3.2 5(c).  
Define a relation $V$ on $\mathbb{R}$ by $V = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y \text{ or } xy = 1\}$.

Claim: $V$ is an equivalence relation on $\mathbb{R}$.

Proof. We must show that $V$ is reflexive on $\mathbb{R}$, symmetric, and transitive.

Reflexive on $\mathbb{R}$: Let $x \in \mathbb{R}$. Since $x = x$, we have $(x, x) \in V$.

Symmetric: Let $x, y \in \mathbb{R}$. Assume that $(x, y) \in V$. Then $x = y$ or $xy = 1$.

This implies that $y = x$ or $yx = 1$, hence $(y, x) \in V$.

Transitive: Let $x, y, z \in \mathbb{R}$. Assume that $(x, y) \in V$ and $(y, z) \in V$.

Then we have $x = y$ or $xy = 1$. And we have $y = z$ or $yz = 1$.

Case 1: Assume that $x = y$ and $y = z$. Then $x = z$, hence $(x, z) \in V$.

Case 2: Assume that $x = y$ and $yz = 1$. Then $x = z$, hence $(x, z) \in V$.

Case 3: Assume that $xy = 1$ and $y = z$. Then $xz = 1$, hence $(x, z) \in V$.

Case 4: Assume that $xy = 1$ and $yz = 1$. Then $x = y = 1$, hence $(x, z) \in V$.

This finishes the proof that $V$ is an equivalence relation on $\mathbb{R}$. □

The equivalence class of 3 is $3/V = \{x \in \mathbb{R} \mid (x, 3) \in V\} = \{x \in \mathbb{R} \mid x = 3 \text{ or } 3x = 1\} = \{3, \frac{1}{3}\}$.

The equivalence class of $\frac{-2}{3}$ is $(\frac{-2}{3})/V = \{x \in \mathbb{R} \mid (x, \frac{-2}{3}) \in V\} = \{x \in \mathbb{R} \mid x = \frac{-2}{3} \text{ or } \frac{-2}{3}x = 1\} = \{-\frac{2}{3}, \frac{3}{2}\}$.

The equivalence class of 0 is $0/V = \{x \in \mathbb{R} \mid (x, 0) \in V\} = \{x \in \mathbb{R} \mid x = 0 \text{ or } 0x = 1\} = \{0\}$.

3.2 7.

Reflexive relations: (b), (c), (d).

Symmetric relations: (b), (c).

Transitive relations: (a), (b), (c).

3.2 12.

Let $A$ be a set and let $R$ and $S$ be equivalence relations on $A$.

Claim: $R \cap S$ is an equivalence relation on $A$.

Proof. Since $R \subset A \times A$ and $S \subset A \times A$, it follows that $R \cap S \subset A \times A$. Therefore $R \cap S$ is a relation on $A$.

We must show that $R \cap S$ is reflexive on $A$, symmetric, and transitive.

Reflexive: Let $x \in A$. Since $R$ is reflexive, we have $(x, x) \in R$. Since $S$ is reflexive, we have $(x, x) \in S$. It follows that $(x, x) \in R \cap S$.

Symmetric: Let $x, y \in A$. Assume that $(x, y) \in R \cap S$. Since $R$ is symmetric and $(x, y) \in R$, we have $(y, x) \in R$. Since $S$ is symmetric and $(x, y) \in S$, we have $(y, x) \in S$. It follows that $(y, x) \in R \cap S$.

Transitive: Let $x, y, z \in A$. Assume that $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$. Since $R$ is transitive and $(x, y) \in R$ and $(y, z) \in R$, we have $(x, z) \in R$. Since $S$ is transitive and $(x, y) \in S$ and $(y, z) \in S$, we have $(x, z) \in S$. It follows that $(x, z) \in R \cap S$.

This completes the proof that $R \cap S$ is an equivalence relation on $A$. □

3.3 3(a). To be very descriptive, we need the following Lemma.

Lemma $\forall x \in \mathbb{R} : (x - 1, x] \cap \mathbb{Z} \neq \emptyset$. 

1
Proof. Let \( x \in \mathbb{R} \).
Choose \( N \in \mathbb{Z} \) so large that \( N > x \).
Set \( S = \{ m \in \mathbb{Z} \mid m \geq N - x \} \).
Then \( S \subseteq \mathbb{N} \) and \( S \neq \emptyset \).
By WOP, \( S \) contains a smallest element \( m_0 \).
Since \( m_0 \in S \) we have \( N - m_0 \leq x \).
Since \( m_0 - 1 \notin S \) we have \( N - m_0 > x - 1 \).
It follows that \( N - m_0 \in (x - 1, x] \cap \mathbb{Z} \).

Define \( Q = \{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid x - y \in \mathbb{Z} \} \).
The exercise tells us that \( Q \) is an equivalence relation on \( \mathbb{R} \); I will not prove this.
The corresponding partition of \( \mathbb{R} \) is the family of subsets \( \mathbb{R}/Q = \{ x/Q \mid x \in \mathbb{R} \} \).
Set \( I = [0, 1) \subset \mathbb{R} \).
For \( z \in I \), set \( A_z = \{ z + m \mid m \in \mathbb{Z} \} = \{ y \in \mathbb{R} \mid z - y \in \mathbb{Z} \} \).
Define the family \( \mathcal{P} = \{ A_z \mid z \in I \} \).

Theorem: \( \mathbb{R}/Q = \mathcal{P} \).

Proof. Let \( S \in \mathbb{R}/Q \).
Then we can choose \( x \in \mathbb{R} \) such that \( S = x/Q \).
By the Lemma, we may choose \( n \in (x - 1, x] \cap \mathbb{Z} \).
Set \( z = x - n \). Then \( z \in I \).
Since \( x - z \in \mathbb{Z} \) we obtain
\[
S = x/Q = \{ y \in \mathbb{R} \mid (x, y) \in Q \} = \{ y \in \mathbb{R} \mid x - y \in \mathbb{Z} \} = \{ y \in \mathbb{R} \mid z - y \in \mathbb{Z} \} = A_z.
\]
It follows that \( S \in \mathcal{P} \).
Now let \( S \in \mathcal{P} \).
Then we can choose \( z \in I \) such that \( S = A_z \).
Since \( S = A_z = z/Q \), we obtain \( S \in \mathbb{R}/Q \).

3.3 3(c).
Define \( R = \{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid \sin(x) = \sin(y) \} \).
I will not show that this is an equivalence relation on \( \mathbb{R} \).
Set \( I = [-1, 1] \).
For \( z \in I \), set \( B_z = \{ y \in \mathbb{R} \mid \sin(y) = z \} \).
From calculus we know that the restriction of \( \sin(x) \) to the interval \( [-\pi/2, \pi/2] \)
has an inverse function \( \sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2] \).
For each \( z \in I \) we then have
\[
B_z = \{ 2\pi m + \sin^{-1}(z) \mid m \in \mathbb{Z} \} \cup \{ \pi/2 + 2\pi m - \sin^{-1}(z) \mid m \in \mathbb{Z} \}.
\]
I will not prove this.
Set \( \mathcal{P} = \{ B_z \mid z \in I \} \).

Theorem: \( \mathbb{R}/R = \mathcal{P} \).

Proof. Let \( S \in \mathbb{R}/R \).
Choose \( x \in \mathbb{R} \) such that \( S = x/R \).
Set \( z = \sin(x) \).
Then \( z \in I \) and \( S = x/R = \{ y \in \mathbb{R} \mid \sin(x) = \sin(y) \} = B_z \).
Therefore \( S \in \mathcal{P} \).
Let \( S \in \mathcal{P} \).
Choose \( z \in I \) such that \( S = B_z \).
Then $S = B_z = z/R$.
Therefore $S \in \mathbb{R}/R$. □

3.3 6(e).

Let $\mathcal{P} = \{A, B\}$ where $A = \{x \in \mathbb{Z} \mid x < 3\}$ and $B = \mathbb{Z} - A$.
Then $\mathcal{P}$ is a partition of $\mathbb{Z}$ (this will not be proved).
The corresponding equivalence relation is defined by:
$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid \exists S \in \mathcal{P} : x \in S \text{ and } y \in S\}$.
Set $Q = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid (x < 3 \text{ and } y < 3) \text{ or } (x \geq 3 \text{ and } y \geq 3)\}$.

**Theorem:** $R = Q$.

**Proof.** Let $(x, y) \in R$.

Choose $S \in \mathcal{P}$ such that $x \in S$ and $y \in S$.
By definition of $\mathcal{P}$ we must have $S = A$ or $S = B$.
Case 1: Assume that $S = A$.
Then $x < 3$ and $y < 3$, so $(x, y) \in Q$.
Case 2: Assume that $S = B$.
Then $x \geq 3$ and $y \geq 3$, so $(x, y) \in Q$.
This proves that $R \subseteq Q$.
The proof that $Q \subseteq R$ is similar, by considering the same two cases. □

3.3 7(b).

For $a \in \mathbb{R}$, set $A_a = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = a - x^2\}$.
Set $\mathcal{P} = \{A_a \mid a \in \mathbb{R}\}$.

**Theorem:** $\mathcal{P}$ is a partition of $\mathbb{R} \times \mathbb{R}$.

**Proof.** According to the definition of a partition, we must prove claims 1-3 below.

Claim 1: $\forall S \in \mathcal{P} : S \neq \emptyset$.
Let $S \in \mathcal{P}$.
Choose $a \in \mathbb{R}$ such that $S = A_a$.
Since $(0, a) \in A_a$, it follows that $S \neq \emptyset$.

Claim 2: $\bigcup_{S \in \mathcal{P}} S = \mathbb{R} \times \mathbb{R}$
Let $(x, y) \in \bigcup_{S \in \mathcal{P}} S$.
Choose $S \in \mathcal{P}$ such that $(x, y) \in S$.
Choose $a \in \mathbb{R}$ such that $S = A_a$.
Since $(x, y) \in A_a$ and $A_a \subseteq \mathbb{R} \times \mathbb{R}$, we obtain $(x, y) \in \mathbb{R} \times \mathbb{R}$.
Let $(x, y) \in \mathbb{R} \times \mathbb{R}$.
Set $a = x^2 + y$.
Then $(x, y) \in A_a$.
Since $A_a \in \mathcal{P}$, this implies that $(x, y) \in \bigcup_{S \in \mathcal{P}} S$.

Claim 3: $\forall S, T \in \mathcal{P} : S = T \text{ or } S \cap T = \emptyset$.
Let $S, T \in \mathcal{P}$.
Choose $a, b \in \mathbb{R}$ such that $S = A_a$ and $T = A_b$.
Case 1: If $a = b$ then $S = T$ holds.
Case 2: Assume that $a \neq b$.
In this case I will show that $S \cap T = \emptyset$.
If this is false, then choose $(x, y) \in S \cap T$.
Since $(x, y) \in A_a$ we have $a = x^2 + y$.
Since $(x, y) \in A_b$ we have $b = x^2 + y$.
It follows that $a = b$, a contradiction.
We conclude that Claim 3 is true. \hfill \square

3.3 \textbf{7(c)}. Let \(Q\) be the equivalence relation corresponding to the partition \(\mathcal{P}\). Then \(Q\) is a relation on the set \(\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}\), i.e. \(Q \subset \mathbb{R}^2 \times \mathbb{R}^2\). It is given by:

\[
Q = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \exists a \in \mathbb{R} : (x_1, y_1) \in A_a \text{ and } (x_2, y_2) \in A_a\}
= \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \exists a \in \mathbb{R} : y_1 + x_2^2 = a \text{ and } y_2 + x_1^2 = a\}
= \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid y_1 + x_2^2 = y_2 + x_1^2\}.
\]