1; \qquad P_n(q) = P_{n-1}(q) + q^{n-1} P_{n-2}(q) \quad \text{if } n \geq 2. \tag{3.4}$$

Note that the polynomials in (3.4), which are q-analogs of the Fibonacci numbers, are sometimes called the *Schur polynomials* because they were employed by Schur in his proof of the Rogers-Ramanujan identities. Indeed, Schur [22] showed that

$$P_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \begin{bmatrix} n \\ \lfloor \frac{n+5j+1}{2} \rfloor \end{bmatrix}_q, \tag{3.5}$$

while elsewhere MacMahon [17] showed that

$$P_n(q) = \sum_{j \geq 0} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q, \tag{3.6}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix}_q := \begin{cases} (q;q)_A (q;q)_B^{-1} (q;q)_{A-B}^{-1} & \text{if } 0 \leq B \leq A \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by combining (3.6) and (3.5), we may observe, as Andrews did in [1], that we have a polynomial identity which generalizes the first Rogers-Ramanujan identity and that we may recover (1.1) by letting $n \to \infty$.

In [24], the third author used these two-variable generalizations $f(t,q)$ of Rogers-Ramanujan type series to find polynomial generalizations of all 130 identities in Slater's list [27]. Previously, Santos [23] had studied a large number of polynomial sequences associated with two-variable generalizations of series in Slater's list. Indeed, the primary use of the $f(t,q)$ has been as a generating function in t for sequences of polynomials.

However, here we wish to turn our attention to a different use of the $f(t,q)$ by following up on an observation made by Andrews [2, p. 89]. Letting $f(q,t)$ and $\Sigma(q)$ be as above, Andrews noted the following: not only do we have, as required,

$$\lim_{t \to 1^-} (1-t) f(t,q) = \Sigma(q),$$

but also

$$\lim_{t \to -1^+} (1-t) f(t,q) = f_0(q),$$

where

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n},$$

one of Ramanujan's fifth order mock theta functions (cf. [30], [31]).

Elsewhere [2, p. 90–91], Andrews notes that if we take Σ to the be the Rogers-Ramanujan type series associated with Eq. (46) on Slater's list [27], and define its two variable generalization as

$$f(t,q) := \sum_{n=0}^{\infty} \frac{t^{3n} q^{n(3n-1)/2}}{(t;q)_{n+1}(t^2q;q^2)_n},$$

we find that $\lim_{t\to -1^+} f(t,q)$ is not a mock theta function, but rather a *false* theta function studied by Rogers [21, p. 333(2)].

Andrews later comments [2, p. 93]: "Now if we view this as a curve $y = f(t)$ the points of which are functions of q, we find that frequently if $f(1)$ is a modular form, $f(-1)$ is a mock or false theta function. Is there some general structure possible in which this seemingly amazing occurrence becomes more explicable?" While we do not have an answer to Andrews' question, we have observed that there is a third possibility. Namely, that $f(1)$ is a modular form and $f(-1)$ neither a mock nor false theta function, but rather a sum of modular forms.

Consider the following family of Rogers-Ramanujan type identities related to the modulus 27 which are due to Dyson [8, p. 433, Eqs. (B1)–(B4)] and reproved by Slater [27, p. 161–2, Eqs. (90)–(93)].

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n-1}} = \frac{(q^{12},q^{15},q^{27};q^{27})_\infty}{(q;q)_\infty} \tag{3.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+1}} = \frac{(q^9;q^9)_\infty}{(q;q)_\infty} \tag{3.8}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+2}} = \frac{(q^6,q^{21},q^{27};q^{27})_\infty}{(q;q)_\infty} \tag{3.9}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+3)}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+2}} = \frac{(q^3,q^{24},q^{27};q^{27})_\infty}{(q;q)_\infty} \tag{3.10}$$

The relevant two-variable generalizations (see [24, p. 15, Thm. 2.2]) of (3.7)–(3.10) are

$$f_{3.7}(t,q) := \frac{1}{1-t} + \sum_{n=1}^{\infty} \frac{t^{2n} q^{n^2}(t^3q^3;q^3)_{n-1}}{(t;q)_{n+1}(t^2q;q)_{2n-1}} \tag{3.11}$$

$$f_{3.8}(t,q) := \sum_{n=0}^{\infty} \frac{t^{2n} q^{n(n+1)}(t^3q^3;q^3)_n}{(t;q)_{n+1}(t^2q;q)_{2n+1}} \tag{3.12}$$

$$f_{3.9}(t,q) := \sum_{n=0}^{\infty} \frac{t^{2n} q^{n(n+2)}(t^3q^3;q^3)_n}{(t;q)_{n+1}(t^2q;q)_{2n+2}} \tag{3.13}$$

$$f_{3.10}(t,q) := \sum_{n=0}^{\infty} \frac{t^{2n} q^{n(n+3)}(t^3q^3;q^3)_n}{(t;q)_{n+1}(t^2q;q)_{2n+2}}. \tag{3.14}$$

We believe that the four identities related to the modulus 108 recorded below, which arise from the $t=-1$ cases of (3.11)–(3.14), are new.

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q^3;q^3)_{n-1}}{(-q;q)_n(q;q)_{2n-1}}= \frac{(q^{12},q^{15},q^{27};q^{27})_\infty-2q^2(-q^{33},-q^{75},q^{108};q^{108})_\infty+2q^7(-q^{15},-q^{93},q^{108};q^{108})_\infty}{(q;q)_\infty} \tag{3.15}$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q^3;q^3)_{n}}{(-q;q)_n(q;q)_{2n+1}}= \frac{(q^{9},q^{18},q^{27};q^{27})_\infty-2q^3(-q^{27},-q^{81},q^{108};q^{108})_\infty+2q^9(-q^{9},-q^{99},q^{108};q^{108})_\infty}{(q;q)_\infty} \tag{3.16}$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q^3;q^3)_{n}}{(-q;q)_n(q;q)_{2n+2}}= \frac{(q^{6},q^{21},q^{27};q^{27})_\infty-2q^4(-q^{21},-q^{87},q^{108};q^{108})_\infty+2q^{11}(-q^{3},-q^{105},q^{108};q^{108})_\infty}{(q;q)_\infty} \tag{3.17}$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n+3)}(-q^3;q^3)_{n}}{(-q;q)_n(q;q)_{2n+2}} = \frac{(q^{3},q^{24},q^{27};q^{27})_\infty-2q^5(-q^{15},-q^{93},q^{108};q^{108})_\infty+2q^{13}(-q^{-3},-q^{111},q^{108};q^{108})_\infty}{(q;q)_\infty} \tag{3.18}$$

The identities (3.15)–(3.18) may be proved using Bailey pairs (see Sec. 2.3). Although less well known than (2.9), the following characterization of Bailey pairs [2, p. 29, Eq. (3.40) with $a=1$] is equivalent to (2.9) with $a=1$:

$$\alpha_n=(1-q^{2n})\sum_{k=0}^{n}\frac{(-1)^{n-k}q^{\binom{n-k}{2}}(q;q)_{n+k-1}}{(q;q)_{n-k}}\beta_k \tag{3.19}$$

Furthermore, it is straightforward to show that (3.19) may be rewritten as

$$\alpha_n=(-1)^nq^{\binom{n}{2}}(1+q^n)\sum_{k=0}^{n}(q^n;q)_k(q^{-n};q)_kq^k\beta_k, \tag{3.20}$$

which is the form we shall employ.

Lemma 3.1. *If, for n a nonnegative integer,*

$$\alpha_n=\begin{cases} 1 & \text{if } n=0\\ (-1)^rq^{\frac92 r^2-\frac32 r}(1+q^{3r}) & \text{if } n=3r>0\\ -2q^{18r^2+9r+1} & \text{if } n=6r+1\\ 2q^{18r^2+15r+3} & \text{if } n=6r+2\\ 2q^{18r^2+21r+6} & \text{if } n=6r+4\\ -2q^{18r^2+27r+10} & \text{if } n=6r+5 \end{cases} \tag{3.21}$$

and

$$\beta_n = \begin{cases} \frac{(-q^3;q^3)_{n-1}}{(-q;q)_n(q;q)_{2n-1}} & \textit{if } n>0 \\ 1 & \textit{if } n=0, \end{cases} \tag{3.22}$$

then (α_n, β_n) *forms a Bailey pair.*

Proof. By inserting (3.21) and (3.22) into (3.20), it is clear that we will be done once we show

$$\begin{multline} (-1)^r q^{\frac{9}{2}r^2-\frac{3}{2}r}(1+q^{3r}) \\ = (-1)^r q^{\binom{3r}{2}}(1+q^{3r})\left\{1+\sum_{k=1}^{3r}\frac{(q^{3r};q)_k(q^{-3r};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}}\right\}, \end{multline} \tag{3.23}$$

$$\begin{multline} -2q^{18r^2+9r+1} \\ = -q^{\binom{6r+1}{2}}(1+q^{6r+1})\left\{1+\sum_{k=1}^{6r+1},\frac{(q^{6r+1};q)_k(q^{-6r-1};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}}\right\}, \end{multline} \tag{3.24}$$

$$\begin{multline} 2q^{18r^2+15r+3} \\ = q^{\binom{6r+2}{2}}(1+q^{6r+2})\left\{1+\sum_{k=1}^{6r+2}\frac{(q^{6r+2};q)_k(q^{-6r-2};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}}\right\}, \end{multline} \tag{3.25}$$

$$\begin{multline} 2q^{18r^2+21r+6} \\ = q^{\binom{6r+4}{2}}(1+q^{6r+4})\left\{1+\sum_{k=1}^{6r+4}\frac{(q^{6r+4};q)_k(q^{-6r-4};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}}\right\}, \end{multline} \tag{3.26}$$

and

$$\begin{multline} -2q^{18r^2+27r+10} \\ = -q^{\binom{6r+5}{2}}(1+q^{6r+5})\left\{1+\sum_{k=1}^{6r+5}\frac{(q^{6r+5};q)_k(q^{-6r-5};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}}\right\}, \end{multline} \tag{3.27}$$

for nonnegative integers r. Using elementary algebra, the equations (3.23)–(3.27) are easily shown to be equivalent to

$$\sum_{k=1}^{3r}\frac{(q^{3r};q)_k(q^{-3r};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}} = 0, \tag{3.28}$$

$$\frac{q^{6r+1}+1}{q^{6r+1}-1}\sum_{k=1}^{6r+1}\frac{(q^{6r+1};q)_k(q^{-6r-1};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}} = 1, \tag{3.29}$$

$$\frac{q^{6r+2}+1}{q^{6r+2}-1}\sum_{k=1}^{6r+2}\frac{(q^{6r+2};q)_k(q^{-6r-2};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}} = 1, \tag{3.30}$$

$$\frac{1+q^{6r+4}}{1-q^{6r+4}}\sum_{k=1}^{6r+4}\frac{(q^{6r+4};q)_k(q^{-6r-4};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}} = 1, \tag{3.31}$$

$$\frac{1+q^{6r+5}}{1-q^{6r+5}}\sum_{k=1}^{6r+5}\frac{(q^{6r+5};q)_k(q^{-6r-5};q)_k(-q^3;q^3)_kq^k}{(-q;q)_k(q;q)_{2k-1}}=1. \tag{3.32}$$

Each of equations (3.28)–(3.32) may be verified using the WZ method [18, Chapter 7] or induction on r. □

Theorem 3.2. *Identity* (3.15) *is valid.*

Proof. Recall that the weak form of Bailey's lemma [2, p. 27, Eq. (3.33) with $a = 1$] states that

$$\sum_{n=0}^{\infty} q^{n^2}\beta_n = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{n^2}\alpha_n \tag{3.33}$$

for any Bailey pair (α_n, β_n). Inserting the Bailey pair established in Lemma 3.1 into (3.33) yields

$$\begin{aligned}
&1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q^3;q^3)_{n-1}}{(-q;q)_n(q;q)_{2n-1}} \\
&\quad=\frac{1}{(q;q)_\infty}\Bigg(1+\sum_{r=1}^{\infty}(-1)^r q^{\frac{27}{2}r^2-\frac{3}{2}r}(1+q^{3r})-2\sum_{r=0}^{\infty}q^{54r^2+21r+2} \\
&\qquad+2\sum_{r=0}^{\infty}q^{54r^2+39r+7}+2\sum_{r=0}^{\infty}q^{54r^2+69r+22}-2\sum_{r=0}^{\infty}q^{54r^2+87r+35}\Bigg) \\
&\quad=\frac{1}{(q;q)_\infty}\Bigg(\sum_{r=-\infty}^{\infty}(-1)^r q^{\frac{27}{2}r^2-\frac{3}{2}r}-2\sum_{r=0}^{\infty}q^{54r^2+21r+2}+2\sum_{r=0}^{\infty}q^{54r^2+39r+7} \\
&\qquad+2\sum_{r=1}^{\infty}q^{18r^2-39r+7}-2\sum_{r=1}^{\infty}q^{54r^2-21r+2}\Bigg) \\
&\quad=\frac{1}{(q;q)_\infty}\Bigg(\sum_{r=-\infty}^{\infty}(-1)^r q^{\frac{27}{2}r^2-\frac{3}{2}r}-2\sum_{r=-\infty}^{\infty}q^{54r^2-21r+2}+2\sum_{r=-\infty}^{\infty}q^{54r^2-39r+7}\Bigg) \\
&\quad=\frac{(q^{12},q^{15},q^{27};q^{27})_\infty-2q^2(-q^{33},-q^{75},q^{108};q^{108})_\infty+2q^7(-q^{15},-q^{93};q^{108};q^{108})_\infty}{(q;q)_\infty}.
\end{aligned} \tag{3.34}$$

□

The other identities (3.16)–(3.18) may be proved similarly.

The identity (3.16) deserves special attention because its right hand side may be expressed as a single infinite product whereas it appears that of the other three can not be simplified beyond a sum of three infinite products.

Theorem 3.3.

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q^3;q^3)_n}{(-q;q)_n(q;q)_{2n+1}}=\frac{(q^3;q^3)_\infty(q^3;q^{18})_\infty(q^{15};q^{18})_\infty}{(q;q)_\infty} \tag{3.35}$$

Proof. We shall require Fricke's quintuple product identity [12]

$$(z^3q,z^{-3}q^2,q^3;q^3)_\infty+z(z^{-3}q,z^3q^2,q^3;q^3)_\infty=(-z^{-1}q,-z,q;q)_\infty(z^{-2}q,z^2q;q^2)_\infty \tag{3.36}$$

and an identity due to Bailey [10, p. 220, Eq. (4.1)]

$$(-z^2q, -z^{-2}q^3, q^4; q^4)_\infty + z(-z^2q^3, -z^{-2}q, q^4; q^4)_\infty = (-z, -z^{-1}q, q; q)_\infty. \tag{3.37}$$

By (3.16), we have

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^3;q^3)_n}{(-q;q)_n(q;q)_{2n+1}}$$
$$= \frac{(q^9, q^{18}, q^{27}; q^{27})_\infty - 2q^3(-q^{27}, -q^{81}, q^{108}; q^{108})_\infty + 2q^9(-q^9, -q^{99}, q^{108}; q^{108})_\infty}{(q;q)_\infty}.$$

Expanding the first triple product in the numerator by (3.37) with q replaced by q^{27} and $z = -q^9$ yields

$$\frac{(-q^{45}, -q^{63}, q^{108}; q^{108})_\infty - 2q^3(-q^{27}, -q^{81}, q^{108}; q^{108})_\infty + q^9(-q^9, -q^{99}, q^{108}; q^{108})_\infty}{(q;q)_\infty}.$$

Two further applications of (3.37) shows that the preceding expression is equal to

$$\begin{aligned} &\frac{(-q^9, -q^{18}, q^{27}; q^{27})_\infty - q^3(-1, -q^{27}, q^{27}; q^{27})_\infty}{(q;q)_\infty} \\ &= \frac{(q^3, q^6, q^9; q^9)_\infty (q^3, q^{15}; q^{18})_\infty}{(q;q)_\infty} \\ &= \frac{(q^3;q^3)_\infty (q^3, q^{15}; q^{18})_\infty}{(q;q)_\infty}, \end{aligned}$$

where the penultimate equality follows from (3.36). □

We close this section by recalling that once a given Bailey pair is established, it may be utilized in connection with limiting cases of Bailey's lemma other than (3.33), thus yielding additional Rogers-Ramanujan type identities. For instance, if we insert the Bailey pair established in Lemma 3.1 into [2, p. 26, Eq. (3.28) with $n, \rho_1 \to \infty$ and $\rho_2 = -\sqrt{q}$], we obtain the identity related to the modulus 144:

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_n(-q^6;q^6)_{n-1}}{(-q^2;q^2)_n(q^2;q^2)_{2n-1}} \tag{3.38}$$
$$= \frac{(q^{15}, q^{21}, q^{36}; q^{36})_\infty - 2q^3(-q^{42}, -q^{102}, q^{144}; q^{144})_\infty + 2q^{10}(-q^{18}, -q^{126}, q^{144}; q^{144})_\infty}{(q;q^2)_\infty (q^4;q^4)_\infty}.$$

A partner of (3.38) is

$$\sum_{n=0}^{\infty} \frac{q^{n^2+4n}(-q;q^2)_{n+1}(-q^6;q^6)_n}{(-q^2;q^2)_n(q^2;q^2)_{2n+2}} \tag{3.39}$$
$$= \frac{(q^3, q^{33}, q^{36}; q^{36})_\infty - 2q^7(-q^{18}, -q^{126}, q^{144}; q^{144})_\infty + 2q^{12}(-q^6, -q^{138}, q^{144}; q^{144})_\infty}{(q;q^2)_\infty (q^4;q^4)_\infty}.$$

4. Concluding Remarks

For quite a long time we were convinced that there must exist a general transformation of the type found in Watson's theorem (see (2.5)), a transformation which would give the result in Theorem 2.7 as a special case for particular values of its parameters.

One reason we thought this transformation had to exist was the appearance of the $(-q;q^5)_\infty(-q^4;q^5)_\infty(q^5;q^5)_\infty$ term on the product side, which can be represented as an infinite series via the Jacobi triple product. This in turn brought to mind Watson's proof of the Rogers-Ramanujan identities, where he showed that these followed as special cases of (2.5).

However, we could not find such a transformation, but possibly our search was incomplete. Does the identity in Theorem 2.7 follow as a special case of some known transformation, perhaps some known transformation between basic hypergeometric series? Is this identity a special case of some as yet undiscovered general transformation?

As remarked at the end of the introduction, varying the form of the series S in (1.3) may lead to other new identities of the Rogers-Ramanujan-Slater type. In particular, one might hope for the discovery of new identities which are not readily proved within the framework of our present understanding of identities of the Rogers-Ramanujan-Slater type.

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