Puzzles for Projections
from 3-step flag varieties

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Schubert varieties in 3-step flag manifold

\[ X = \text{Fl}(a_1, a_2, a_3; n) = \{(A_1 \subset A_2 \subset A_3 \subset \mathbb{C}^n) \mid \dim(A_k) = a_k\} \]

**Def:** A **Schubert string** for \( X \) is a permutation of \( 0^{a_1} 1^{a_2-a_1} 2^{a_3-a_2} 3^{n-a_3} \).
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\( \mathbb{C}^n \) has basis \( \{e_1, e_2, \ldots, e_n\} \). \( u = (u_1, u_2, \ldots, u_n) \) Schubert string for \( X \).

**Def:** \( A^u = (A_1^u \subset A_2^u \subset A_3^u) \in X \) by \( A_k^u = \text{Span}_\mathbb{C}\{e_i : u_i < k\} \).

**Example:** \( X = \text{Fl}(1, 3, 4; 6) \)

\[ A^{130123} = (\mathbb{C}e_3 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5) \]
Schubert varieties in 3-step flag manifold

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\( \mathbf{B}^+ \subset \text{GL}(\mathbb{C}^n) \) upper triangular ; \( \mathbf{B}^- \subset \text{GL}(\mathbb{C}^n) \) lower triangular.

**Schubert varieties:** \( X_u = \overline{\mathbf{B}^+.A^u} \) ; \( X^u = \overline{\mathbf{B}^-.A^u} \subset X \)

\( \dim(X_u) = \text{codim}(X^u, X) = \ell(u) = \#\{i < j \mid u_i > u_j\} \)
Projection to Grassmannian

\[ \pi : X = \text{Fl}(a_1, a_2, a_3; n) \longrightarrow Y = \text{Gr}(a_2, n) ; \quad \pi(A_1 \subset A_2 \subset A_3) = A_2 \]

Simple labels for \( X \): 0, 1, 2, 3

Simple labels for \( Y \): 01, 23  

Merged !!
Projection to Grassmannian

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Simple labels for \( X \): 0, 1, 2, 3

Simple labels for \( Y \): 01, 23  Merged !!

Schubert string for \( Y \): \( w = (w_1, \ldots, w_n), \ w_i \in \{01, 23\}, \ a_2 = \# 01 \)

Def: \( V^w = \text{Span}\{e_i \mid w_i = 01\} \in Y \)

Note: \( \pi(A^u) = A_2^u = V^w \) where \( w_i = 01 \Leftrightarrow u_i \in \{0, 1\} \)
Projection to Grassmannian

\[ \pi : X = \text{Fl}(a_1, a_2, a_3; n) \rightarrow Y = \text{Gr}(a_2, n) \ ; \ \pi(A_1 \subset A_2 \subset A_3) = A_2 \]

Simple labels for \( X \): 0, 1, 2, 3
Simple labels for \( Y \): 01, 23 Merged !!

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Note: \( \pi(A^u) = A^u_2 = V^w \) where \( w_i = 01 \Leftrightarrow u_i \in \{0, 1\} \)

Schubert varieties: \( Y_w = \overline{B^+}.V^w \ ; \ Y^w = \overline{B^-}.V^w \subset Y \)

Translation to Young diagrams for \( Y = \text{Gr}(3, 8) \):

\( w = 23-01-23-23-01-23-01-23 \leftrightarrow \)

\[
\begin{array}{cccccccc}
\text{Young Diagram 1} & & & & & & & \\
\text{Young Diagram 2} & & & & & & & \\
\end{array}
\]
Projection to Grassmannian

\( \pi : X = \text{Fl}(a_1, a_2, a_3; n) \longrightarrow Y = \text{Gr}(a_2, n) \); \( \pi(A_1 \subset A_2 \subset A_3) = A_2 \)

Simple labels for \( X \): 0, 1, 2, 3

Simple labels for \( Y \): 01, 23 \hspace{1cm} \text{Merged !!}

Schubert string for \( Y \): \( w = (w_1, \ldots, w_n) \), \( w_i \in \{01, 23\} \), \( a_2 = \#01 \)

Def: \( V^w = \text{Span}\{e_i \mid w_i = 01\} \in Y \)

Note: \( \pi(A^u) = A_2^u = V^w \) \hspace{1cm} \text{where} \hspace{1cm} w_i = 01 \Leftrightarrow u_i \in \{0, 1\} \)

Schubert varieties: \( Y_w = \overline{B^+.V^w} \); \( Y^w = \overline{B^-.V^w} \subset Y \)

Goal: \( \int_X [X^u] \cdot [X^v] \cdot \pi^*[Y^w] = \# u v \)
Product with pullback:

\[ [X^u] \cdot \pi^*[Y^w] = \sum_v \left( \int_X [X^u] \cdot [X^v] \cdot \pi^*[Y^w] \right) [X^v] \quad \text{in} \quad H^*(X; \mathbb{Z}) \]

Pushforward of product:

\[ \pi_*( [X^u] \cdot [X^v] ) = \sum_w \left( \int_X [X^u] \cdot [X^v] \cdot \pi^*[Y^w] \right) [Y^w] \quad \text{in} \quad H^*(Y; \mathbb{Z}) \]
Simple puzzle pieces:

Composed puzzle pieces:

Definition of composed pieces:

\[
\begin{align*}
\triangle & \quad \triangle & \quad \triangle & \quad \triangle \\
\begin{array}{c}
0 \quad 0 \\
01
\end{array} & \quad \begin{array}{c}
1 \quad 1 \\
01
\end{array} & \quad \begin{array}{c}
2 \quad 2 \\
23
\end{array} & \quad \begin{array}{c}
3 \quad 3 \\
23
\end{array} \\
\begin{array}{c}
0(23) \\
12 \\
0(12)
\end{array} & \quad \begin{array}{c}
0(23) \\
13 \\
0(12)
\end{array} & \quad \begin{array}{c}
0((12)3) \\
12 \\
0((12)3)
\end{array} & \quad \begin{array}{c}
0(12) \\
13 \\
0(12)
\end{array}
\end{align*}
\]

\[
\begin{align*}
(b, a) & \quad \frac{b}{a} \\
& \quad \frac{b}{a} \quad \text{are edges} \quad \text{AND} \quad \max(a) < \min(b) \quad \text{AND} \quad \text{a and b are not merged.}
\end{align*}
\]
Theorem: \[ \int_X [X^u] \cdot [X^v] \cdot \pi^* [Y^w] = \# \]

Example: \[ \pi : X = \text{Fl}(1, 2, 3; 5) \longrightarrow \text{Gr}(2, 5) = Y \]
\[ \pi_* ([X^{10323}] \cdot [X^{10332}]) = ? \]
Theorem: \( \int_X [X^u] \cdot [X^v] \cdot \pi^*[Y^w] = \# u \ra v \ra w \)

Example: \( \pi : X = \text{Fl}(1, 2, 3; 5) \ra \text{Gr}(2, 5) = Y \)

\( \pi_*([X^{10323}] \cdot [X^{10332}]) = [Y^{\square}] + [Y^{\square}] \)
Theorem: \[ \int_X [X^u] \cdot [X^v] \cdot \pi^*[Y^w] = \# u v w \]

Example: \[ \pi: X = Fl(1, 2, 3; 5) \rightarrow Gr(2, 5) = Y \]

\[ \pi_*([X^{10323}] \cdot [X^{10332}]) = [Y^{\begin{array}{c} \text{\Box} \\ \text{\Box}\end{array}}] + [Y^{\begin{array}{c} \text{\Box} \\ \text{\Box}\end{array}}] \]
Quantum cohomology

Gromow-Witten invariants of $Y = \text{Gr}(m,n)$:

$$\langle Y^u, Y^v, Y_w \rangle_d = \# \text{ rational curves } C \subset Y \text{ of degree } d$$

meeting $Y^u$, $g.Y^v$, and $Y_w$

where $g \in GL_n$ is a fixed general element.

$$\langle Y^u, Y^v, Y_w \rangle_d = 0 \quad \text{if infinitely many curves exist.}$$

Small quantum cohomology ring

$$QH(Y) = H^*(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$$

$$[Y^u] * [Y^v] = \sum_{w} \langle Y^u, Y^v, Y_w \rangle_d q^d [Y^w]$$
Quantum = classical

\[ X = \text{Fl}(m - d, m, m + d; n) \xrightarrow{\pi} Y = \text{Gr}(m, n) \]
\[ \downarrow \phi \]
\[ Z = \text{Fl}(m - d, m + d; n) \]

**Theorem (B-Kresch-Tamvakis)**

\[ \langle Y^u, Y^v, Y_w \rangle_d = \# \phi \pi^{-1}(Y^u) \cap \phi \pi^{-1}(g.Y^v) \cap \phi \pi^{-1}(Y_w) \]

\[ C \leftrightarrow \left( \text{Ker}(C), \text{Span}(C) \right) := \left( \bigcap_{V \in C} V, \sum_{V \in C} V \right) \in Z \]
Quantum = classical

\[ X = \text{Fl}(m - d, m, m + d; n) \xrightarrow{\pi} Y = \text{Gr}(m, n) \]

\[ \phi \]

\[ Z = \text{Fl}(m - d, m + d; n) \]

**Theorem (B-Kresch-Tamvakis)**

\[ \langle Y^u, Y^v, Y^w \rangle_d = \# \phi \pi^{-1}(Y^u) \cap \phi \pi^{-1}(g.Y^v) \cap \phi \pi^{-1}(Y^w) \]

\[ C \leftrightarrow (\text{Ker}(C), \text{Span}(C)) := \left( \bigcap_{V \in C} V, \sum_{V \in C} V \right) \in Z \]

**Theorem (B-Mihalcea)**  
Equivariant generalization:

\[ \langle Y^u, Y^v, Y^w \rangle_d^T = \int_Z \phi_* \pi^*[Y^u] \cdot \phi_* \pi^*[Y^v] \cdot \phi_* \pi^*[Y^w] \]

\[ = \int_X \phi^* \phi_* \pi^*[Y^u] \cdot \phi^* \phi_* \pi^*[Y^v] \cdot \pi^*[Y^w] \]
Example

\[ Z = \text{Fl}(1, 3; 5) \xleftarrow{\phi} X = \text{Fl}(1, 2, 3; 5) \xrightarrow{\pi} Y = \text{Gr}(2, 5) \]

Compute coefficient of \( q^1 \) in quantum product \( [Y^\square] \star [Y^\square] \in QH(Y) \)

Quantum = classical implies:

\[
([Y^\square] \star [Y^\square])_1 = \pi_* (\phi^* \phi_* \pi^*[Y^\square] \cdot \phi^* \phi_* \pi^*[Y^\square]) \]

\[
= \pi_* ([X^{10323}] \cdot [X^{10332}]) = [Y^\square] + [Y^\square]
\]
Example

\[ Z = \text{Fl}(1, 3; 5) \overset{\phi}{\leftarrow} X = \text{Fl}(1, 2, 3; 5) \overset{\pi}{\rightarrow} Y = \text{Gr}(2, 5) \]

Compute coefficient of \( q^1 \) in quantum product \( [Y^{\square}] \ast [Y^{\square\square}] \in QH(Y) \)

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([Y^{\square}] \ast [Y^{\square\square}])_1 = \pi_* (\phi_* \phi_* \pi_* [Y^{\square}] \cdot \phi_* \phi_* \pi_* [Y^{\square\square}])
\]

\[
= \pi_*([X^{10323}] \cdot [X^{10332}]) = [Y^{\square\square}] + [Y^{\square}]
\]

Now use 2-step puzzle formula:

\[
([Y^{\square}] \ast [Y^{\square\square}])_1 = \pi_* \phi^* (\phi_* \pi_* [Y^{\square}] \cdot \phi_* \pi_* [Y^{\square\square}])
\]

\[
= \pi_* \phi^* ([Z^{10212}] \cdot [Z^{10221}]) = [Y^{\square\square}] + [Y^{\square}]
\]
Projection to 2-step flag manifold

\[ \pi : \text{Fl}(a_1, a_2, a_3; n) \rightarrow \text{Fl}(a_1, a_3; n) \]

**Simple puzzle pieces:**

Simple labels:

\[ \text{Fl}(a_1, a_2, a_3; n): 0, 1, 2, 3 \]
\[ \text{Fl}(a_1, a_3; n): 0, 12, 3 \]

**Composed puzzle pieces:**

Composed labels:

\[
\begin{align*}
A &= 01 & B &= 02 & C &= 03 & D &= 0(12) & E &= 13 & F &= 23 & G &= (12)3 & H &= 0(13) \\
J &= 0(23) & K &= 0((12)3) & L &= 1(23) & M &= (01)2 & N &= (01)3 & P &= (02)3 & R &= (0(12))3 \\
S &= 0(1(23)) & T &= (01)(23) & U &= ((01)2)3 & V &= 0(((01)2)3) & W &= (0(1(23)))3
\end{align*}
\]

Rule: \((a, b)\) can be a label only if \(\max(a) < \min(b)\) OR \(\max(a) = \min(b)\) AND repetition separated by 3 parentheses.
Puzzle formula for projections

Let \( \pi : X \rightarrow Y \) be a projection of partial flag manifolds. Assume \( X \) has at most 3 steps, \( Y \) has at most 2 steps.

**Theorem:**
\[
\int_X [X^u] \cdot [X^v] \cdot \pi^* [Y^w] = \# u v w
\]
Puzzle formula for projections

Let $\pi : X \to Y$ be a projection of partial flag manifolds. Assume $X$ has at most 3 steps, $Y$ has at most 2 steps.

**Theorem:**

\[
\int_X [X^u] \cdot [X^v] \cdot \pi^*[Y^w] = \# \triangle uwv
\]

**Known cases:**

- Puzzle rule for $H^*(\text{Gr}(m, n))$  
  (Knutson, Tao, Woodward)
- Puzzle rule for $H^*_T(\text{Gr}(m, n))$  
  (Knutson, Tao)
- Puzzle rule for $H^*(\text{Fl}(a, b; n))$  
  (conjectured by Knutson, proof in [B-Kresch-Purbhoo-Tamvakis], different positive formula by Coskun.)

**Conjecture (Knutson, Buch) / Theorem (Knutson - Zinn-Justin)**

Formula holds for $X = Y = \text{Fl}(a_1, a_2, a_3; n)$. 
The mutation algorithm
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The mutation algorithm
The mutation algorithm

Diagram of a triangular lattice with labels and numbers.
The mutation algorithm
The mutation algorithm
The mutation algorithm

32
1 1 1
F 1 F
12
1 1 1 10
2 2 3 1 1 0
12
E
1111
A 3
111
G
3A
12
0 D 0 0
12
12
0 0 1 1
12
The mutation algorithm
The mutation algorithm
The mutation algorithm
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Resolutions of temporary puzzle piece

Def: Two gashes are equivalent if one can be propagated to the other.

Def: A gash is opposite to \( \frac{a}{b} \) \( \iff \) it is equivalent to \( \frac{b}{a} \).
Resolutions of temporary puzzle piece

Def: Two gashes are equivalent if one can be propagated to the other.

Def: A gash is opposite to $a/b \iff$ it is equivalent to $b/a$.

Def: A resolution of a temporary piece is a puzzle piece that creates two opposite gashes on replacement.

Example:

Resolutions of $G^E_E$:
**Resolutions of temporary puzzle piece**

**Def:** Two gashes are **equivalent** if one can be propagated to the other.

**Def:** A gash is **opposite** to \( \frac{a}{b} \) \( \iff \) it is equivalent to \( \frac{b}{a} \).

**Def:** A **resolution** of a temporary piece is a puzzle piece that creates two opposite gashes on replacement.

**Example:**

Resolutions of \( \begin{array}{c} \triangle \ \text{G} \ \text{E} \ \text{E} \end{array} \):

\[ \begin{align*}
&\begin{array}{c}
3 \ \text{E} \\
\text{E} \\
\end{array} \quad \begin{array}{c}
\text{G} \ 3 \ \text{E} \\
\text{E} \\
\end{array} \quad \begin{array}{c}
\text{G} \ 3 \ \text{E} \\
\text{E} \\
\end{array} \\
&\begin{array}{c}
\text{E} \\
12 \\
\end{array} \\
\text{G} \ 3 \ \text{E} \\
\text{E} \\
\end{align*} \]

**Fact:** Each temporary piece has exactly 3 resolutions.

**Note:** Every gash is either a **left gash** or a **right gash**.
The mutation algorithm
The mutation algorithm
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The mutation algorithm

Diagram showing a triangular grid with numbers and letters.
The mutation algorithm
The mutation algorithm
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The mutation algorithm
Component of the mutation graph
Borel construction for puzzle pieces

Def: A scab is a small rhombus consisting of two distinct puzzle pieces.
Borel construction for puzzle pieces

Def: A **scab** is a small rhombus consisting of two distinct puzzle pieces.

Def: A **resolution** of a scab is a symmetric rhombus that creates two opposite gashes on replacement, with left gash in left side, right gash in right side.
**Borel construction for puzzle pieces**

**Def:** A *scab* is a small rhombus consisting of two distinct puzzle pieces.

![Diagram of scab pieces]

**Def:** A *resolution* of a scab is a symmetric rhombus that creates two opposite gashes on replacement, with left gash in left side, right gash in right side.

![Diagram of resolutions]

**Fact:** Any scab has at most one resolution.

**Def:** A resolution of a scab is also called an *equivariant puzzle piece.*
All puzzle pieces for $\pi : \text{Fl}(a_1, a_2, a_3; n) \to \text{Gr}(a_2, n)$
Equivariant cohomology

\( T \subset \text{GL}(\mathbb{C}^n) \)  max torus of diagonal matrices.

\[ \Lambda = H^*_T(\text{pt}; \mathbb{Z}) = \mathbb{Z}[y_1, y_2, \ldots, y_n] ; \ y_i = -c_1(\mathbb{C}e_i) \]

\[ H^*_T(X; \mathbb{Z}) = \bigoplus_u \Lambda[X^u] \] is a \( \Lambda \)-algebra.
Equivariant cohomology

\( T \subset \text{GL}(\mathbb{C}^n) \) max torus of diagonal matrices.

\[ \Lambda = H^*_T(\text{pt}; \mathbb{Z}) = \mathbb{Z}[y_1, y_2, \ldots, y_n] ; \quad y_i = -c_1(\mathbb{C}e_i) \]

\[ H^*_T(X; \mathbb{Z}) = \bigoplus \Lambda[X^u] \text{ is a } \Lambda\text{-algebra.} \]

\[ [X^u] \cdot [X^v] = \sum_w C^w_{u,v} [X^w] ; \quad C^w_{u,v} = \int_X^T [X^u] \cdot [X^v] \cdot [X^w] \in \Lambda \]

where \( \int_X^T : H^*_T(X; \mathbb{Z}) \to \Lambda \) is pushforward along \( X \to \{\text{pt}\} \).

**Theorem (Graham):** \( C^w_{u,v} \in \mathbb{Z}_{\geq 0}[y_2 - y_1, \ldots, y_n - y_{n-1}] \)
Equivariant cohomology

$T \subset \text{GL}(\mathbb{C}^n)$ max torus of diagonal matrices.

$\Lambda = H^*_T(\text{pt}; \mathbb{Z}) = \mathbb{Z}[y_1, y_2, \ldots, y_n]$ ; $y_i = -c_1(\mathbb{C}e_i)$

$H^*_T(X; \mathbb{Z}) = \bigoplus \Lambda [X^u]$ is a $\Lambda$-algebra.

$[X^u] \cdot [X^v] = \sum_w C_{u,v}^w [X^w]$ ; $C_{u,v}^w = \int_X^T [X^u] \cdot [X^v] \cdot [X^w] \in \Lambda$

where $\int_X^T : H^*_T(X; \mathbb{Z}) \to \Lambda$ is pushforward along $X \to \{\text{pt}\}$.

**Theorem (Graham):** $C_{u,v}^w \in \mathbb{Z}_{\geq 0}[y_2 - y_1, \ldots, y_n - y_{n-1}]$

**Def:** weight($\triangle$) = $y_j - y_i$ where $i < j$ are defined by
Equivariant puzzle formula

Let $\pi : X \to Y$ be a projection of partial flag manifolds.

Assume $X$ has at most 3 steps, $Y$ has at most 2 steps.

Let $\alpha, \beta \in H^*_T(X)$ and $\gamma \in H^*_T(Y)$ be Schubert classes, such that one of $\alpha, \beta, \gamma$ is $B^+$-stable, the other two are $B^-$-stable.

Consider puzzles with all equivariant pieces pointing to $B^+$-stable side:

$$P = \alpha \beta \gamma$$

**Theorem:** If all scabs pointing to $B^+$-stable side have resolutions, then

$$\int_X T \alpha \cdot \beta \cdot \pi^*(\gamma) = \sum_{P} \prod_{\triangledown \in P} \text{weight}(\triangledown)$$
Example \[ \pi : X = \text{Fl}(1, 2, 3; 4) \rightarrow Y = \text{Fl}(1, 3; 4) \]

\[ [X^{2013}] \cdot \pi^* [Y_{12-0-3-12}] = ? \]
Example \[ \pi : X = \text{Fl}(1, 2, 3; 4) \rightarrow Y = \text{Fl}(1, 3; 4) \]

\[ [X^{2013}] \cdot \pi^* [Y^{12-0-3-12}] = ? \]
Example \( \pi : X = \text{Fl}(1, 2, 3; 4) \rightarrow Y = \text{Fl}(1, 3; 4) \)

\[ [X^{2013}] \cdot \pi^* [Y^{12-0-3-12}] = \]

\[
\begin{array}{ccc}
\begin{array}{ccc}
2 & 2 & 12 \\
1 & 1 & 0 \\
12 & 3 & 0 \\
\end{array} & \begin{array}{ccc}
2 & 2 & 12 \\
3 & 1 & 0 \\
3 & 0 & 12 \\
\end{array} & \begin{array}{ccc}
3 & 2 & 0 \\
1 & 0 & 0 \\
12 & 0 & 0 \\
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{ccc}
0 & 0 & 12 \\
0 & 0 & 12 \\
3 & 3 & 3 \\
\end{array} & \begin{array}{ccc}
0 & 0 & 12 \\
1 & 1 & 3 \\
3 & 0 & 12 \\
\end{array} & \begin{array}{ccc}
2 & 2 & 3 \\
3 & 3 & 0 \\
12 & 3 & 0 \\
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{ccc}
R & 0 & 0 \\
D & 3 & 3 \\
0 & 12 \\
\end{array} & \begin{array}{ccc}
E & 0 & 0 \\
C & 3 & 1 \\
3 & 0 \\
\end{array} & \begin{array}{ccc}
F & 0 & 0 \\
F & 0 & 0 \\
F & 0 & 0 \\
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{ccc}
A & 1 & 1 \\
0 & 0 \\
12 \\
\end{array} & \begin{array}{ccc}
A & 1 & 1 \\
0 & 0 \\
12 \\
\end{array} & \begin{array}{ccc}
A & 1 & 1 \\
0 & 0 \\
12 \\
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{ccc}
12 & 3 & 0 \\
12 & 3 & 0 \\
12 & 3 & 0 \\
\end{array} & \begin{array}{ccc}
12 & 3 & 0 \\
12 & 3 & 0 \\
12 & 3 & 0 \\
\end{array} & \begin{array}{ccc}
12 & 3 & 0 \\
12 & 3 & 0 \\
12 & 3 & 0 \\
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{ccc}
\frac{y_2 - y_1}{X^{2031}} & \frac{y_2 - y_1}{X^{3012}}
\end{array}
\end{array}
\]
Existence of equivariant puzzle pieces

Results: Formula for \([X^u] \cdot \pi^*[Y^v]\) in \(H^*_T(X; \mathbb{Z})\) for every \(\pi : X \to Y\)

Formula for \(\pi_*([X^u] \cdot [X^v])\) in \(H^*_T(Y; \mathbb{Z})\) for every \(\pi : X \to Y\)

except \(\pi : \text{Fl}(a_1, a_2, a_3; n) \to \text{Fl}(a_1, a_3; n)\)
Existence of equivariant puzzle pieces

Results: Formula for $[X^u] \cdot \pi^*[Y^v]$ in $H_T^*(X; \mathbb{Z})$ for every $\pi : X \rightarrow Y$

Formula for $\pi_*( [X^u] \cdot [X^v] )$ in $H_T^*(Y; \mathbb{Z})$ for every $\pi : X \rightarrow Y$

except $\pi : \text{Fl}(a_1, a_2, a_3; n) \rightarrow \text{Fl}(a_1, a_3; n)$

Reason: The scab $\begin{array}{c}
3 \\
3 \\
W \\
3
\end{array}$ has no resolution!
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except $\pi : \text{Fl}(a_1, a_2, a_3; n) \to \text{Fl}(a_1, a_3; n)$

Reason: The scab $\begin{array}{c}
3 \\
3 \\
\text{W} \\
8
\end{array}$ has no resolution!

Example: $\pi : \text{Fl}(a_1, a_2, a_3; n) \to \text{Fl}(a_1, a_2; n)$

The scab $\begin{array}{c}
3 \\
3 \\
\text{C} \\
\text{E}
\end{array}$ has no resolution

$\Rightarrow$ No formula for $\pi^*(\gamma) \cdot \alpha$:
Existence of equivariant puzzle pieces

**Results:**

Formula for $[X^u] \cdot \pi^*[Y^v]$ in $H_T^*(X; \mathbb{Z})$ for every $\pi : X \to Y$

Formula for $\pi_*([X^u] \cdot [X^v])$ in $H_T^*(Y; \mathbb{Z})$ for every $\pi : X \to Y$

except $\pi : \text{Fl}(a_1, a_2, a_3; n) \to \text{Fl}(a_1, a_3; n)$

**Reason:** The scab $\triangle$ has no resolution!

**Example:** $\pi : \text{Fl}(a_1, a_2, a_3; n) \to \text{Fl}(a_1, a_2; n)$

The scab $\triangle$ has no resolution

$\Rightarrow$ No formula for $\pi^*(\gamma) \cdot \alpha$

All scabs $\triangle$ have resolutions

$\Rightarrow$ Obtain formula for $\beta \cdot \pi^*(\gamma)$:
Projection to a point \( \pi : \text{Fl}(n) \to \{ \text{pt} \} \)

\[
\int_X [X^u] \cdot [X^v] \cdot \pi^*[\text{pt}] = \begin{cases} 
[X^{uv^{-1}w_0}]_{w_0} & \text{if } \ell(uv^{-1}w_0) = \ell(u) - \ell(v^{-1}w_0) \\
0 & \text{otherwise.}
\end{cases}
\]

Puzzle pieces: \( a a \) for \( 1 \leq a \leq n \)

Equivariant pieces: \( a b \) \( b a \) for \( 1 \leq a < b \leq n \)
Projection to a point \( \pi : \text{Fl}(n) \rightarrow \{ \text{pt} \} \)

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Puzzle pieces: \( a \quad a \quad \) for \( 1 \leq a \leq n \)

Equivariant pieces: \( a \ b \ b \ a \) for \( 1 \leq a < b \leq n \)

Puzzle formula specializes to pipe dream formula for double Schubert polynomials

Billey-Jockusch-Stanley
Billey-Bergeron
Fomin-Kirillov
Highlights from proof $X = G/P$ ; Fix $T \subset B \subset P \subset G$

Weyl groups

$W = N_G(T)/T$ ; $W_P = N_P(T)/T$

$W^P \subset W$ subset of minimal representatives for cosets in $W/W_P$

Schubert varieties

$X_u = B_u.P$ , $X^u = B^{-u}.P$ for $u \in W$.

$\dim(X_u) = \operatorname{codim}(X^u, X) = \ell(u)$ whenever $u \in W^P$.

Schubert structure constants

$C_{u,v}^w = \int_X^T [X^u] \cdot [X^v] \cdot [X_w] \in \Lambda = H^*_T(\text{pt}; \mathbb{Z})$

$[X^u] \cdot [X^v] = \sum_w C_{u,v}^w [X^w] \text{ in } H^*_T(X; \mathbb{Z})$
Chevalley formula

Let $D \in H^2_T(X; R)$, $R$ commutative ring.

Write $u \rightarrow u'$ for covering relation in $W^P$: $u' = us_\alpha$ and $\ell(u') = \ell(u) + 1$

Define $(D, \frac{u'}{u}) := "(D, \alpha^\vee)" = \int_{C_\alpha} T D \in H^*_T(pt; R)$

where $C_\alpha \subset X$ is the $T$-stable curve through $1.P$ and $s_\alpha.P$

Chevalley: $D \cdot [X^u] = D_u [X^u] + \sum_{u \rightarrow u'} (D, \frac{u'}{u}) [X^{u'}]$ in $H^*_T(X; R)$
Molev-Sagan equations

Lemma: If $\eta \in R$ satisfies $\eta^2 + \eta + 1 = 0$, then

$(-\eta^2 D_u - \eta D_v - D_w) C_{u,v}^w =$

$\eta^2 \sum_{u \rightarrow u'} (D, \frac{u'}{u}) C_{u',v}^w + \eta \sum_{v \rightarrow v'} (D, \frac{v'}{v}) C_{u,v'}^w + \sum_{w' \rightarrow w} (D, \frac{w}{w'}) C_{u,v}^{w'}$

Proof: Expand and integrate

$\eta^2 (D \cdot [X^u]) [X^v] [X_w] + \eta [X^u] (D \cdot [X^v]) [X_w] + [X^u][X^v] (D \cdot [X_w]) = 0$
Molev-Sagan equations

Lemma: If \( \eta \in R \) satisfies \( \eta^2 + \eta + 1 = 0 \), then

\[
(-\eta^2 D_u - \eta D_v - D_w) C_{u,v}^w = \\
\eta^2 \sum_{u \to u'} (D, \frac{u'}{u}) C_{u',v}^w + \eta \sum_{v \to v'} (D, \frac{v'}{v}) C_{u,v'}^w + \sum_{w' \to w} (D, \frac{w}{w'}) C_{u,v}^{w'}
\]

Proof: Expand and integrate

\[
\eta^2 (D \cdot [X^u]) [X^v] [X^w] + \eta [X^u] (D \cdot [X^v]) [X^w] + [X^u][X^v] (D \cdot [X^w]) = 0
\]

Application:

Take \( R = \mathbb{C}[\delta_\beta \mid \beta \in \Delta - \Delta_P] \), \( D = \sum_{\beta \in \Delta - \Delta_P} \delta_\beta [X^{s\beta}] \), \( \eta = \exp\left(\frac{2\pi i}{3}\right) \)

Note: \( (-\eta^2 D_u - \eta D_v - D_w) = 0 \iff u = v = w \)
Molev-Sagan recursion for $\pi: X = G/P \rightarrow Y = G/Q$

Want to compute $C^{w}_{u,v}$ for which $u \in W^{Q}$ or $v \in W^{Q}$ or $w \in W^{Q,\text{max}}$

Use $D' = \sum_{\beta \in \Delta - \Delta_{Q}} \delta_{\beta}[X^{s_{\beta}}]$ divisor pulled back from $Y$ !!

$(-\eta^{2}D'_{u} - \eta D'_{v} - D'_{w})C^{w}_{u,v} =$

$\eta^{2} \sum_{u \rightarrow u'} (D', \frac{u'}{u})C^{w}_{u',v} + \eta \sum_{v \rightarrow v'} (D', \frac{v'}{v})C^{w}_{u,v'} + \sum_{w' \rightarrow w} (D', \frac{w'}{w})C^{w'}_{u,v}$

Recursion involves only $C^{w'}_{u',v'}$ with $u' \in W^{Q}$ or $v' \in W^{Q}$ or $w' \in W^{Q,\text{max}}$. 
Molev-Sagan recursion for \( \pi : X = G/P \longrightarrow Y = G/Q \)

Want to compute \( C_{w,v}^u \) for which \( u \in W^Q \) or \( v \in W^Q \) or \( w \in W^{Q,\text{max}} \).

Use \( D' = \sum_{\beta \in \Delta - \Delta_Q} \delta_\beta [X^{s_\beta}] \) divisor pulled back from \( Y \) !!

\[
(-\eta^2 D'_u - \eta D'_v - D'_w) C_{u,v}^w = \\
\eta^2 \sum_{u \to u'} (D', \frac{u'}{u}) C_{u',v}^w + \eta \sum_{v \to v'} (D', \frac{v'}{v}) C_{u,v'}^w + \sum_{w' \to w} (D', \frac{w'}{w}) C_{u,v}^{w'}
\]

Recursion involves only \( C_{u',v'}^{w'} \) with \( u' \in W^Q \) or \( v' \in W^Q \) or \( w' \in W^{Q,\text{max}} \).

But: \( (-\eta^2 D'_u - \eta D'_v - D'_w) = 0 \iff u^Q = v^Q = w^Q \in W^Q \)

Here \( u = u^Q u_Q \) is parabolic factorization: \( u^Q \in W^Q \) and \( u_Q \in W_Q \)
Molev-Sagan recursion for $\pi : X = G/P \rightarrow Y = G/Q$

Want to compute $C_{w,v}^u$ for which $u \in W^Q$ or $v \in W^Q$ or $w \in W^{Q,\text{max}}$.

Use $D' = \sum_{\beta \in \Delta - \Delta_Q} \delta_{\beta}[X^s\beta]$ divisor pulled back from $Y$ !!

$$\begin{align*}
-\eta^2 D'_u - \eta D'_v - D'_w)C_{u,v}^w &= \\
\eta^2 \sum_{u \rightarrow u'} (D', \frac{u'}{u}) C_{u',v}^w + \eta \sum_{v \rightarrow v'} (D', \frac{v'}{v}) C_{u,v'}^w + \sum_{w' \rightarrow w} (D', \frac{w'}{w}) C_{u,v}^{w'}
\end{align*}$$

Recursion involves only $C_{u',v'}^w$ with $u' \in W^Q$ or $v' \in W^Q$ or $w' \in W^{Q,\text{max}}$.

But: $(-\eta^2 D'_u - \eta D'_v - D'_w) = 0 \iff u^Q = v^Q = w^Q \in W^Q$

Here $u = u^Q u_Q$ is parabolic factorization: $u^Q \in W^Q$ and $u_Q \in W_Q$

Theorem: Let $u, v, w \in W^P$ satisfy $u^Q = v^Q = w^Q = \kappa \in W^Q$.

Then $C_{u,v}^w(X) = C_{\kappa,\kappa}(Y) \kappa( C_{u_Q,v_Q}^{w_Q}(F))$ where $F = \pi^{-1}(1.Q) = Q/P$. 
Example \[ \pi : X = \text{Fl}(2, 4, 6; 7) \longrightarrow Y = \text{Gr}(4, 7) \]

\[ F = \pi^{-1}(\text{pt}) = \text{Gr}(2, 4) \times \text{Gr}(2, 3) \]

\[ u = 1301220, \quad v = 1201320, \quad w = 01-23-01-01-23-23-01 \]

\[ u_Q = 1010322, \quad v_Q = 1010232, \quad w_Q = 01-01-01-01-23-23-23 \]

\[ C^w_{u,v}(X) = C^w_{w,w}(Y) \kappa(C^w_{u_Q,v_Q}(F)) = C^w_{w,w}(Y) \kappa(C^{01-01-01-01}_{1010,1010}) \kappa(C^{23-23-23}_{322,232}) \]

where \[ \kappa = 1347256 \in S_7 \]
\[ C_{322,23}^{3-23-23} = \]
\[ + \]
\[ C_{1010,1010}^{01-01-01-01} = \]
\[ C_{w,w}^w(Y) = \]
\[ C^w_{u,v}(X) = \]

\[ \begin{array}{c}
\begin{array}{cccccccc}
0 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
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\begin{array}{cccccccc}
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2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
\end{array}
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\begin{array}{cccccccc}
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1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
\end{array}
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\end{array} \]