**Solution to HW 8**

2.5 1(b).  
*Theorem:* \( \forall n \in \mathbb{N}: \ n > 33 \Rightarrow (\exists s, t \in \mathbb{Z} : s \geq 3 \land t \geq 2 \land n = 4s + 5t) \).  

*Proof.* Define the predicate  
\[ P(n) : n > 33 \Rightarrow (\exists s, t \in \mathbb{Z} : s \geq 3 \land t \geq 2 \land n = 4s + 5t) \]  
We must prove: \( \forall n \in \mathbb{N} : P(n) \).  
By the PCI, it is enough to show:  
\[ (*) \forall n \in \mathbb{N} : ( P(1) \land P(2) \land \cdots \land P(n-1) ) \Rightarrow P(n) \]  
Let \( n \in \mathbb{N} \).  
Assume \( P(1) \land P(2) \land \cdots \land P(n-1) \).  
We must show that \( P(n) \) is true.  
Assume that \( n > 33 \).  
We consider two cases.  
Case 1: Assume that \( 34 \leq n \leq 37 \).  
Set \( s = 40 - n \) and \( t = n - 32 \).  
Then \( s \geq 40 - 37 = 3 \) and \( t \geq 34 - 32 = 2 \).  
Furthermore, we have \( 4s + 5t = 4(40 - n) + 5(n - 32) = 5n - 4n + 4 \cdot 40 - 5 \cdot 32 = n \).  
It follows that \( P(n) \) is true.  
Case 2: Assume that \( n \geq 38 \).  
By assumption we know that \( P(n - 4) \) is true.  
Since \( P(n - 4) \) holds and \( n - 4 > 33 \), we may choose \( s, t \in \mathbb{Z} \) such that:  
\( s \geq 3 \) and \( t \geq 2 \) and \( n - 4 = 4s + 5t \).  
Set \( s' = s + 1 \) and \( t' = t \).  
Then we have \( s', t' \in \mathbb{Z} \), \( s' \geq 3 \), \( t' \geq 2 \), and \( 4s' + 5t' = (4s + 5t) + 4 = n \).  
It follows that \( P(n) \) is true.  
We deduce that \( (*) \) is true, hence the theorem is true by the PCI.  

2.5 2. Let \( a_1 = 2, a_2 = 4 \), and \( a_{n+2} = 5a_{n+1} - 6a_n \) for all \( n \geq 1 \).  
*Theorem:* \( \forall n \in \mathbb{N}: a_n = 2^n \).  

*Proof.* Define the predicate  
\[ P(n) : a_n = 2^n \]  
We must prove: \( \forall n \in \mathbb{N} : P(n) \).  
By the PCI, it is enough to show:  
\[ (*) \forall n \in \mathbb{N} : ( P(1) \land P(2) \land \cdots \land P(n-1) ) \Rightarrow P(n) \]  
Let \( n \in \mathbb{N} \).  
Assume \( P(1) \land P(2) \land \cdots \land P(n-1) \).  
We must show that \( P(n) \) is true.  
We consider 3 cases.  
Case 1: If \( n = 1 \), then \( a_n = 2 = 2^n \).  
Case 2: If \( n = 2 \), then \( a_n = 4 = 2^n \).  
Case 3: Assume that \( n \geq 3 \).  
Then \( P(n - 2) \) holds by assumption, so we have \( a_{n-2} = 2^{n-2} \).  
And \( P(n - 1) \) holds by assumption, so we have \( a_{n-1} = 2^{n-1} \).  
We therefore obtain:  
\[ a_n = 5a_{n-1} - 6a_{n-2} = 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2} = 5 \cdot 2^{n-1} - 3 \cdot 2^{n-1} = 2 \cdot 2^{n-1} = 2^n \]  
This shows that \( P(n) \) is true.  
We deduce that \( (*) \) is true, hence the theorem is true by the PCI.
2.5 4(b).
\[ f_1 = 1, \ f_2 = 1, \ f_3 = 2, \ f_4 = 3, \ f_5 = 5, \]
\[ f_6 = 8, \ f_7 = 13, \ f_8 = 21, \ f_9 = 34, \ f_{10} = 55. \]

2.5 5(b).
Theorem: \( \forall n \in \mathbb{N}: \gcd(f_n, f_{n+1}) = 1. \)

Proof. (i) Basis step:
For \( n = 1 \) we have \( \gcd(f_n, f_{n+1}) = \gcd(f_1, f_2) = \gcd(1, 1) = 1. \)

(ii) Inductive step: Let \( n \in \mathbb{N}. \)
Assume that \( \gcd(f_n, f_{n+1}) = 1. \)
Then we obtain
\[ \gcd(f_{n+1}, f_{n+2}) = \gcd(f_{n+1}, f_n + f_{n+1}) = \gcd(f_{n+1}, f_n) = 1. \]
(iii) Conclude by PMI: \( \forall n \in \mathbb{N}: \gcd(f_n, f_{n+1}) = 1. \)

2.5 8.
Theorem:
\( \forall a, b \in \mathbb{Z}: (a, b) \neq (0, 0) \Rightarrow \) (there is a smallest positive linear comb. of \( a \) and \( b \)).

Proof. Let \( a, b \in \mathbb{Z}. \)

Assume that \( (a, b) \neq (0, 0). \)
Consider the set of positive linear combinations of \( a \) and \( b: \)
\[ S = \{ n \in \mathbb{N} \mid \exists s, t \in \mathbb{Z}: n = sa + tb \}. \]
Since \( (a, b) \neq (0, 0), \) we must have \( a \neq 0 \) or \( b \neq 0. \)
It follows that \( |a| + |b| > 0, \) hence \( |a| + |b| \in \mathbb{N}. \)
Notice that \( |a| + |b| \) is a linear combination of \( a \) and \( b. \)
In fact, we may choose \( s \in \{ 1, -1 \} \) such that \( |a| = sa. \)
And we may choose \( t \in \{ 1, -1 \} \) such that \( |b| = tb. \)
Then we have \( |a| + |b| = sa + tb. \)
We deduce that \( |a| + |b| \in S. \)
This shows that \( S \) is not empty.
Since \( S \) is a non-empty subset of \( \mathbb{N}, \)
it follows from the WOP that \( S \) has a smallest element \( m. \)
This integer \( m \) is the smallest linear combination of \( a \) and \( b. \)

3.1 5(g,h).
Define the relations
\[ R = \{(1, 5), (2, 2), (3, 4), (5, 2)\}, \]
\[ S = \{(2, 4), (3, 4), (3, 1), (5, 5)\}, \]
and
\[ T = \{(1, 4), (3, 5), (4, 1)\}. \]
Then \( S \circ T = \{(3, 5)\} \) and \( R \circ (S \circ T) = \{(3, 2)\}. \)
And we have \( R \circ S = \{(3, 5), (5, 2)\} \) and \( (R \circ S) \circ T = \{(3, 2)\}. \)

3.1 9.
Let \( R \subset A \times B \) and \( S \subset B \times C \) be relations.
Then \( S \circ R \subset A \times C \) is a relation from \( A \) to \( C. \)

(a) Claim: \( \text{Dom}(S \circ R) \subset \text{Dom}(R). \)

Let \( x \in \text{Dom}(S \circ R). \)
By definition of the domain of a relation,
we may choose \( z \in C \) such that \( (x, z) \in S \circ R. \)
By definition of the composition of two relations,
we may choose \( y \in B \) such that \((x, y) \in R \) and \((y, z) \in S\).

Since \((x, y) \in R\), it follows that \(x \in \text{Dom}(R)\).

(b) Take \( A = B = C = \{1, 2\} \).

Set \( R = I_{\{1,2\}} = \{(1, 1), (2, 2)\} \) and \( S = \{(1, 1)\} \).

Then \( S \circ R = \{(1, 1)\} \).

We have \( \text{Dom}(S \circ R) = \{1\} \subsetneq \{1, 2\} = \text{Dom}(R) \).

(c) We always have \( \text{Rng}(S \circ R) \subset \text{Rng}(S) \).

The opposite inclusion is not true in the following example.

Take \( A = B = C = \{1, 2\} \).

Set \( R = \{(1, 1)\} \) and \( S = I_{\{1,2\}} = \{(1, 1), (2, 2)\} \).

Then \( S \circ R = \{(1, 1)\} \).

We have \( \text{Rng}(S \circ R) = \{1\} \subsetneq \{1, 2\} = \text{Rng}(S) \).