2.2 9(g).
**Theorem:** \( \forall \text{ sets } A, B, C : (A \cup B) \cap C \subseteq A \cup (B \cap C) \)

**Proof.** Let \( A, B, C \) be sets.
Let \( x \in (A \cup B) \cap C \).
Then \( x \in A \cup B \) and \( x \in C \).
In particular, we have \( x \in A \) or \( x \in B \).
If \( x \in A \), then \( x \in A \cup (B \cap C) \).
If \( x \in B \), then \( x \in B \cap C \), hence \( x \in A \cup (B \cap C) \).
We conclude that \( x \in A \cup (B \cap C) \), as required. \( \square \)

2.2 10(d).
**Theorem:** \( \forall \text{ sets } A, B, C, D : (C \subseteq A \text{ and } D \subseteq B) \Rightarrow (D - A \subseteq B - C) \).

**Proof.** Let \( A, B, C, D \) be sets.
Assume that \( C \subseteq A \) and \( D \subseteq B \).
Let \( x \in D - A \).
Then \( x \in D \) and \( x \notin A \).
Since \( x \in D \) and \( D \subseteq B \), we have \( x \in B \).
Since \( x \notin A \) and \( C \subseteq A \), we have \( x \notin C \).
Therefore \( x \in B - C \). \( \square \)

2.2 11(d).
**Theorem:** Let \( A = \{1, 2\} \) and \( B = \{2\} \). Then \( P(A) - P(B) \nsubseteq P(A - B) \).

**Proof.** We have \( P(A) - P(B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} - \{\emptyset, \{2\}\} = \{\{1\}, \{1, 2\}\} \)
and \( P(A - B) = P(\{1\}) = \{\emptyset, \{1\}\} \).
It follows that \( P(A) - P(B) \nsubseteq P(A - B) \). \( \square \)

2.2 11(f).
**Theorem:** Let \( A = \{1, 2, 3\}, B = \{2, 3\}, C = \{3\} \). Then \( A - (B - C) \neq (A - B) - C \).

**Proof.** We have \( A - (B - C) = \{1, 2, 3\} - \{2\} = \{1, 3\} \) and \( (A - B) - C = \{1\} - \{3\} = \{1\} \), hence \( A - (B - C) \neq (A - B) - C \). \( \square \)

2.3 1(h). Set \( \Delta = (0, \infty) \). For \( r \in \Delta \), set \( A_r = [-\pi, r) \). Set \( A = \{A_r : r \in \Delta\} \).
**Theorem:** \( \bigcup_{r \in \Delta} A_r = [-\pi, \infty) \) and \( \bigcap_{r \in \Delta} A_r = [-\pi, 0] \).

**Proof.** The theorem is a consequence of the following four claims.
Claim 1: \( \bigcup_{r \in \Delta} A_r \subseteq [-\pi, \infty) \).
Let \( x \in \bigcup_{r \in \Delta} A_r \).
By definition of the union over \( A \), we may choose \( r \in \Delta \) s.t. \( x \in A_r = [-\pi, r) \).
Since \( [-\pi, r) \subseteq [-\pi, \infty) \), it follows that \( x \in [-\pi, \infty) \).
Claim 2: \( [-\pi, \infty) \subseteq \bigcup_{r \in \Delta} A_r \).
Let \( x \in [-\pi, \infty) \).
Set \( r = x + 4 \).
Then \( r \in \Delta \) and \( x \in A_r \).
It follows that \( x \in \bigcup_{r \in \Delta} A_r \).
Claim 3: \( \bigcap_{r \in \Delta} A_r \subseteq [-\pi, 0] \).
Let \( x \in \bigcap_{r \in \Delta} A_r \).
Then \( x \in A_1 = [-\pi, 1) \), so we must have \( x \geq -\pi \).
We prove by contradiction that $x \leq 0$.

Suppose that $x > 0$.

Set $r = x/2$.

Since $x \in A$, we obtain $x < x/2$, a contradiction.

We conclude that $-\pi \leq x \leq 0$, so $x \in [-\pi, 0]$.

Claim 4: $[-\pi, 0] \subset \bigcap_{r \in \Delta} A_r$.

Let $x \in [-\pi, 0]$.

We will show that: $\forall r \in \Delta : x \in A_r$.

Let $r \in \Delta$.

Then $-\pi \leq x \leq 0 < r$, so we have $x \in (-\pi, r) = A_r$.

It follows that $x \in \bigcap_{r \in \Delta} A_r$.

We conclude that $-\pi \leq x \leq 0$, so $x \in [-\pi, 0]$.

Claim 4: $[-\pi, 0] \subset \bigcap_{r \in \Delta} A_r$.

Let $x \in [-\pi, 0]$.

We will show that: $\forall r \in \Delta : x \in A_r$.

Let $r \in \Delta$.

Then $-\pi \leq x \leq 0 < r$, so we have $x \in (-\pi, r) = A_r$.

It follows that $x \in \bigcap_{r \in \Delta} A_r$.

$\square$

2.3 5(b). Let $A = \{ A_\alpha : \alpha \in \Delta \}$ be an indexed family of sets.

Theorem: $(\bigcup_{\alpha \in \Delta} A_\alpha)^c = \bigcap_{\alpha \in \Delta} A_\alpha^c$

Proof. Let $x$ be any element of the universe. [Notice that a universe must be given, since otherwise the complement of a set has no meaning.]

The following list of statements are equivalent:

$x \in (\bigcup_{\alpha \in \Delta} A_\alpha)^c$ $\iff$

$x \notin \bigcup_{\alpha \in \Delta} A_\alpha$ $\iff$

$\sim (\exists \alpha \in \Delta : x \in A_\alpha)$ $\iff$

$\forall \alpha \in \Delta : x \notin A_\alpha$ $\iff$

$\forall \alpha \in \Delta : x \in A_\alpha^c$ $\iff$

$x \in \bigcap_{\alpha \in \Delta} A_\alpha^c$.

Since $x$ was arbitrary, we conclude that $(\bigcup_{\alpha \in \Delta} A_\alpha)^c = \bigcap_{\alpha \in \Delta} A_\alpha^c$.

$\square$

2.3 12.

Theorem: For each $n \in \mathbb{N}$ set $A_n = (0, 1/n)$. Then we have:

(1) $\forall n \in \mathbb{N} : A_n \subset (0, 1)$.

(2) $\forall n, m \in \mathbb{N} : A_n \cap A_m \neq \emptyset$.

(3) $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$.

Proof of (1). Let $n \in \mathbb{N}$. Since $1/n \leq 1$, it follows that $A_n = (0, 1/n) \subset (0, 1)$.

Proof of (2). Let $n, m \in \mathbb{N}$.

Case 1: If $n \leq m$, then $A_m \subset A_n$, hence $A_n \cap A_m = A_m \neq \emptyset$.

Case 2: If $n > m$, then $A_n \subset A_m$, hence $A_n \cap A_m = A_n \neq \emptyset$.

Proof of (3). Assume that $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$.

Then we may choose $x \in \bigcap_{n \in \mathbb{N}} A_n$.

This implies: $\forall n \in \mathbb{N} : x \in A_n = (0, 1/n)$.

Since $x \in A_1$, we must have $0 < x < 1$.

Choose $n \in \mathbb{N}$ so that $n > 1/x$.

Since $x > 1/n$, it follows that $x \notin (0, 1/n) = A_n$.

This contradiction shows that our initial assumption was false.

We conclude that $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$.

$\square$

2.3 14. Let $A$ and $B$ be two families of pairwise disjoint sets. [Make sure to review exactly what a pairwise disjoint family is!]

Set $\mathcal{C} = A \cap B$ and $\mathcal{D} = A \cup B$.

Theorem (a): $\mathcal{C}$ is a pairwise disjoint family of sets.
Proof. Let $X, Y \in \mathcal{C}$. We must show that $X = Y$ or $X \cap Y = \emptyset$.

Since $\mathcal{C} \subset \mathcal{A}$, we have $X, Y \in \mathcal{A}$.

Since $\mathcal{A}$ is pairwise disjoint, we deduce that $X = Y$ or $X \cap Y = \emptyset$, as required. □

**Theorem (c):** $(\bigcup_{A \in \mathcal{A}} A) \cap (\bigcup_{B \in \mathcal{B}} B) = \emptyset \Rightarrow \mathcal{D}$ is pairwise disjoint.

Proof. Assume that $(\bigcup_{A \in \mathcal{A}} A) \cap (\bigcup_{B \in \mathcal{B}} B) = \emptyset$.

Let $X, Y \in \mathcal{D}$. We must show that $X = Y$ or $X \cap Y = \emptyset$.

Since $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$, we have $(X \in \mathcal{A}$ or $X \in \mathcal{B})$ and $(Y \in \mathcal{A}$ or $Y \in \mathcal{B}$).

Case 1: Assume that $X \in \mathcal{A}$ and $Y \in \mathcal{A}$.

Since $\mathcal{A}$ is pairwise disjoint, we must have $X = Y$ or $X \cap Y = \emptyset$.

Case 2: Assume that $X \in \mathcal{B}$ and $Y \in \mathcal{B}$.

Since $\mathcal{B}$ is pairwise disjoint, we must have $X = Y$ or $X \cap Y = \emptyset$.

Case 3: Assume that $X \in \mathcal{A}$ and $Y \in \mathcal{B}$.

Then $X \subset (\bigcup_{A \in \mathcal{A}} A)$ and $Y \subset (\bigcup_{B \in \mathcal{B}} B)$.

Since $(\bigcup_{A \in \mathcal{A}} A) \cap (\bigcup_{B \in \mathcal{B}} B) = \emptyset$, we deduce that $X \cap Y = \emptyset$.

Case 4: Assume that $X \in \mathcal{B}$ and $Y \in \mathcal{A}$.

By interchanging $X$ and $Y$, it follows from Case 3 that $X \cap Y = \emptyset$.

Since we have exhausted all possibilities, we conclude $X = Y$ or $X \cap Y = \emptyset$. □

**Theorem:** Let $\mathcal{A} = \{\{1\}\}$ and $\mathcal{B} = \{\{1, 2\}\}$, and set $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$. Then $\mathcal{A}$ and $\mathcal{B}$ are both pairwise disjoint families of sets, but $\mathcal{D}$ is not pairwise disjoint.

Proof. It follows directly from the definition that both $\mathcal{A}$ and $\mathcal{B}$ are pairwise disjoint families of sets.

We have $\mathcal{D} = \{\{1\}, \{1, 2\}\}$.

This family is not pairwise disjoint, since the members $X = \{1\}$ and $Y = \{1, 2\}$ of $\mathcal{D}$ do not satisfy that $X = Y$ or $X \cap Y = \emptyset$. □