A Partition Bijection Related to the Rogers-Selberg Identities and Gordon's Theorem

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Abstract

We provide a bijective map from the partitions enumerated by the series side of the Rogers-Selberg mod 7 identities onto partitions associated with a special case of Basil Gordon's combinatorial generalization of the Rogers-Ramanujan identities. The implications of applying the same map to a special case of David Bressoud's even modulus analog of Gordon's theorem are also explored.

Key words: partitions; Rogers-Ramanujan identities; Rogers-Selberg identities; Gordon's theorem; Andrews-Gordon-Bressoud theorem.

1 Introduction

The celebrated Rogers-Ramanujan identities were first discovered by L.J. Rogers in 1894 [15], but received little attention until they were rediscovered independently by S. Ramanujan [13, Ch. III, pp. 33 ff.] and I. Schur [17] some two decades later. They are as follows:

Theorem 1.1 (The Rogers-Ramanujan Identities)

$$\sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q;q)_j} = \prod_{\substack{j \geqq 1 \\ j \not\equiv 0, \pm 1 (\mathrm{mod}\ 5)}} \frac{1}{1-q^j} \tag{1.1}$$

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and

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(q;q)_j} = \prod_{\substack{j \geq 1 \\ j \not\equiv 0, \pm 2 (\text{mod } 5)}} \frac{1}{1 - q^j}, \qquad (1.2)$$

where $(a;q)_j := \prod_{h=0}^{j-1}(1 - aq^h)$ *for* j *a positive integer and* $(a;q)_0 := 1$.

The Rogers-Ramanujan identities and the other q-series identities mentioned in this paper may be considered identities of analytic functions that are valid if and only if $|q| < 1$. Since our interest here is combinatorial, convergence conditions will not be mentioned again.

Rogers presented a large number of q-series identities which resemble (1.1) and (1.2) in [15] and [16]. Among them were a set of three identities associated with the modulus 7, which received little attention until they were independently rediscovered by A. Selberg [18] and then re-proved by F.J. Dyson in [9].

Theorem 1.2 (The Rogers-Selberg Identities)

$$\sum_{j=0}^{\infty} \frac{q^{2j^2+2j}(-q^{2j+2};q)_\infty}{(q^2;q^2)_j} = \prod_{\substack{j \geq 1 \\ j \not\equiv 0, \pm 1 (\text{mod } 7)}} \frac{1}{1 - q^j}, \qquad (1.3)$$

$$\sum_{j=0}^{\infty} \frac{q^{2j^2+2j}(-q^{2j+1};q)_\infty}{(q^2;q^2)_j} = \prod_{\substack{j \geq 1 \\ j \not\equiv 0, \pm 2 (\text{mod } 7)}} \frac{1}{1 - q^j}, \qquad (1.4)$$

and

$$\sum_{j=0}^{\infty} \frac{q^{2j^2}(-q^{2j+1};q)_\infty}{(q^2;q^2)_j} = \prod_{\substack{j \geq 1 \\ j \not\equiv 0, \pm 3 (\text{mod } 7)}} \frac{1}{1 - q^j}, \qquad (1.5)$$

where $(a;q)_\infty = \prod_{h=0}^{\infty}(1 - aq^h)$.

Both MacMahon [13, Ch. III, pp. 33 ff.] and Schur [17] showed that (1.1) and (1.2) could be interpreted as identities in the theory of integer partitions. (See §2 for the definition of partition and related terms.)

Theorem 1.3 (The Rogers-Ramanujan Identities–Combinatorial Version)
For $i = 1, 2$, *the number of partitions of* n *into parts which are nonconsecutive integers greater than* $i - 1$ *and in which no part is repeated equals the number of partitions of* n *into parts* $\not\equiv 0, \pm i$ (mod 5).

In 1961, Basil Gordon [10] gave the following generalization of the combinatorial generalization of the Rogers-Ramanujan identities:

Theorem 1.4 (Gordon's theorem) *Let* $G_{k,i}(n)$ *denote the number of partitions of* n *into parts such that* 1 *appears as a part at most* $i - 1$ *times and*

the total number of appearances of any two consecutive integers is at most $k-1$. Let $C_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i$ (mod $2k+1$). Then $G_{k,i}(n) = C_{k,i}(n)$ for $1 \leqq i \leqq k$ and all integers n.

Indeed, it is an elementary exercise to see that (1.1) and (1.2) correspond to the $i = 1$ and $i = 2$ cases, respectively, of the $k = 2$ case of Gordon's theorem (see, e.g., [11, p. 290 ff.]).

The right hand sides of (1.3), (1.4), and (1.5) are clearly the generating functions for the partitions enumerated by $C_{3,1}(n)$, $C_{3,2}(n)$, and $C_{3,3}(n)$ respectively. Nonetheless, relating the partitions enumerated by the $G_{3,i}(n)$ to the left hand sides of (1.3), (1.4), and (1.5) is *not* a straightforward matter.

In a recent paper, George Andrews [5] provided the following partition theoretic interpretation of (1.4):

Theorem 1.5 (Andrews) *Let $A_2(n)$ denote the number of partitions of n such that if $2j$ is the largest repeated even part, then all positive even integers less than $2j$ also appear at least twice, no odd part less than $2j$ appears, and no part greater than $2j$ is repeated. Then $A_2(n) = C_{3,2}(n)$ for all n.*

PROOF. Note that

$$\frac{q^{2j^2+2j}(-q^{2j+1};q)_\infty}{(q^2;q^2)} = \frac{q^{2+2+4+4+\cdots+2j+2j}}{(q^2;q^2)} \times (-q^{2j+1};q)_\infty.$$

By the methods of Euler (cf. [3, p. 4 ff.]),

$$\frac{q^{2+2+4+4+\cdots+2j+2j}}{(q^2;q^2)}$$

is the generating function for partitions into 2's, 4's, 6's, $\ldots, 2j$'s with each part appearing at least twice, and $(-q^{2j+1};q)_\infty$ is the generating function for partitions into distinct parts with each part at least $2j+1$. Thus, by summing over all nonnegative j, it follows that the left hand side of (1.4) is the generating function for $A_2(n)$. Again by Euler's method, it is immediate that the right hand side of (1.4) is the generating function for $C_{3,2}(n)$.

The analogous partition theoretic interpretation of (1.3) is given next.

Theorem 1.6 *Let $A_1(n)$ denote the number of partitions of n such that if $2j$ is the largest repeated even part, then all positive even integers less than $2j$ also appear at least twice, no odd part less than $2j+2$ appears, and no part greater than $2j$ is repeated. Then $A_1(n) = C_{3,1}(n)$ for all n.*

PROOF. The proof parallels that of Theorem 1.5, except that the role of $(-q^{2j+1}; q)_\infty$ in Theorem 1.5 is played by $(-q^{2j+2}; q)_\infty$, which is the generating function for partitions into distinct parts with each part at least $2j + 2$, and that the right hand side of (1.3) is the generating function for $C_{3,1}(n)$.

The purpose of this paper is provide an explicit bijection between the partitions enumerated by Andrews' $A_2(n)$ and those of Gordon's $G_{3,2}(n)$. As we shall see, the same map also provides a bijection between the partitions enumerated by $G_{3,1}(n)$ and those enumerated by $A_1(n)$, as well as a new partition theorem related to a special case of David Bressoud's even modulus analog of Gordon's theorem.

2 Definitions

2.1 Standard Definitions in Partition Theory

The definitions and symbols introduced in this subsection are all standard (cf. [12]).

A *partition* π of an integer n is a nonincreasing sequence of nonnegative integers

$$\pi = \{\pi_1, \pi_2, \pi_3, \dots\}$$

such that $\sum_{i=1}^\infty \pi_i = n$. Each nonzero term in $\{\pi_1, \pi_2, \pi_3, \dots\}$ is called a *part* of the partition π. The number of parts in π is called the *length* $\ell(\pi)$ of π. Since the tail $\{\pi_{\ell(\pi)+1}, \pi_{\ell(\pi)+2}, \dots\}$ of any partition π must be $\{0, 0, 0, 0, \dots\}$, it will be convenient to suppress the infinite string of zeros when writing a specific partition. The *empty partition*, $\emptyset = \{0, 0, 0, 0, \dots\} = \{\}$, has length zero, i.e. no parts.

For two partitions, π and λ, we may write $\pi \geqq \lambda$ if $\pi_i \geqq \lambda_i$ for all i. The *multiplicity* of the integer j in π, denoted $m_j(\pi)$, is the number of times j appears in π.

At times it will be convenient to express π in the alternate notation

$$\pi = \langle 1^{m_1(\pi)} 2^{m_2(\pi)} 3^{m_3(\pi)} \cdots \rangle$$

meaning that π contains $m_1(\pi)$ 1's, $m_2(\pi)$ 2's, etc. In this notation it is customary to omit the term $j^{m_j(\pi)}$ when $m_j(\pi) = 0$.

The *union* of two partitions π and λ, denoted $\pi \cup \lambda$, is the partition whose parts

are those of π and λ together, arranged in nonincreasing order. For example,

$$\{8, 3, 3, 2, 1\} \cup \{9, 7, 5, 3, 1\} = \{9, 8, 7, 5, 3, 3, 3, 2, 1, 1\}.$$

The *sum* of two partitions π and λ is

$$\pi + \lambda := \{\pi_1 + \lambda_1, \pi_2 + \lambda_2, \dots\}.$$

If $\pi \geqq \lambda$, then the *difference* of π and λ is given by

$$\pi - \lambda := \{\pi_1 - \lambda_1, \pi_2 - \lambda_2, \dots\}.$$

2.2   Definitions of Special Symbols

In the standard notation, the ith largest part of the partition π is denoted π_i, where $1 \leq i \leq \ell(\pi)$. Accordingly, the ith smallest part of π is $\pi_{\ell(\pi)-i+1}$. However, there will be occasions where a less cumbersome notation for the ith smallest part will be useful, so let us define

$$\pi_{[i]} := \pi_{\ell(\pi)-i+1} \text{ for } i = 1, 2, \dots, \ell(\pi).$$

For two partitions π and λ, let us write $\pi \succ \lambda$ if the smallest part of π is greater than the largest part of λ, i.e.,

$$\pi_{[1]} = \pi_{\ell(\pi)} > \lambda_1.$$

Let $D(\pi)$ denote the number of different parts in π which appear at least twice, i.e.
$$D(\pi) := \mathrm{Card}\{j \in \pi \mid m_j(\pi) > 1\},$$
thus $D(\{21, 15, 15, 12, 11, 9, 9, 6, 5, 5, 2\}) = 3$ since three integers (15, 9, and 5) each appear more than once as parts.

Let $R_k(\pi)$ denote the k-th largest repeated part in π, i.e.

$$R_1(\pi) := \max\{j \mid m_j(\pi) > 1\},$$

$$R_k(\pi) := \max\{j \mid m_j(\pi) > 1 \text{ and } j < R_{k-1}(\pi)\} \text{ for } k = 2, 3, \dots, D(\pi),$$
$$R_{D(\pi)+1}(\pi) = 0.$$

Let $\mathcal{G}_1$ denote the set of partitions enumerated by $G_{3,1}(n)$ in Gordon's theorem, i.e. partitions π such that
$$m_1(\pi) = 0 \tag{2.1}$$

and
$$m_j(\pi) + m_{j+1}(\pi) \leqq 2 \tag{2.2}$$
for all $j \geqq 1$.

Let $\mathcal{G}_2$ denote the set of partitions enumerated by $G_{3,2}(n)$ in Gordon's theorem, i.e. partitions π such that
$$m_1(\pi) \leqq 1 \tag{2.3}$$
and
$$m_j(\pi) + m_{j+1}(\pi) \leqq 2 \tag{2.4}$$
for all $j \geqq 1$.

Let $\mathcal{A}_1$ denote the set of partitions enumerated by $A_1(n)$ in Theorem 1.6, i.e. partitions π such that
$$m_j(\pi) \leqq 1 \text{ if } j \text{ is odd,} \tag{2.5}$$
$$m_j(\pi) = 0 \text{ if } j \text{ is odd and } j < R_1(\pi) + 2, \text{ and} \tag{2.6}$$
$$m_j(\pi) \geqq 2 \text{ if } j \text{ is even and } j < R_1(\pi). \tag{2.7}$$

Let $\mathcal{A}_2$ denote the set of partitions enumerated by $A_2(n)$ in Theorem 1.5, i.e. partitions π such that
$$m_j(\pi) \leqq 1 \text{ if } j \text{ is odd,} \tag{2.8}$$
$$m_j(\pi) = 0 \text{ if } j \text{ is odd and } j < R_1(\pi), \text{ and} \tag{2.9}$$
$$m_j(\pi) \geqq 2 \text{ if } j \text{ is even and } j < R_1(\pi). \tag{2.10}$$

For any partition $\pi \in \mathcal{G}_2$, let $\pi^{(0)}$ denote the subpartition of π whose parts are greater than $R_1(\pi)$, and $\pi^{(k)}$ denote the subpartition of π whose parts are less than $R_k(\pi)$ and greater than $R_{k+1}(\pi)$, for $1 \leqq k \leqq D(\pi)$. Notice that by Condition (2.4), no part of $\pi \in \mathcal{G}_2$ may appear more than twice. Thus any $\pi \in \mathcal{G}_2$ may be decomposed uniquely as

$$\pi = \pi^{(0)} \cup \langle R_1(\pi)^2 \rangle \cup \pi^{(1)} \cup \langle R_2(\pi)^2 \rangle \cup \pi^{(2)} \cup \cdots \cup \langle R_{D(\pi)}(\pi)^2 \rangle \cup \pi^{(D(\pi))},$$

where each $\pi^{(k)}$ is a partition into distinct parts, and

$$\pi^{(0)} \succ \langle R_1(\pi)^2 \rangle \succ \pi^{(1)} \succ \langle R_2(\pi)^2 \rangle \succ \pi^{(2)} \succ \cdots \succ \langle R_{D(\pi)}(\pi)^2 \rangle \succ \pi^{(D(\pi))}.$$

Of course, any of the $\pi^{(k)}$ may be empty.

Let $P(\pi)$ denote the subpartition of π consisting of parts greater than $R_1(\pi)$, and $S(\pi)$ denote the subpartition of π consisting of parts less than $R_1(\pi)$. Thus,
$$P(\{21, 15, 15, 12, 11, 9, 9, 7, 5, 5, 2\}) = \{21\}$$
and
$$S(\{21, 15, 15, 12, 11, 9, 9, 7, 5, 5, 2\}) = \{12, 11, 9, 9, 7, 5, 5, 2\}.$$

6

For any partition $\lambda \in \mathcal{A}_2$, λ can be decomposed uniquely into

$$\lambda = P(\lambda) \cup L(\lambda),$$

where $L(\lambda)$ is the subpartition of λ consisting of all parts of λ less than or equal to $R_1(\lambda)$. Clearly, $P(\lambda)$ is a partition with distinct parts, $R_1(\lambda)$ is a partition into parts which are even and repeated, and

$$P(\lambda) \succ L(\lambda).$$

Remark 2.1 *Note that $P(\lambda) = \lambda^{(0)}$. One reason for introducing the P notation is to emphasize that for $\lambda \in \mathcal{A}_2$, the partition λ naturally decomposes into just two subpartitions, $P(\lambda)$ and $L(\lambda)$, whereas for $\pi \in \mathcal{G}_2$, the partition π naturally decomposes into $2D(\pi) + 1$ subpartitions, of which all of the $\pi^{(k)}$ are partitions with distinct parts. Also, when P is applied to a partition $\pi \in \mathcal{G}_2$, it is used in conjunction with S, where P and S can be thought of as a "prefix" and "suffix" respectively.*

3   The Bijection

3.1   Mapping $\mathcal{G}_1$ onto $\mathcal{A}_1$.

Since $\mathcal{G}_1 \subsetneq \mathcal{G}_2$ and $\mathcal{A}_1 \subsetneq \mathcal{A}_2$, let us initially turn our attention to $\mathcal{G}_1$ and $\mathcal{A}_1$.

Definition 3.1 *Define the map f on the set $\mathcal{G}_1$ by*

$$f(\pi) := \bigcup_{i=1}^{D(\pi)} \langle (2i)^{R_i(\pi) - R_{i+1}(\pi) - \ell(\pi^{(i)})} \rangle \cup \bigcup_{i=0}^{D(\pi)} \left(\pi^{(i)} + \langle (2i)^{\ell(\pi^{(i)})} \rangle \right), \qquad (3.1)$$

for $\pi \in \mathcal{G}_1$. Equivalently, f may be defined recursively by

$$f(\pi) := P(\pi) \cup \left[\langle 2^{R_1(\pi)} \rangle + f(S(\pi)) \right]$$

with initial condition

$$f(\emptyset) = \emptyset.$$

Example 3.2 *If*

$$\pi = \{40, 37, 36, 22, 22, 20, 19, 17, 17, 15, 13, 12, 10, 8, 8, 4, 4, 2\},$$

then

$$f(\pi) = \{40, 37, 36, 22, 21, 19, 17, 16, 14, 10\} \cup \langle 2^3 4^5 6^4 8^3 \rangle. \qquad (3.2)$$

7

In more detail, the mapping is

$$\{40, 37, 36, 22, 22, 20, 19, 17, 17, 15, 13, 12, 10, 8, 8, 4, 4, 2\}$$

$$= \{40, 37, 36\} \cup \{22, 22\} \cup \{20, 19\} \cup \{17, 17\} \cup \{15, 13, 12, 10\} \cup \{8, 8\} \cup \{4, 4\} \cup \{2\}$$

$$\xrightarrow{f} \langle 2^{22-17-2} \rangle \cup \langle 4^{17-8-4} \rangle \cup \langle 6^{8-4-0} \rangle \cup \langle 8^{4-0-1} \rangle$$

$$\cup \{40, 37, 36\} \cup \{20 + 2, 19 + 2\} \cup \{15 + 4, 13 + 4, 12 + 4, 10 + 4\} \cup \emptyset \cup \{2 + 8\}$$

$$= \{40, 37, 36, 22, 21, 19, 17, 16, 14, 10\} \cup \langle 2^3 4^5 6^4 8^3 \rangle.$$

The following diagram helps to demonstrate how the the recursive formulation of f works. In each iteration, the largest repeated part R is underlined, and then converted to R 2's in the next iteration. At the last step, the columns are added to form the image of π under f.

40 37 36 <u>22</u> <u>22</u> 20 19 17 17 15 13 12 10 8 8 4 4 2

↓

40 37 36   2   2   2   2   2   2   2   2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

20 19 <u>17</u> <u>17</u> 15 13 12 10 8 8 4 4 2

↓

40 37 36   2   2   2   2   2   2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

20 19   2   2   2   2 2 2 2 2 2 2 2 2 2 2 2 2

15 13 12 10 <u>8</u> <u>8</u> 4 4 2

↓

40 37 36   2   2   2   2   2   2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

20 19   2   2   2   2 2 2 2 2 2 2 2 2 2 2 2 2

15 13 12 10 2 2 2 2 2 2 2 2

<u>4</u> <u>4</u> 2

↓

40 37 36   2   2   2   2   2   2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

+ 20 19   2   2   2   2 2 2 2 2 2 2 2 2 2 2 2 2

+ 15 13 12 10   2 2 2 2 2 2 2 2

+   2 2 2 2

+   2

_____

40 37 36 22 21 19 17 16 14 10 8 8 8 6 6 6 6 4 4 4 4 4 2 2 2

(3.3)

8

Thus the parts of both π and $f(\pi)$ are encoded in the matrix

$$
\begin{bmatrix}
40 & 37 & 36 & 2 & 2 & 2 & 2 & 2 & 2\,2\,2\,2\,2\,2\,2\,2\,2\,2\,2\,2\,2\,2\,2\,2\,2\,2 \\
0 & 0 & 0 & 20 & 19 & 2 & 2 & 2 & 2\,2\,2\,2\,2\,2\,2\,2\,2\,2\,2\,2\,2\,0\,0\,0\,0 \\
0 & 0 & 0 & 0 & 0 & 15 & 13 & 12 & 10\,2\,2\,2\,2\,2\,2\,2\,2\,0\,0\,0\,0\,0\,0\,0\,0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\,2\,2\,2\,2\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\,2\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0
\end{bmatrix}. \qquad (3.4)
$$

Display (3.4) may be the best way of simultaneously displaying π and $f(\pi)$; the distinct parts of π appear in the matrix as nonzero, nonrepeated entries in a given row, parts $R_i(\pi)$ which appear twice are represented as a sequence of $R_i(\pi)$ 2's in the ith row, and the parts of $f(\pi)$ are the sums of the columns.

Accordingly, (3.4) motivates the following definition. Let $\lambda := f(\pi)$.

Definition 3.3 *Let $\pi \in \mathcal{G}_2$. The $\mathcal{S}$-diagram of the partitions π and λ is the $(D(\pi)+1) \times \left(\ell(\pi^{(0)}) + R_1(\pi)\right)$ matrix (or, equivalently, the $(1 + R(\lambda)/2) \times \ell(\lambda)$ matrix) whose first row consists of the parts of $\pi^{(0)}$ in nonincreasing order followed by $R_1(\pi)$ 2's, and whose ith row consists of $\sum_{k=0}^{i-1} \ell(\pi^{(k)})$ 0's followed by the parts of $\pi^{(i-1)}$ in nonincreasing order followed by $R_i(\pi)$ 2's, followed by 0's, for $2 \leqq i \leqq D(\pi)+1$. The parts of λ are then given by the sums of the columns.*

Clearly, there is a unique $\mathcal{S}$-diagram for each partition $\pi \in \mathcal{G}_2$. Next, let us examine how to determine the $\mathcal{S}$-diagram given only a partition in $\mathcal{A}_2$.

Again, $\lambda = f(\pi)$. Observe that

$$\lambda_j = \pi_j, \text{ if } 1 \leqq j \leqq \ell(\pi^{(0)}),$$

$$\ell(\lambda) = \ell(\pi^{(0)}) + R_1(\pi),$$

and

$$\lambda_1 > \lambda_2 > \cdots > \lambda_{\ell(\pi^{(0)})}$$

since $\pi^{(0)}$ is a partition with distinct parts. Accordingly,

$$\ell(\lambda) - j + 1 < \lambda_j, \text{ if } 1 \leqq j \leqq \ell(\pi^{(0)})$$

However,

$$\lambda_{\ell(\pi^{(0)})} = 2 + \pi_{\ell(\pi^{(0)})+3} \leqq 2 + 2 + R_1(\pi)$$

and so

$$\ell(\lambda) - (\ell(\pi^{(0)} + 1)) + 1 - \lambda_{\ell(\pi^{(0)})+1} \geqq \ell(\pi^{(0)} + 1)$$

9

which means that

$$R_1(\pi) = \ell(\lambda) - (\ell(\pi^{(0)} + 1)) + 1 - \lambda_{\ell(\pi^{(0)}+1)} = \max_{1 \leqq j \leqq \ell(\lambda)} \left\{ j \mid j \geqq \lambda_{[j]} \right\}. \quad (3.5)$$

In order to clarify what is being asserted, let us examine the above statements in terms of Example 3.2. The three largest parts of $\lambda = f(\pi)$, namely 40, 37, and 36, are respectively the 25th, 24th, and 23th *smallest* parts. These parts (40, 37, and 36), are all larger than the number of parts less than equal to them (25, 24, and 23 respectively). However the fourth largest (i.e. 22nd smallest) part of λ, $\lambda_4 = \lambda_{[22]} = 22$ is such that it has 22 parts less than or equal to it, and since the part in question, 22, does not exceed the number of parts less than or equal to it (i.e. 22), it must be that $R_1(\pi) = 22$.

In other words, knowing the parts of λ (with no foreknowledge of the parts of π), (3.5) allows us to recover the first row of the $\mathcal{S}$-diagram.

Analogously, knowing the parts of λ only, we can recover the ith row of the $\mathcal{S}$-diagram via

$$R_i(\pi) = \max_{1 \leqq j \leqq \ell(\lambda)} \left\{ j - \sum_{h=1}^{i-1} m_{2h}(\lambda) \mid j - \sum_{h=1}^{i-1} m_{2h}(\lambda) \geqq \lambda_{[j]} - 2(i-1) \right\} \quad (3.6)$$

for $i = 1, 2, \ldots, 1 + \frac{R_1(\lambda)}{2}$.

Theorem 3.4 *The map f is a bijection of $\mathcal{G}_1$ onto $\mathcal{A}_1$.*

PROOF. First, we need to show that for any $\pi \in \mathcal{G}_1$, $f(\pi) \in \mathcal{A}_1$. Let $1 \leqq k \leqq D(\pi)$. Since $R_{k+1}(\pi)$ appears twice in π, the smallest part in $\pi^{(k)}$ must be at least two less than $R_{k+1}(\pi)$ (by (2.4)), and thus the largest part in $\pi^{(k+1)}$ must be at least two larger than $R_{k+1}(\pi)$, (again by (2.4)). Thus, the smallest part in $\pi^{(k)}$ must exceed the largest part in $\pi^{(k+1)}$ by at least four. Therefore,

$$\left(\pi^{(k)} + \langle (2k)^{\ell(\pi^{(k)})} \rangle \right) \succ \left(\pi^{(k+1)} + \langle (2k+2)^{\ell(\pi^{(k+1)})} \rangle \right),$$

and thus all parts of $P(f(\pi))$ are distinct. Furthermore,

$$P(f(\pi)) \succ L(f(\pi))$$

since

$$\begin{aligned}
P(f(\pi))_{[1]} &= \pi_{[1]} + 2D(\pi) \\
&= \pi_{[1]} + L(f(\pi))_1 \\
&\geqq 2 + L(f(\pi))_1 \quad \text{(by (2.3))} \\
&> L(f(\pi))_1.
\end{aligned}$$

10

Also, by (3.1), $L(f(\pi))$ contains the numbers $2i$ for $i = 1, 2, \ldots, D(\pi)$ as parts. The number of appearances of $2i$ in $f(\pi)$ is $R_i(\pi) - R_{i+1}(\pi) - \ell(\pi^{(i)})$. Notice that Condition (2.2) forces $R_i(\pi) - R_{i+1}(\pi) - \ell(\pi^{(i)}) \geqq 2$, and so $f(\pi) \in \mathcal{A}_1$.

Next, we demonstrate the invertibility of f. Again, let $\lambda = f(\pi)$. Notice that $L(\lambda)_1 = R_1(\lambda) = 2D(\pi)$ and that $\ell(P(\lambda)) = \sum_{k=0}^{D(\pi)} \ell(\pi^{(k)})$. From the definition (3.1) of $f(\pi)$,

$$m_{2j}(\lambda) = R_j(\pi) - R_{j+1}(\pi) - \ell(\pi^{(j)}) \tag{3.7}$$

for $j = 1, 2, \ldots, D(\pi)$. Thus, once the $R_j(\pi)$ and $\ell(\pi^{(j)})$ are recovered from λ using (3.6), this will be sufficient information to reconstruct $\pi = f^{-1}(\lambda)$ from (3.1).

Once the $R_j(\pi)$ are known, the rest of π can be easily constructed by subtracting appropriate multiples of two from the parts of $P(\lambda)$ with the aid of (3.1) and (3.7).

Example 3.5 *Let us now consider $\lambda = \{40, 37, 36, 22, 21, 19, 17, 16, 14, 10\} \cup \langle 2^3 4^5 6^4 8^3 \rangle \in \mathcal{A}_1$ in order to show that it will map to the π from Example 3.2.*

Since the initial repeated even parts in λ are $2, 4, 6,$ and 8, a total of 4 different parts, π must contain 4 different repeated parts, $R_1(\pi) > R_2(\pi) > R_3(\pi) > R_4(\pi)$. Other important quantities which can be simply "read off" are $\ell(\lambda) = 25$, $m_2(\lambda) = 3$, $m_4(\lambda) = 5$, $m_6(\lambda) = 4$, and $m_8(\lambda) = 3$. Now consider the computation of $R_1(\pi)$ using (3.6). The 25 parts of λ are labeled from smallest to largest.

25 24 23 22 21 20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

40 37 36 22 21 19 17 16 14 10 8 8 8 6 6 6 6 4 4 4 4 4 2 2 2

$\uparrow$

Reading from the left, the first instance of a part not exceeding the number of parts less than or equal to it is $\lambda_{[22]} = 22$, thus, $R_1(\pi) = 22$, and $\pi^{(0)} = \{40, 37, 36\}$. Therefore, the first row of the $\mathcal{S}$-diagram must be 40, 37, 36, followed by twenty-two 2's.

Since λ has 25 parts, three of which are 2's, the partition $\lambda - \langle 2^{25} \rangle$ contains $25 - 3 = 22$ parts. Furthermore, the three largest parts of λ have already been determined in the previous step, so they can be removed from further consideration. Labeling the 19 parts of $\{\lambda_4, \lambda_5, \ldots, \lambda_{22}\} - \langle 2^{19} \rangle$ from smallest

to largest,

$$19 \; 18 \; 17 \; 16 \; 15 \; 14 \; 13 \; 12 \; 11 \; 10 \; 9 \; 8 \; 7 \; 6 \; 5 \; 4 \; 3 \; 2 \; 1$$

$$20 \; 19 \; 17 \; 15 \; 14 \; 12 \;\; 8 \;\; 6 \;\; 6 \;\; 6 \; 4 \; 4 \; 4 \; 4 \; 2 \; 2 \; 2 \; 2 \; 2$$

$$\uparrow$$

which lets us conclude that $R_2(\pi) = 17$ and $\pi^{(1)} = \{22 - 2, 21 - 2\}$. Therefore, we now know that the second row of the $\mathcal{S}$-diagram begins with $\ell(\pi^{(0)}) = 3$ zeros, followed by $22 - 2, 21 - 2$, followed by seventeen 2's, followed by $m_2(\lambda) = 3$ zeros.

To find the next row, number the parts of $\{\lambda_6, \lambda_7, \ldots, \lambda_{17}\} - \langle 4^{12} \rangle$ from smallest to largest:

$$12 \; 11 \; 10 \;\; 9 \; 8 \; 7 \; 6 \; 5 \; 4 \; 3 \; 2 \; 1$$

$$15 \; 13 \; 12 \; 10 \; 6 \; 4 \; 4 \; 4 \; 2 \; 2 \; 2 \; 2$$

$$\uparrow$$

Thus, $R_3(\pi) = 8$ and $\pi^{(2)} = \{19 - 4, 17 - 4, 16 - 4, 14 - 4\}$.

Next,

$$4 \; 3 \; 2 \; 1$$

$$4 \; 2 \; 2 \; 2$$

$$\uparrow$$

and so $R_4(\pi) = 4$ and $\pi^{(3)} = \emptyset$.

Finally, all that remains is the single part $10 - 8 = 2$, so this 2 must be placed in column $9 = 1 + \ell(\pi^{(0)}) + \ell(\pi^{(1)}) + \ell(\pi^{(2)}) + \ell(\pi^{(3)})$ of the last row of the $\mathcal{S}$-diagram.

Thus, the full $\mathcal{S}$-diagram is given by (3.4), and we conclude that

$$\pi = f^{-1}(\lambda) = \{40, 37, 36, 22, 22, 20, 19, 17, 17, 15, 13, 12, 10, 8, 8, 4, 4, 2\}.$$

3.2   Extending the Map to $\mathcal{G}_2 \to \mathcal{A}_2$

Theorem 3.6 *The map f given in Definition 3.1 also provides a bijection from $\mathcal{G}_2$ to $\mathcal{A}_2$.*

PROOF. Notice that $\mathcal{G}_1 \subsetneq \mathcal{G}_2$ and $\mathcal{A}_1 \subsetneq \mathcal{A}_2$. The set $\mathcal{G}_2 \setminus \mathcal{G}_1$ consists of precisely those partitions in $\mathcal{G}_2$ which contain a one. The set $\mathcal{A}_2 \setminus \mathcal{A}_1$ consists of precisely those partitions λ in $\mathcal{A}_2$ for which the smallest part of $P(\lambda)$ is

exactly one more than the largest part in $L(\lambda)$. Observe that any partition $\pi \in \mathcal{G}_2 \setminus \mathcal{G}_1$ has an $\mathcal{S}$-diagram whose $\left(R(\pi) + 1, \sum_{h=0}^{D(\pi)} \ell(\pi^{(h)})\right)$ entry is a 1. This condition is equivalent to the sum of the $\left(\sum_{h=0}^{D(\pi)} \ell(\pi^{(h)})\right)$th column being exactly one more than the next column, i.e. the smallest part of $P(\lambda)$ is exactly one more than the largest part of $L(\lambda)$.

3.3   The Other Rogers-Selberg Identity (1.5)

Obviously, it would be desirable to use the f map, or some generalization of it, to produce a bijection from the set of partitions $\mathcal{G}_3$ enumerated by the $G_{3,3}(n)$ of Gordon's theorem onto a set of partitions enumerated by the left hand side of (1.5), say $\mathcal{A}_3$. In particular, we would like to have $\mathcal{A}_2 \subsetneq \mathcal{A}_3$, just as we have $\mathcal{A}_1 \subsetneq \mathcal{A}_2$.

Difficulties arise immediately, however. Firstly, the unaltered f-map will not produce a bijection from the partitions enumerated by $G_{3,3}(n)$; observe that $f(\{3, 1, 1\}) = f(\{3, 2\}) = \{3, 2\}$. Furthermore, finding a natural partition theoretic interpretation of the left hand side of (1.5) analogous to that of Theorem 1.5 is not as straightforward as one might first suppose. In (1.3) and (1.4), the $2j^2 + 2j = 2 + 2 + 4 + 4 + \cdots + 2j + 2j$ in the exponent of q over the expression $(q^2; q^2)_j$ allows the expression

$$\frac{q^{2j^2+2j}}{(q^2; q^2)_j}$$

to be dealt with neatly as partitions into 2's, 4's, ..., $2j$'s with each part appearing at least twice. In (1.5), we instead have q^{2j^2}, but how shall the $2j^2$ be split up?

We will allow ourselves to be guided by a simple bijection of $\mathcal{G}_3$ onto $\mathcal{G}_1$. Consider the map

$$g : \mathcal{G}_3 \to \mathcal{G}_1$$

where $g(\{\pi_1, \pi_2, \ldots, \pi_{\ell(\pi)}\}) = \{\pi_1 + 1, \pi_2 + 1, \ldots, \pi_{\ell(\pi)} + 1\}$. It is clear that $\ell(\pi) = \ell(g(\pi))$ and that if π is a partition of n, then $g(\pi)$ is a partition of $n + \ell(\pi)$. It is also clear that g is a bijection. By Theorem 3.4, f maps $\mathcal{G}_1$ bijectively onto $\mathcal{A}_1$. Thus, all that is needed is to map $\mathcal{A}_1$ bijectively onto $\mathcal{A}_3$ in as simple a manner as possible, so that a partition $f(g(\pi))$ of $n + \ell(\pi)$ in $\mathcal{A}_1$ maps to a partition of n in $\mathcal{A}_3$. Define $h : \mathcal{A}_1 \to \mathcal{A}_3$ as follows. For $\lambda \in \mathcal{A}_1$, subtract 1 from each part in $P(\lambda)$ and subtract 1 from two 2's, two 4's, two 6's, ..., two $(R_1(\lambda))$'s. Letting $\bar{f} := h \circ f \circ g$, we have a bijection

$$\bar{f} : \mathcal{G}_3 \to \mathcal{A}_3.$$

13

To view $\bar{f}$ as a single operation, rather than a composition of three maps, let us define the following modified $\mathcal{S}$-diagram.

Definition 3.7 *Let $\pi \in \mathcal{G}_3$ and $\lambda = \bar{f}(\pi)$. The $\bar{\mathcal{S}}$-diagram of the partitions π and λ is the $(D(\pi) + 1) \times \left(\ell(\pi^{(0)}) + R_1(\pi) + 1\right)$ matrix (or, equivalently, the $(1 + R(\lambda)/2) \times \ell(\lambda)$ matrix) whose first row consists of the parts of $\pi^{(0)}$ in nonincreasing order followed by $R_1(\pi) - 1$ 2's, then two 1's and whose ith row consists of $\sum_{k=0}^{i-1} \ell(\pi^{(k)})$ 0's, followed by the parts of $\pi^{(i-1)}$ in nonincreasing order, followed by $R_i(\pi) - 1$ 2's, followed by two 1's, and the rest 0's, for $2 \leqq i \leqq D(\pi) + 1$. The parts of λ are then given by the sums of the columns.*

Example 3.8 $\bar{f}(\{16, 14, 12, 12, 7, 5, 5, 3, 2, 1\}) = \{16, 14, 9, 7, 6, 5, 4, 3, 3, 2, 2, 2, 2, 1, 1\}$. *The corresponding $\bar{\mathcal{S}}$-diagram is*

$$
\begin{bmatrix}
16 & 14 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\
0 & 0 & 7 & 2 & 2 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

The partition theorem analogous to Theorems 1.5 and 1.6 is therefore as follows:

Theorem 3.9 *Let $A_3(n)$ denote the number of partitions of n such that if $2j - 1$ is the largest repeated odd part, then all positive odd integers less than $2j$ appear exactly twice, and no part greater than $2j$ may be repeated. Then $A_3(n) = C_{3,3}(n)$ for all n.*

PROOF. Note that

$$
\frac{q^{2j^2}(-q^{2j+1}; q)_\infty}{(q^2; q^2)} = \frac{q^{1+1+3+3+\cdots+(2j-1)+(2j-1)}}{(q^2; q^2)} \times (-q^{2j+1}; q)_\infty.
$$

By the methods of Euler (cf. [3, p. 4 ff.]), the expression

$$
\frac{q^{1+1+3+3+\cdots+(2j-1)+(2j-1)}}{(q^2; q^2)}
$$

is the generating function for partitions into exactly two 1's, two 3's, two 5's, ..., two $(2j - 1)$'s with 2's, 4's, ..., $2j$'s allowed to appear any number of times (or not at all). The product $(-q^{2j+1}; q)_\infty$ is the generating function for partitions into distinct parts with each part at least $2j + 1$. Thus, by summing over all nonnegative j, it follows that the left hand side of (1.5) is the generating function for $A_3(n)$. Again by Euler's method, it is immediate that the right hand side of (1.5) is the generating function for $C_{3,3}(n)$.

14

Remark 3.10 *It should be noted that Andrews gave a different partition-theoretic interpretation of the Rogers-Selberg identities in [4]. This interpretation was studied further by Bressoud [8].*

4   The Modulus 6 Case of Bressoud's Theorem

Bressoud [7] gave an analog of Gordon's partition theorem for even moduli.

Theorem 4.1 (Bressoud) *Let $B_{k,i}(n)$ denote the number of partitions π of n into parts such that*

$$m_1(\pi) \leqq i - 1, \tag{4.1}$$
$$m_j(\pi) + m_{j+1}(\pi) \leqq k - 1 \text{ for } j = 1, 2, 3, \ldots, \text{ and} \tag{4.2}$$
$$\text{if } \pi_j - \pi_{j+k-2} \leqq 1, \text{ then } \sum_{h=0}^{k-2} \pi_{j+h} \equiv (i-1) \pmod{2}. \tag{4.3}$$

Let $D_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i \pmod{2k}$. Then $B_{k,i}(n) = D_{k,i}(n)$ for $1 \leqq i < k$ and all integers n.

The $k = 3$, $i = 1$ case of Bressoud's theorem first appeared in Andrews [1, p. 432, Thm. 1]. This will be the case of most interest to us. It may be stated as follows.

Theorem 4.2 (Andrews) *The number of partitions of n into parts greater than 1, in which no consecutive integers appear, and in which no part appears more than twice equals the number of partitions of n into parts $\not\equiv 0, \pm 1 \pmod{6}$.*

The "mod 6 analog" of the Rogers-Selberg identity (1.3) may be stated as

$$\sum_{j=0}^{\infty} \frac{q^{2j^2+2j}(q; q^2)_j(-q^{2j+2}; q)_\infty}{(q^2; q^2)_j} = \prod_{\substack{j \geq 1 \\ j \not\equiv 0, \pm 1 (\mathrm{mod}\ 6)}} \frac{1}{1 - q^j}. \tag{4.4}$$

Equation (4.4) is due to Slater [20, p. 154, Eq. (27), with q replaced by $-q$].

The $k = 3$, $i = 2$ case of Bressoud's theorem corresponds to Euler's classic theorem that the number of partitions into odd parts equals the number of partitions into distinct parts [3, p. 5, Cor. 1.2], and so the "mod 6 analog" of the Rogers-Selberg identity (1.4) is simply

$$(-q;q)_\infty = \prod_{\substack{j \geq 1 \\ j \not\equiv 0,\pm 2 (\mathrm{mod}\ 6)}} \frac{1}{1-q^j} \left(= \prod_{\substack{j=1 \\ j \equiv 1 (\mathrm{mod}\ 2)}}^\infty \frac{1}{1-q^j} \right). \tag{4.5}$$

Since the f map restricted to partitions with distinct parts is the identity map, this case will not be of interest here.

A third partner also exists, but the infinite product on the right hand side is not as neat as that of (4.4) or (4.5) because $-3 \equiv 3 \pmod 6$:

$$\sum_{j=0}^\infty \frac{q^{2j^2}(q;q^2)_j(-q^{2j+1};q)_\infty}{(q^2;q^2)_j} = \prod_{j \geq 1} \frac{(1-q^{3j})(1-q^{6j-3})}{1-q^j}. \tag{4.6}$$

An identity equivalent to (4.6) was found independently by James McLaughlin in a computer search [14].

Notice that for a given k and i, the conditions restricting the appearance of parts in partitions enumerated by $G_{k,i}(n)$ in Gordon's theorem are the same as those restricting the appearance of parts in partitions enumerated by $B_{k,i}(n)$ in Bressoud's theorem, except that Bressoud's theorem contains condition (4.3), while Gordon's theorem does not. Thus if $\mathcal{B}_1$ denotes the set of partitions enumerated by $B_{3,1}(n)$ in Bressoud's theorem, it is immediate that

$$\mathcal{B}_1 \subsetneq \mathcal{G}_1.$$

Therefore f maps $B_{3,1}(n)$ onto some proper subset of $\mathcal{A}_1$. It fairly straightforward to see that the subset of $\mathcal{A}_1$ in question is those partitions in which the distinct parts greater than the largest repeated even part are all nonconsecutive integers. Formally, the f map supplies the following theorem.

Theorem 4.3 *Let $B_{3,1}(n)$ be as in Bressoud's theorem. Let $T(n)$ denote the number of partitions of n into parts such that if $2j$ is the largest repeated even part, then all positive even integers less than $2j$ also appear at least twice, no odd part less than $2j+2$ appears, no part greater than $2j$ is repeated, and no two consecutive integers appear. Then $B_{3,1}(n) = T(n)$ for all n.*

While it must be admitted that the preceding theorem is not the most elegant identity in the theory of partitions, it would nonetheless be desirable, in the present context, to be able to identify the partitions enumerated by $T(n)$ with the left hand side of (4.4). Unfortunately, the presence of the factor $(q;q^2)_j$ in the numerator of the left hand side of (4.4) (which is absent from the Rogers-Selberg identities) creates an inclusion-exclusion situation; i.e. unlike Eqs. (1.3), (1.4), and (1.5), Eq. (4.4) is *not* positive term-by-term. Accordingly, any hope of getting an immediate result like that of Theorem 1.5 is dashed.

The author tried to find a direct combinatorial proof of Theorem 4.3, but was unable to do so.

Undeterred, we proceed by defining the following two-variable analogs of Eqs. (4.4)–(4.6).

$$F_1(a) := F_1(a; q) := \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n(n+1)} (q; q^2)_n (-aq^{2n+2}; q)_\infty}{(q^2; q^2)_n} \tag{4.7}$$

$$F_2(a) := F_2(a; q) := (-aq; q)_\infty \tag{4.8}$$

$$F_3(a) := F_3(a; q) := \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n^2} (q; q^2)_n (-aq^{2n+1}; q)_\infty}{(q^2; q^2)_n}. \tag{4.9}$$

Next, define

$$E_1(a) := E_1(a; q) := \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n(n+1)}}{(q^2; q^2)_n} \sum_{m=0}^{\infty} \frac{a^m q^{m(m+2n+1)}}{(q; q)_m} \tag{4.10}$$

$$E_2(a) := E_2(a; q) := (-aq; q)_\infty \tag{4.11}$$

$$E_3(a) := E_3(a; q) := \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n^2}}{(q^2; q^2)_n} \sum_{m=0}^{\infty} \frac{a^m q^{m(m+2n)}}{(q; q)_m}. \tag{4.12}$$

Lemma 4.4 *The functions $F_1(a)$, $F_2(a)$, $F_3(a)$ satisfy the following system of q-difference equations:*

$$F_1(a) = F_3(aq) \tag{4.13}$$
$$F_2(a) = (1 + aq) F_2(aq) \tag{4.14}$$
$$F_3(a) = F_1(a) + aq(1 + aq) F_1(aq) \tag{4.15}$$

17

PROOF. Equations (4.13) and (4.14) are immediate from the definitions (4.7)–(4.9).

$$
\begin{aligned}
F_3(a) - F_1(a) &= \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n^2} (q;q^2)_n (-aq^{2n+2};q)_\infty}{(q^2;q^2)_n} (1 + aq^{2n+1} - q^{2n}) \\
&= \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n^2} (q;q^2)_n (-aq^{2n+2};q)_\infty}{(q^2;q^2)_n} \\
&\qquad \times \left((1 - q^{2n})(1 + aq^{2n+1}) + aq^{4n+1} \right) \\
&= \sum_{n=1}^{\infty} \frac{a^{2n} q^{2n^2} (q;q^2)_n (-aq^{2n+1};q)_\infty}{(q^2;q^2)_{n-1}} \\
&\qquad + \sum_{n=0}^{\infty} \frac{a^{2n+1} q^{2n^2+4n+1} (q;q^2)_n (-aq^{2n+2};q)_\infty}{(q^2;q^2)_n} \\
&= \sum_{n=0}^{\infty} \frac{a^{2n+2} q^{2n^2+4n+2} (q;q^2)_{n+1} (-aq^{2n+3};q)_\infty}{(q^2;q^2)_n} \\
&\qquad + \sum_{n=0}^{\infty} \frac{a^{2n+1} q^{2n^2+4n+1} (q;q^2)_n (-aq^{2n+2};q)_\infty}{(q^2;q^2)_n} \\
&= aq \sum_{n=0}^{\infty} \frac{a^{2n+1} q^{2n^2+4n+1} (q;q^2)_n (1 - q^{2n+1})(-aq^{2n+3};q)_\infty}{(q^2;q^2)_n} \\
&\qquad + \sum_{n=0}^{\infty} \frac{a^{2n+1} q^{2n^2+4n+1} (q;q^2)_n (-aq^{2n+2};q)_\infty}{(q^2;q^2)_n} \\
&= \sum_{n=0}^{\infty} \frac{a^{2n+1} q^{2n^2+4n+1} (q;q^2)_n (-aq^{2n+3};q)_\infty}{(q^2;q^2)_n} \\
&\qquad \times \left(aq(1 - q^{2n+1}) + (1 + aq^{2n+2}) \right) \\
&= (1 + aq) \sum_{n=0}^{\infty} \frac{a^{2n+1} q^{2n^2+4n+1} (q;q^2)_n (-aq^{2n+3};q)_\infty}{(q^2;q^2)_n} \\
&= aq(1 + aq) F_1(aq),
\end{aligned}
$$

and thus (4.15) is established.

Lemma 4.5 *The functions $E_1(a)$, $E_2(a)$, $E_3(a)$ satisfy the following system of q-difference equations:*

$$
\begin{aligned}
E_1(a) &= E_3(aq) & (4.16) \\
E_2(a) &= (1 + aq) E_2(aq) & (4.17) \\
E_3(a) &= E_1(a) + aq(1 + aq) E_1(aq) & (4.18)
\end{aligned}
$$

18

PROOF. Equations (4.16) and (4.17) are immediate from the definitions (4.10)–(4.12).

$$
\begin{aligned}
E_3(a) - E_1(a) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a^{2n+m}q^{2n^2+m^2+2mn}}{(q^2;q^2)_n(q;q)_m}(1-q^{2n+m}) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a^{2n+m}q^{2n^2+m^2+2mn}}{(q^2;q^2)_n(q;q)_m}\left((1-q^{2n}) + q^{2n}(1-q^m)\right) \\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{a^{2n+m}q^{2n^2+m^2+2mn}}{(q^2;q^2)_{n-1}(q;q)_m} + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{a^{2n+m}q^{2n^2+m^2+2mn+2n}}{(q^2;q^2)_n(q;q)_{m-1}} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a^{2n+m+2}q^{2n^2+2n+m^2+2mn+2m+2}}{(q^2;q^2)_n(q;q)_m} \\
&\qquad + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a^{2n+m+1}q^{2n^2+m^2+2m+2mn+4n+1}}{(q^2;q^2)_n(q;q)_m} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a^{2n+m+1}q^{2n^2+m^2+2m+2mn+4n+1}}{(q^2;q^2)_n(q;q)_m}(1+aq) \\
&= aq(1+aq)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a^{2n+m}q^{2n^2+m^2+2m+2mn+4n}}{(q^2;q^2)_n(q;q)_m} \\
&= aq(1+aq)E_1(aq),
\end{aligned}
$$

and thus (4.18) is established.

Theorem 4.6

$$
\sum_{n=0}^{\infty} T(n)q^n = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(q;q^2)_n(-q^{2n+2};q)_\infty}{(q^2;q^2)_n} \tag{4.19}
$$

PROOF. By a standard argument (see, e.g. [2, p. 442 ff., Lemma 1 and the remark following Eq. (5.9)]), the system (4.13)–(4.15) has unique solutions with $F_1(0) = F_2(0) = F_3(0)$. Since by Lemma 4.5, (4.10)–(4.12) satisfy the same system of q-difference equations and $E_1(0) = E_2(0) = E_3(0)$, it follows that $E_i(a) = F_i(a)$, for $i = 1,2,3$. In particular, $E_1(1) = F_1(1)$. Further, observing that the inner sum in $E_1(1)$, by (1.1), generates partitions into parts which are distinct, nonconsecutive integers greater than $2n+1$,

$$
\sum_{n=0}^{\infty} T(n)q^n = F_1(1),
$$

and thus the result follows.

Remark 4.7 *The motivation behind using the set of q-difference equations (4.13)–(4.15) comes from the fact that it is known (see [2], [7], [19]) that the functions arising in $k = 3$ case of Bressoud's theorem must satisfy (4.13)–(4.15).*

Namely, it can be shown that if $Q_i(a) := (-aq; q)_\infty J_{1,i/2}(; a^2; q^2)$, where $J_{k,i}(a; x; q)$ is defined in [2] and [3, p. 106, Eq. (7.2.2)], the $Q_i(a)$ for $i = 1, 2, 3$ also satisfy (4.13)–(4.15).

5   Discussion

The f map was created specifically to map the set $\mathcal{G}_2$ onto the set $\mathcal{A}_2$. As was shown, f was also useful in a variety of other closely related contexts. It might be interesting to explore whether this map, or a generalization of it, might be applicable in additional settings.

This having been said, it is not clear what these additional settings might be. Recall that our work began with the observation that

$$\sum_{n=0}^{\infty} G_{3,2}(n)q^n = \prod_{\substack{j \geq 1 \\ j \not\equiv 0, \pm 2 (\text{mod } 7)}} \frac{1}{1 - q^j} = \sum_{j=0}^{\infty} \frac{q^{2j^2 + 2j}(-q^{2j+1}; q)_\infty}{(q^2; q^2)_j}, \qquad (5.1)$$

where the first equality follows from Gordon's theorem, and the second equality is the second Rogers-Selberg identity. Our f map provides a bijection between the partitions enumerated on the left hand side with those enumerated in Andrews' combinatorial interpretation of the right. The middle member of (5.1) serves only as a bridge between the left and right and does not play a role in the bijection.

A natural modulus 5 analog of (5.1) is

$$\sum_{n=0}^{\infty} G_{2,1}(n)q^n = \prod_{\substack{j \geq 1 \\ j \not\equiv 0, \pm 1 (\text{mod } 5)}} \frac{1}{1 - q^j} = \sum_{j=0}^{\infty} \frac{(-1)^j q^{3j^2 - 2j}(-q^{2j+1}; q)_\infty}{(q^2; q^2)_j}, \qquad (5.2)$$

where the first equality follows from Gordon's theorem and the second equality is an identity due to Rogers [15, p. 339, Ex. 2] which also appears in Slater's list [20, p. 154, Eq. (19)]. We are confronted with several difficulties immediately. First, the partitions enumerated by $G_{2,1}(n)$ do not have any repeated parts and therefore our f map becomes the identity map in this context. Next, even though the right hand side of (5.2) resembles that of (5.1), the presence of the factor $(-1)^j$ in (5.2) precludes the possibility of a combinatorial interpretation as simple as that of the right hand side of (5.1).

Perhaps the following modulus 9 analog of (5.3) would be more amenable to

study via the methods of this paper:

$$\sum_{n=0}^{\infty} G_{4,1}(n)q^n = \prod_{\substack{j \geq 1 \\ j \not\equiv 0, \pm 1 \,(\mathrm{mod}\ 9)}} \frac{1}{1 - q^j}$$

$$= \sum_{j=0}^{\infty} \frac{q^{3j^2+3j}(-q^{3j+3}; q^3)_\infty}{(q^3; q^3)_j(1 - q^{3j+2})} \prod_{h=j+1}^{\infty} \left(1 + \frac{q^{3h+1}}{1 - q^{3h+1}} + \frac{q^{3h+2}}{1 - q^{3h+2}}\right), \quad (5.3)$$

where the first equality follows from Gordon's theorem and the second equality is an identity due to Bailey [6, p. 422, Eq. (1.7)] which also appears in Slater [20, p. 156, Eq. (40)]. Andrews provides a (somewhat complicated) combinatorial interpretation of the last member of (5.3) in [5, §5]. It should be noted, however, that in the $k = 4$ case of Gordon's theorem, parts may be repeated up to three times (with certain additional restrictions) and thus it is not obvious how to adapt or generalize the f map to this situation.

Acknowledgments

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