Mutations of Puzzles and
Equivariant cohomology of two-step flag varieties

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arXiv:1401.3065
Two-step flag varieties

Fix $0 \leq a \leq b \leq n$.

$X = \text{Fl}(a, b; n) = \{(A, B) \mid A \subset B \subset \mathbb{C}^n; \dim(A) = a; \dim(B) = b\}$
Two-step flag varieties

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Def: A 012-string for $X$ is a permutation of $0^a 1^{b-a} 2^{n-b}$.

$\mathbb{C}^n$ has basis $\{e_1, e_2, \ldots, e_n\}$. $u = (u_1, u_2, \ldots, u_n)$ 012-string.

Def. $(A_u, B_u) \in X$ by $A_u = \text{Span}\{e_i : u_i = 0\}$ and $B_u = \text{Span}\{e_i : u_i \leq 1\}$.

Example: $X = \text{Fl}(1, 3; 5)$. $u = 10212$. $(A_u, B_u) = (\mathbb{C}e_2, \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_4)$. 
Two-step flag varieties

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Def: A **012-string** for $X$ is a permutation of $0^a 1^{b-a} 2^{n-b}$.

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Def. $(A_u, B_u) \in X$ by $A_u = \text{Span}\{e_i : u_i = 0\}$ and $B_u = \text{Span}\{e_i : u_i \leq 1\}$.

Example: $X = \text{Fl}(1, 3; 5)$. $u = 10212$. $(A_u, B_u) = (\mathbb{C}e_2, \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_4)$.

**Schubert variety:** $X_u = \overline{B.(A_u, B_u)}$ ; $B \subset \text{GL}(\mathbb{C}^n)$ lower triangular.

$\text{codim}(X_u, X) = \ell(u) = \#\{i < j \mid u_i > u_j\}$
Equivariant cohomology

$T \subset \text{GL}(\mathbb{C}^n)$ maximal torus of diagonal matrices.

$H^*_T(\text{point}) = \mathbb{Z}[y_1, \ldots, y_n]$, where $y_i = -c_1(\mathcal{C}e_i)$.

$H^*_T(X) = \bigoplus_u \mathbb{Z}[y_1, \ldots, y_n] \cdot [X_u]$ is an algebra over $H^*_T(\text{point})$. 
Equivariant cohomology

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$H_T^*(X) = \bigoplus_u \mathbb{Z}[y_1, \ldots, y_n] \cdot [X_u]$ is an algebra over $H_T^*(\text{point})$.

The **equivariant Schubert structure constants** of $X$ are the polynomials $C_{u,v}^w \in \mathbb{Z}[y_1, \ldots, y_n]$ defined by

$$[X_u] \cdot [X_v] = \sum_w C_{u,v}^w [X_w]$$
Equivariant cohomology

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The equivariant Schubert structure constants of $X$ are the polynomials $C^w_{u,v} \in \mathbb{Z}[y_1, \ldots, y_n]$ defined by

$$[X_u] \cdot [X_v] = \sum_w C^w_{u,v} [X_w]$$

$H^*_T(X)$ graded ring $\Rightarrow C^w_{u,v}$ homogeneous of degree $\ell(u) + \ell(v) - \ell(w)$.

$\ell(w) = \ell(u) + \ell(v) \Rightarrow C^w_{u,v} = \#(g_1.X_u \cap g_2.X_v \cap g_3.X_w^\vee); g_i \in \text{GL}(\mathbb{C}^n)$.

Theorem (Graham) $C^w_{u,v} \in \mathbb{Z}_{\geq 0}[y_2 - y_1, \ldots, y_n - y_{n-1}]$
Puzzle pieces

Simple labels: 0, 1, 2

Composed labels: 3 = 10, 4 = 21, 5 = 20, 6 = 2(10), 7 = (21)0
Equivariant puzzles
Equivariant puzzles

Note: The composed labels are uniquely determined by the simple labels.
Equivariant puzzles

\[ P = \begin{array}{cccccc}
2 & 0 & 2 & 0 & 2 & 0 \\
5 & 0 & 0 & 2 & 2 & 0 \\
0 & 2 & 0 & 0 & 2 & 2 \\
2 & 0 & 2 & 2 & 2 & 1 \\
0 & 4 & 0 & 7 & 0 & 2 \\
0 & 0 & 2 & 2 & 2 & 5 \\
1 & 1 & 3 & 4 & 1 & 1 \\
1 & 1 & 2 & 1 & 2 & 2 \\
1 & 1 & 1 & 4 & 1 & 1 \\
1 & 1 & 1 & 1 & 4 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 0 & 2 & 0 \\
1 & 1 & 2 & 1 & 1 & 1 \\
1 & 1 & 2 & 1 & 1 & 0 \\
1 & 1 & 2 & 1 & 1 & 0 \\
1 & 1 & 2 & 1 & 1 & 0 \\
1 & 1 & 2 & 1 & 1 & 0 \\
\end{array} = \begin{array}{cccccc}
2 & 0 & 2 & 0 & 2 & 0 \\
0 & 2 & 0 & 0 & 2 & 2 \\
2 & 0 & 2 & 2 & 2 & 1 \\
0 & 0 & 2 & 2 & 2 & 5 \\
1 & 1 & 3 & 4 & 1 & 1 \\
1 & 1 & 2 & 1 & 2 & 2 \\
0 & 0 & 2 & 2 & 2 & 0 \\
0 & 2 & 0 & 0 & 2 & 2 \\
1 & 1 & 3 & 4 & 1 & 1 \\
1 & 1 & 2 & 1 & 2 & 2 \\
1 & 1 & 1 & 4 & 1 & 1 \\
1 & 1 & 1 & 1 & 4 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 2 & 0 & 2 & 0 \\
1 & 1 & 2 & 1 & 1 & 1 \\
1 & 1 & 2 & 1 & 1 & 0 \\
1 & 1 & 2 & 1 & 1 & 0 \\
1 & 1 & 2 & 1 & 1 & 0 \\
\end{array} \]

**Note:** The composed labels are uniquely determined by the simple labels.

**Boundary:** \( \partial P = \nabla_{u,v}^{u,v} \) where \( u = 110202 \), \( v = 021210 \), \( w = 120210 \).
Equivariant puzzle formula

**Theorem**

\[
C_{u,v}^w = \sum_{\partial P = \triangle_{w,v}} \prod_{\diamond \in P} \text{weight}(\diamond)
\]

\[
\text{weight}(\diamond) = y_j - y_i
\]
Equivariant puzzle formula

**Theorem**

\[
C_{w, v}^{u} = \sum_{\partial P = \Delta_{w}^{u, v}} \prod_{\star \in P} \text{weight}(\star)
\]

**Known cases:**

- Puzzle rule for \(H^*(Gr(m, n))\) (Knutson, Tao, Woodward)
- Puzzle rule for \(H^*_T(Gr(m, n))\) (Knutson, Tao)
- Puzzle rule for \(H^*(Fl(a, b; n))\) (conjectured by Knutson, proof in [B-Kresch-Purbhoo-Tamvakis], different positive formula by Coskun.)
Example: Let $X = \text{Fl}(2, 4; 5)$. In $H^*_T(X)$ we have:

$[X_{01201}] \cdot [X_{10102}] = \, ?$
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$[X_{01201}] \cdot [X_{10102}] = $

$[X_{12010}] + [X_{11200}] + (y_4 - y_1)[X_{12001}]$

$+ (y_5 + y_4 - y_3 - y_1)[X_{10210}] + (y_4 - y_3)(y_4 - y_1)[X_{10201}]$
Quantum cohomology of Grassmannians

\[ X = \text{Gr}(m, n) = \{ V \subset \mathbb{C}^n \mid \dim(V) = m \} = \text{Fl}(m, m; n) \]

Schubert varieties $\leftrightarrow$ 02-strings $\leftrightarrow$ Young diagrams

$X_{0222020220} \leftrightarrow 0222020220 \leftrightarrow \begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}$
Quantum cohomology of Grassmannians

\[ X = \text{Gr}(m, n) = \{ V \subset \mathbb{C}^n \mid \dim(V) = m \} = \text{Fl}(m, m; n) \]

Schubert varieties \( \iff \) 02-strings \( \iff \) Young diagrams

\[ X_{0222020220} \iff 0222020220 \iff \begin{array}{ccccccc}
0 & 2 & 2 & 2 & 0 & 2 & 0 \\
2 & 2 & 2 & 0 & 2 & 0 & 0 \\
2 & 2 & 2 & 0 & 2 & 0 & 0 \\
\end{array} \]
Quantum cohomology of Grassmannians

\[ X = Gr(m, n) = \{ V \subset \mathbb{C}^n \mid \dim(V) = m \} = Fl(m, m; n) \]

Schubert varieties \( \leftrightarrow \) 02-strings \( \leftrightarrow \) Young diagrams

\[ X_{0220202020} \leftrightarrow 0220202020 \leftrightarrow \]

(Small) equivariant quantum ring:

\[ QH^*_T(X) = H^*_T(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q] = \bigoplus \mathbb{Z}[y_1, \ldots, y_n, q] \cdot [X_\lambda] \]

Ring structure is defined by equivariant Gromov-Witten invariants

\[ N_{\lambda, \mu}^{\nu, d} \in \mathbb{Z}[y_1, \ldots, y_n] : \]

\[ [X_\lambda] \ast [X_\mu] = \sum_{\nu, d \geq 0} N_{\lambda, \mu}^{\nu, d} q^d [X_\nu] \]
Gromov-Witten invariants of $X = \text{Gr}(m, n)$

\[[X_\lambda] \ast [X_\mu] = \sum_{\nu, d \geq 0} N_{\lambda, \mu}^{\nu, d} q^d [X_\nu]\]

$N_{\lambda, \mu}^{\nu, 0} = C_{\lambda, \mu}^{\nu}$ \hspace{1cm} \text{(} QH_T(X) \text{ is a deformation of } H^*_T(X). \text{)}$

N_{\lambda, \mu}^{\nu, d} \in \mathbb{Z}[y_1, \ldots, y_n]$ is homogeneous of degree $|\lambda| + |\mu| - |\nu| - nd$.

$|\lambda| + |\mu| = |\nu| + nd \Rightarrow$

$N_{\lambda, \mu}^{\nu, d} = \# \text{ rational curves } C \subset X \text{ of degree } d \text{ meeting } g_1.X_\lambda, g_2.X_\mu, g_3.X_\nu.$

Thm (Mihalcea) $N_{\lambda, \mu}^{\nu, d} \in \mathbb{Z}_{\geq 0}[y_2 - y_1, \ldots, y_n - y_{n-1}]$
Quantum equals classical theorem

Def: (B) Given curve $C \subset X = \text{Gr}(m, n)$ set

$\text{Ker}(C) = \bigcap_{V \in C} V \subset \mathbb{C}^n$ and $\text{Span}(C) = \sum_{V \in C} V \subset \mathbb{C}^n$
Quantum equals classical theorem

Def: (B) Given curve $C \subset X = \text{Gr}(m, n)$ set

$$\text{Ker}(C) = \bigcap_{V \in C} V \subset \mathbb{C}^n \quad \text{and} \quad \text{Span}(C) = \sum_{V \in C} V \subset \mathbb{C}^n$$

Fix degree $d$. Set $Y = \text{Fl}(m - d, m + d; n)$.

Given a 02-string $\lambda$ for $X$, let $\lambda(d)$ be the 012-string for $Y$ obtained from $\lambda$ by replacing the first $d$ occurrences of 2 and the last $d$ occurrences of 0 with 1.

$\lambda = 022020220$ and $d = 2$ gives $\lambda(d) = 0112021221$. 

\[
\begin{array}{ccccccccc}
0 & 1 & 1 & 2 & 0 & 2 & 1 & 2 & 2 \\
1 & 1 & 20 & 1 & 1 & 1 & 2 & 1 & 1
\end{array}
\]
Quantum equals classical theorem

Def: (B) Given curve \( C \subset X = \text{Gr}(m, n) \) set
\[
\text{Ker}(C) = \bigcap_{V \in C} V \subset \mathbb{C}^n \quad \text{and} \quad \text{Span}(C) = \sum_{V \in C} V \subset \mathbb{C}^n
\]

Fix degree \( d \). Set \( Y = \text{Fl}(m - d, m + d; n) \).

Given a 02-string \( \lambda \) for \( X \), let \( \lambda(d) \) be the 012-string for \( Y \) obtained from \( \lambda \) by replacing the first \( d \) occurrences of 2 and the last \( d \) occurrences of 0 with 1.

\[
\lambda = 022020220 \quad \text{and} \quad d = 2 \quad \text{gives} \quad \lambda(d) = 0112021221.
\]

\[
Y_{\lambda(d)} = \{(A, B) \in Y \mid \exists V \in X_\lambda : A \subset V \subset B\}
\]
\[
= \text{Set of Kernel-Span pairs of general curves of degree } d \text{ meeting } X_\lambda.
\]
Quantum equals classical theorem

Theorem (B–Kresch–Tamvakis) For $|\lambda| + |\mu| = |\nu| + nd$ we have bijection

$$\begin{align*}
\{ \text{rational curves in } X \text{ of degree } d \text{ meeting } & g_1.X_{\lambda}, \ g_2.X_{\mu}, \ g_3.X_{\nu} \} \\
\leftrightarrow & \hspace{1cm} g_1.Y_{\lambda(d)} \cap g_2.Y_{\mu(d)} \cap g_3.Y_{\nu(d)} \\
C \mapsto & \hspace{1cm} (\text{Ker}(C), \ \text{Span}(C))
\end{align*}$$
Quantum equals classical theorem

**Theorem (B–Kresch–Tamvakis)** For $|\lambda| + |\mu| = |\nu| + nd$ we have bijection

\[
\begin{cases}
\text{rational curves in } X \\
\text{of degree } d \text{ meeting} \\
g_1.X_\lambda, \ g_2.X_\mu, \ g_3.X_\nu
\end{cases}
\quad \longleftrightarrow \quad g_1.Y_{\lambda(d)} \cap g_2.Y_{\mu(d)} \cap g_3.Y_{\nu(d)}
\]

\[C \quad \longleftrightarrow \quad (\ker(C), \ \text{Span}(C))\]

**Theorem (B-Mihalcea)**

\[N^{\nu, d}_{\lambda, \mu} = C^{\nu(d), v}_{\lambda(d), \mu(d)} \quad \in \quad \mathbb{Z}[y_1, \ldots, y_n]\]

**Corollary:**

\[N^{\nu, d}_{\lambda, \mu} = \sum_{\partial P = \triangle_{\nu(d)}^{\lambda(d), \mu(d)}} \prod_{\text{weight(\diamond)}} \in P\]
The mutation algorithm

Puzzle:
- Shape is a hexagon.
- All pieces may be rotated.
- Boundary labels are simple.
The mutation algorithm

Flawed puzzle containing the gash pair:
The mutation algorithm

Remove problematic piece.
The mutation algorithm

Replace with:

```
0 0
1 0
```
The mutation algorithm

Replace with \[ \begin{array}{ccc}
1 & \frac{2}{3} & 0 \\
\frac{2}{3} & 1 & 0 \\
0 & 0 & 1
\end{array} \] OR \[ \begin{array}{ccc}
1 & \frac{2}{3} & 0 \\
\frac{2}{3} & 1 & 0 \\
0 & 0 & 1
\end{array} \]
The mutation algorithm

The piece fits. Always at most one choice !!!
The mutation algorithm

But no puzzle piece fits this time.
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm

Flawed puzzle containing the illegal puzzle piece: 4 4 4
The mutation algorithm
The mutation algorithm

Use directed gashes.
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm

Resolution: 

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm

[Diagram of a hexagonal grid with numbers and colors indicating the mutation process]
The mutation algorithm
The mutation algorithm
The mutation algorithm

Flawed puzzle containing a gash pair.
The mutation algorithm
The mutation algorithm

Resolution:
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm

Flawed puzzle containing the illegal puzzle piece:
The mutation algorithm

Resolution:  

\[
\begin{array}{c}
1 \\
6 \\
7 \\
\end{array}
\quad \mapsto \quad
\begin{array}{c}
1 \\
6 \\
1 \\
1 \\
7 \\
\end{array}
\]
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm

Flawed puzzle containing the marked scab:
The mutation algorithm

Resolution:
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm

Resolution: 

\[
\begin{array}{c}
1 \\
6 \\
7
\end{array}
\]

\[
\begin{array}{c}
1 \\
6 \\
4 \\
7
\end{array}
\]
The mutation algorithm
The mutation algorithm
The mutation algorithm
The mutation algorithm

Flawed puzzle containing a marked scab.
Component of the mutation graph:
Resolutions of illegal puzzle pieces:
Resolutions of marked scabs:
Example:
Proof that mutation algorithm works:
Proof that mutation algorithm works:

Consider connected component of the edges:
Proof that mutation algorithm works:

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Proof that mutation algorithm works:

Consider connected component of the edges:
Proof that mutation algorithm works:

Technical result: The two gashes will propagate to the same location.

In particular, the above situation is impossible!!
Aura of semi-labeled edges

An aura is a linear form in $\mathbb{C}[\delta_0, \delta_1, \delta_2]$. $\uparrow \in \mathbb{C}$ is a unit vector.
Aura of semi-labeled edges

An aura is a linear form in \( R = \mathbb{C}[\delta_0, \delta_1, \delta_2] \). \( \uparrow \in \mathbb{C} \) is a unit vector.

Def: \( \mathcal{A}(\text{0}) = \delta_0 \), \( \mathcal{A}(\text{1}) = \delta_1 \), \( \mathcal{A}(\text{2}) = \delta_2 \)

If is a puzzle piece, then \( \mathcal{A}(/x) + \mathcal{A}(y\backslash) + \mathcal{A}(z) = 0 \).
Aura of semi-labeled edges

An aura is a linear form in $R = \mathbb{C}[\delta_0, \delta_1, \delta_2]$. $\mathbb{C}$ is a unit vector.

Def: $A(0) = \delta_0$, $A(1) = \delta_1$, $A(2) = \delta_2$

If $\triangle x y z$ is a puzzle piece, then $A(\triangle x) + A(\triangle y) + A(\triangle z) = 0$.

$A(3) = \delta_1 \to \delta_0$, $A(4) = \delta_2 \to \delta_1$, $A(5) = \delta_2 \to \delta_0$

$A(6) = \delta_2 \to \delta_0$, $A(7) = \delta_2 \to \delta_0$
Aura of gashes

Definition: \( A\left(\frac{x}{y}\right) = A\left(\frac{x}{1}\right) + A\left(\frac{1}{y}\right) \)

Example: \( A\left(\frac{0}{4}\right) = A\left(\frac{0}{1}\right) + A\left(\frac{1}{4}\right) = \delta_0 + \delta_1 + \delta_2 \)
Aura of gashes

Definition: \( A\left(\frac{x}{y}\right) = A\left(\frac{x}{x}\right) + A\left(\frac{y}{y}\right) \)

Example: \( A\left(\frac{0}{4}\right) = A\left(\frac{0}{0}\right) + A\left(\frac{4}{4}\right) = \delta_0 \delta_1 \delta_2 \)

Properties:

- The aura of a gash is invariant under propagations.
Aura of gashes

Definition: \( A\left(\frac{x}{y}\right) = A\left(\frac{x}{1}\right) + A\left(\frac{1}{y}\right) \)

Example: \( A\left(\frac{0}{4}\right) = A\left(\frac{0}{1}\right) + A\left(\frac{1}{4}\right) = \delta_0 \delta_1 \delta_2 \)

Properties:

- The aura of a gash is invariant under propagations.
- Sum of auras of gashes of any resolution is zero.

\[ A\left(\frac{4}{2}\right) + A\left(\frac{1}{2}\right) = 0 \]
Aura of gashes

Definition: \[ \mathcal{A}(\frac{x}{y}) = \mathcal{A}(\frac{x}{y}) + \mathcal{A}(\frac{y}{x}) \]

Example: \[ \mathcal{A}(\frac{0}{4}) = \mathcal{A}(\frac{0}{4}) + \mathcal{A}(\frac{4}{0}) = \]

Properties:

- The aura of a gash is invariant under propagations.
- Sum of auras of gashes of any resolution is zero.
  \[ \mathcal{A}(\frac{4}{2}) + \mathcal{A}(\frac{1}{2}) = 0 \]
- Sum of auras of right gashes of resolutions of illegal puzzle piece is zero.
  \[ \mathcal{A}(\frac{4}{0}) + \mathcal{A}(\frac{3}{5}) + \mathcal{A}(\frac{1}{6}) = 0 \]
Aura of puzzles

Let \( \tilde{P} \) be a resolution of a flawed puzzle \( P \).

Def: \( \mathcal{A}(\tilde{P}) = \mathcal{A}(\text{right gash in } \tilde{P}) \)

\[
\begin{align*}
\mathcal{A}(\begin{array}{ccc}0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{array}) &= \mathcal{A}(\begin{array}{c}0 \ 1 \end{array}) \\
\mathcal{A}(\begin{array}{ccc}2 & 0 & 0 \\
5 & 2 & 0 \\
0 & 0 & 2 \\
\end{array}) &= \mathcal{A}(\begin{array}{c}5 \\
0 \\
\end{array}) \\
\mathcal{A}(\begin{array}{ccc}0 & 2 & 0 \\
2 & 0 & 2 \\
0 & 0 & 0 \\
\end{array}) &= \mathcal{A}(\begin{array}{c}0 \ 0 \\
2 \end{array})
\end{align*}
\]
Aura of puzzles

Let $\tilde{P}$ be a resolution of a flawed puzzle $P$.

**Def:** $\mathcal{A}(\tilde{P}) = \mathcal{A}(\text{right gash in } \tilde{P})$

$\mathcal{A}(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{array}) = \mathcal{A}(\begin{array}{c} / \end{array})$

$\mathcal{A}(\begin{array}{ccc} 2 & 0 & 5 \\ 0 & 2 & 5 \\ 2 & 0 & 0 \end{array}) = \mathcal{A}(\begin{array}{c} \frac{5}{0} \end{array})$

$\mathcal{A}(\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{array}) = \mathcal{A}(\begin{array}{c} \frac{0}{2} \end{array})$

If $\tilde{P}$ is the only resolution of $P$, then set $\mathcal{A}(P) = \mathcal{A}(\tilde{P})$.

**Key identity:** Let $S$ be any finite set of flawed puzzles that is closed under mutations. Then

$$\sum_{P \in S_{scab}} \mathcal{A}(P) + \sum_{P \in S_{gash}} \mathcal{A}(P) = 0$$
From now on:

- All puzzles are triangles.
- All equivariant puzzle pieces and scabs are vertical.

**Def:** For any 012-string $u = (u_1, u_2, \ldots, u_n)$ we set

$$C_u := \sum_{i=1}^{n} \delta_{u_i} y_i \in R[y_1, \ldots, y_n]$$

**Exercise:**

$$\partial P = \bigtriangleup_{w}^{u,v} \quad \Rightarrow \quad \sum_{s \text{ scab in } P} - \text{weight}(s) A(s) = C_u \cdot \downarrow + C_v \cdot \leftarrow + C_w \cdot \uparrow$$
From now on:

- All puzzles are triangles.
- All equivariant puzzle pieces and scabs are vertical.

**Def:** For any 012-string $u = (u_1, u_2, \ldots, u_n)$ we set

$$C_u := \sum_{i=1}^{n} \delta_{u_i} y_i \in R[y_1, \ldots, y_n]$$

**Exercise:**

$$\partial P = \triangle_{\ W_i}^{u_i, v} \Rightarrow$$

$$\sum_{s \text{ scab in } P} - \text{weight}(s) A(s) = C_u \cdot \downarrow + C_v \cdot \leftarrow + C_w \cdot \uparrow$$

Write $u \rightarrow u'$ if $u \leq u'$ in Bruhat order and $\ell(u) + 1 = \ell(u')$.

**Examples:**

- $022221 \rightarrow 122220$ ; $02 \rightarrow 20$ ; $100002 \rightarrow 200001$

Set $\delta(u_{u'}) = \delta_{u_i} - \delta_{u'_{i}}$ where $i$ is minimal with $u_i \neq u'_i$. 
From now on:

- All puzzles are triangles.
- All equivariant puzzle pieces and scabs are vertical.

**Def:** For any 012-string \( u = (u_1, u_2, \ldots, u_n) \) we set

\[
C_u := \sum_{i=1}^{n} \delta_{u_i} y_i \in R[y_1, \ldots, y_n]
\]

**Exercise:**
\[
\partial P = \triangle_{w}^{u,v} \Rightarrow \\
\sum_{s \text{ scab in } P} -\text{weight}(s) A(s) = C_u \cdot \downarrow + C_v \cdot \leftarrow + C_w \cdot \uparrow
\]

Write \( u \rightarrow u' \) if \( u \leq u' \) in Bruhat order and \( \ell(u) + 1 = \ell(u') \).

**Examples:**
022221 \rightarrow 122220 ; 02 \rightarrow 20 ; 100002 \rightarrow 200001

Set \( \delta(\frac{u}{u'}) = \delta_{u_i} - \delta_{u'_i} \) where \( i \) is minimal with \( u_i \neq u'_i \).

**Def:**
\[
\hat{C}_{u,v}^w = \sum_{\partial P = \triangle_{w}^{u,v}} \prod_{\diamond \in P} \text{weight}(\diamond)
\]
Molev–Sagan type recursion:

\[
(C_u \cdot \downarrow + C_v \cdot \leftarrow + C_w \cdot \uparrow) \cdot \hat{C}_{u,v}^w
\]

\[
= \sum_{\partial P = \triangle_{u,v}^w} \sum_{s \text{ scab in } P} -A(s) \text{ weight}(s) \prod_{\diamond \in P} \text{ weight}(\diamond)
\]
Molev–Sagan type recursion:

\[
(C_u \cdot \downarrow + C_v \cdot \leftarrow + C_w \cdot \uparrow) \cdot \hat{C}_{u,v}^w
\]

\[
= \sum_{\partial P = \triangle_{w}^{u,v}} \sum_{s \text{ scab in } P} -A(s) \text{ weight}(s) \prod_{\diamond \in P} \text{ weight}(\diamond)
\]

\[
= \sum_{\partial P = \triangle_{w}^{u,v}} -A(P) \text{ weight}(s) \prod_{\diamond \in P} \text{ weight}(\diamond)
\]

*P has marked scab s*
Molev–Sagan type recursion:

\[
(C_u \cdot \downarrow + C_v \cdot \leftarrow + C_w \cdot \uparrow) \cdot \hat{C}_{u,v}^w
\]

\[
= \sum_{\partial P = \triangle^{u,v}_{w}} \sum_{s \text{ scab in } P} -A(s) \text{ weight}(s) \prod_{\diamond \in P} \text{ weight}(\diamond)
\]

\[
= \sum_{\partial P = \triangle^{u,v}_{w}} -A(P) \text{ weight}(s) \prod_{\diamond \in P} \text{ weight}(\diamond)
\]

\[
= \sum_{\partial P = \triangle^{u,v}_{w}} A(P) \prod_{\diamond \in P} \text{ weight}(\diamond)
\]

- \(P\) has marked scab \(s\)
- \(P\) has gash pair
Molev–Sagan type recursion:

\[
(C_u \cdot \downarrow + C_v \cdot \leftarrow + C_w \cdot \uparrow) \cdot \hat{C}_{u,v}^w
\]

\[
= \sum_{\partial P = \triangle_{uw}} \sum_{s \text{ scab in } P} -A(s) \text{ weight}(s) \prod_{\diamond \in P} \text{ weight}(\diamond)
\]

\[
= \sum_{\partial P = \triangle_{uw}} -A(P) \text{ weight}(s) \prod_{\diamond \in P} \text{ weight}(\diamond)
\]

\[P \text{ has marked scab } s\]

\[
= \sum_{\partial P = \triangle_{uw}} A(P) \prod_{\diamond \in P} \text{ weight}(\diamond)
\]

\[P \text{ has gash pair}\]

\[
= \leftarrow \cdot \sum_{u \rightarrow u'} \delta \left(\frac{u}{u'}\right) \hat{C}_{u',v}^w + \rightarrow \cdot \sum_{v \rightarrow v'} \delta \left(\frac{v}{v'}\right) \hat{C}_{u,v'}^w + \downarrow \cdot \sum_{w' \rightarrow w} \delta \left(\frac{w'}{w}\right) \hat{C}_{u,v}^{w'}
\]
**Theorem** (Method first applied by Molev and Sagan.)

The equivariant Schubert structure constants $C_{w, v}^w \in \mathbb{Z}[y_1, \ldots, y_n]$ of $X = Fl(a, b; n)$ are uniquely determined by

1. \[ C_{w, w}^w = \prod_{i<j: w_i > w_j} (y_j - y_i) \quad \text{(Kostant-Kumar)} \]

2. \[
\left( C_u \cdot \quad + \quad C_v \cdot \quad + \quad C_w \cdot \quad \right) \cdot C_{u, v}^w
\]
   \[ = \quad \cdot \sum_{u \rightarrow u'} \delta\left(\frac{u}{u'}\right) C_{u', v}^w + \quad \cdot \sum_{v \rightarrow v'} \delta\left(\frac{v}{v'}\right) C_{u, v'}^w + \quad \cdot \sum_{w' \rightarrow w} \delta\left(\frac{w'}{w}\right) C_{u, v}^{w'}
\]
Theorem  (Method first applied by Molev and Sagan.)

The equivariant Schubert structure constants \( C_{u,v}^w \in \mathbb{Z}[y_1, \ldots, y_n] \) of \( X = Fl(a, b; n) \) are uniquely determined by

(1) \[
C_{w,w}^w = \prod_{i<j: w_i > w_j} (y_j - y_i)
\]

(Kostant-Kumar)

(2) \[
(C_u \cdot ✔ + C_v \cdot ☐ + C_w \cdot ☒) \cdot C_{u,v}^w
= ✔ \cdot \sum_{u \rightarrow u'} \delta\left(\frac{u}{u'}\right) C_{u',v}^w + ☐ \cdot \sum_{v \rightarrow v'} \delta\left(\frac{v}{v'}\right) C_{u,v'}^w + ☒ \cdot \sum_{w' \rightarrow w} \delta\left(\frac{w'}{w}\right) C_{u,v}^{w'}
\]

Consequence:

\[
C_{u,v}^w = \hat{C}_{u,v}^w = \sum_{\partial P = \triangle_{u,v}^w} \prod_{\text{weight}(\text{図})} \in P
\]