2.4 5(f).
Definition: Let \( \{A_i : i \in \mathbb{N}\} \) be an indexed family of sets.

Set \( U_1 = A_1 \).
For \( n \in \mathbb{N} \), set \( U_{n+1} = U_n \cup A_{n+1} \).
For \( n \in \mathbb{N} \), we define \( \bigcup_{i=1}^{n} A_i = U_n \).

2.4 5(g).
Definition: Let \( x_1, x_2, \ldots \) be a sequence of real numbers.

Set \( p_1 = x_1 \).
For \( n \in \mathbb{N} \), set \( p_{n+1} = p_n \cdot x_{n+1} \).
For \( n \in \mathbb{N} \), we define \( \prod_{i=1}^{n} x_i = p_n \).

2.4 6(c).
Theorem: \( \forall n \in \mathbb{N} : \sum_{i=1}^{n} 2^i = 2^{n+1} - 2 \).
Proof. (i) Basis step: For \( n = 1 \) we have \( \sum_{i=1}^{1} 2^i = 2 = 2^{n+1} - 2 \).
(ii) Inductive step: Let \( n \in \mathbb{N} \).
Assume that \( \sum_{i=1}^{n} 2^i = 2^{n+1} - 2 \).
[Note: We assume the identity for this specific \( n \).]
Then we obtain
\[
\sum_{i=1}^{n+1} 2^i = (\sum_{i=1}^{n} 2^i) + 2^{n+1} = (2^{n+1} - 2) + 2^{n+1} = 2 \cdot 2^{n+1} - 2 = 2^{n+1+1} - 2.
\]
[Note: And now we have proved that the identity holds for \( n+1 \).]
(iii) Conclude by PMI: \( \forall n \in \mathbb{N} : \sum_{i=1}^{n} 2^i = 2^{n+1} - 2 \).

2.4 6(e).
Theorem: \( \forall n \in \mathbb{N} : \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \).
Proof. (i) Basis step: For \( n = 1 \) we have \( \sum_{i=1}^{1} i^3 = 1 = \left( \frac{n(n+1)}{2} \right)^2 \).
(ii) Inductive step: Let \( n \in \mathbb{N} \).
Assume that \( \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \).
Then we obtain
\[
\sum_{i=1}^{n+1} i^3 = (\sum_{i=1}^{n} i^3) + (n+1)^3 = \left( \frac{n(n+1)}{2} \right)^2 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{n^2(n+1)^2}{4} + \frac{(4n+4)(n+1)^2}{4} = \frac{(n+1)(n+2)}{2} \cdot \frac{(n+1)(n+2)}{2}.
\]
(iii) Conclude by PMI: \( \forall n \in \mathbb{N} : \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \).
2.4 7(h).
Theorem: \( \forall n \in \mathbb{N}: 3^n \geq 1 + 2^n. \)

Proof. (i) Basis step: For \( n = 1 \) we have \( 3^1 = 3 = 1 + 2^1. \)
(ii) Inductive step: Let \( n \in \mathbb{N}. \)
Assume that \( 3^n \geq 1 + 2^n. \)
Then we obtain
\[
3^{n+1} = 3 \cdot 3^n \geq 3(1 + 2^n) = 3 + 3 \cdot 2^n \geq 1 + 2 \cdot 2^n = 1 + 2^{n+1}.
\]
(iii) Conclude by PMI: \( \forall n \in \mathbb{N}: 3^n \geq 1 + 2^n. \) \( \square \)

2.4 7(m).
Theorem: \( \forall n \in \mathbb{N}: \frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \in \mathbb{Z}. \)

Proof. (i) Basis step:
For \( n = 1 \) we have \( \frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} = \frac{1}{3} + \frac{1}{5} + \frac{7}{15} = \frac{15}{15} = 1, \) which is an integer.
(ii) Inductive step: Let \( n \in \mathbb{N}. \)
Assume that \( \frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \in \mathbb{Z}. \)
Then we obtain
\[
\left( \frac{n+1}{3} \right)^3 + \left( \frac{n+1}{5} \right)^5 + \frac{7(n+1)}{15}
= \frac{n^3 + 3n^2 + 3n + 1}{3^3} + \frac{n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1}{5^5} + \frac{7n + 7}{15}
= \frac{n^3}{3} + \frac{3n^2}{3} + \frac{3n}{3} + \frac{1}{3} + \frac{n^5}{5} + \frac{5n^4}{5} + \frac{10n^3}{5} + \frac{10n^2}{5} + \frac{5n}{5} + \frac{1}{5} + \frac{7n}{15} + \frac{7}{15}
= \left( \frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \right) + \left( \frac{3n^2}{3} + \frac{3n}{3} + \frac{5n^4}{5} + \frac{10n^3}{5} + \frac{10n^2}{5} + \frac{5n}{5} \right) + \left( \frac{1}{3} + \frac{1}{5} + \frac{7}{15} \right)
= \left( \frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \right) + (n^2 + n + n^4 + 2n^3 + 2n^2 + n) + 1.
\]
The first parenthesis is an integer by the induction hypothesis, and the second is an integer because \( n \) is an integer.
It follows that \( \frac{(n+1)^3}{3} + \frac{(n+1)^5}{5} + \frac{7(n+1)}{15} \in \mathbb{Z}. \)
(iii) Conclude by PMI: \( \forall n \in \mathbb{N}: \frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \in \mathbb{Z}. \) \( \square \)