POSITIVITY OF QUIVER COEFFICIENTS THROUGH THOM POLYNOMIALS

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1. Introduction

Let $(e_0, e_1, \dots, e_n)$ be a dimension vector of non-negative integers. The space $V = \mathrm{Hom}(\mathbb{C}^{e_0}, \mathbb{C}^{e_1}) \oplus \cdots \oplus \mathrm{Hom}(\mathbb{C}^{e_{n-1}}, \mathbb{C}^{e_n})$ of equioriented quiver representations of type A has a natural action of the group $G = \mathrm{GL}(e_0) \times \cdots \times \mathrm{GL}(e_n)$ given by $(g_0, \dots, g_n).(\phi_1, \dots, \phi_n) = (g_1 \phi_1 g_0^{-1}, \dots, g_n \phi_n g_{n-1}^{-1})$. An orbit r of this action is characterized by its set of *rank conditions* $\{r_{ij}\}$ for $0 \leq i < j \leq n$, where r_{ij} is the rank of the composed map $\phi_j \phi_{j-1} \cdots \phi_{i+1}$ for any point in this orbit. In this paper we study the G-equivariant cohomology class of the orbit closure $\overline{r}$. We will call this class for the *Thom polynomial* of the orbit, and we denote it by Tp_r.

This Thom polynomial can be regarded as a formula for the degeneracy locus obtained when the integers r_{ij} are used as rank conditions for a sequence of vector bundles and bundle maps (see e.g. [10] for the translation). Buch and Fulton gave a formula expressing the cohomology class of such a degeneracy locus as a linear combination of products of Schur determinants [4]. When interpreted for Thom polynomials, this formula has the form

$$\mathrm{Tp}_r = \sum_{\lambda} c_\lambda(r)\, s_{\lambda_1}(x^1; x^0)\, s_{\lambda_2}(x^2; x^1) \cdots s_{\lambda_n}(x^n; x^{n-1}) \tag{1}$$

where the sum is over certain sequences of partitions λ, and the symbol x^i denotes the Chern roots $\{x^i_1, \dots, x^i_{e_i}\}$ of the i'th factor of G. The *quiver coefficients* $c_\lambda(r)$ appearing in this formula are integers uniquely determined by (1) in addition to the condition that $c_\lambda(r) = c_\lambda(r + k)$ for all $k \geq 0$, where $r + k$ denotes the rank conditions $\{r_{ij} + k\}$ obtained by adding the integer k to the original rank conditions. Although the formula for quiver coefficients in [4] does not reveal their signs, it was conjectured that all quiver coefficients are non-negative.

Fehér and Rimányi suggested a different method for computing Thom polynomials in [7, 9], which works more generally for all quiver representations associated to Dynkin diagrams. In this approach, the Thom polynomial Tp_r is obtained as the unique solution to a system of linear equations, which say that the Thom polynomial must vanish when restricted to orbits disjoint from the closure of r.

The G-orbits in the representation space V were first classified by Abeasis and Del Fra using *lace diagrams* [1]. An important idea in recent work of Knutson, Miller, and Shimozono [13] was to reinterpret these lace diagrams as sequences

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of permutations, which can be identified with the components of a *Gröbner degeneration* of the orbit closure. In a talk about this work given by E. Miller at the Boston AMS-meeting in October 2002, the following *component formula* was conjectured, which expresses the Thom polynomial Tp_r as a sum of products of Schubert polynomials:

$$\mathrm{Tp}_r = \sum_{(w_1,\dots,w_n)} \mathfrak{S}_{w_1}(x^1;x^0)\,\mathfrak{S}_{w_1}(x^2;x^1)\cdots\mathfrak{S}_{w_n}(x^n;x^{n-1})\,.$$

This sum is over all *minimal* lace diagrams, whose definition is recalled in section 2. This conjecture was subsequently proved independently by the authors of [13] and the third author of the present paper. The main goal of this paper is to present the Hungarian approach, which consists of simply verifying that the component formula satisfies the required equations for being a Thom polynomial.

The component formula also has a stable variant, where the Schubert polynomials are replaced with *Stanley symmetric functions*. This version of the formula was first proved in [13]. Since Stanley symmetric functions are Schur positive [6, 16], the stable component formula implies that quiver coefficients are non-negative. In this paper we give a simple argument that the two versions of the component formula are equivalent, thus obtaining a short proof of the non-negativity of quiver coefficients based on Thom polynomial theory. In comparison, the proof of the component formulas given in [13] relies on two different geometric constructions, one of which is the above mentioned Gröbner degeneration, and the other being a *ratio formula* derived from a geometric study of the Zelevinsky map [19, 14].

Part of our verification of the component formula consists of proving that this formula is symmetric in each set of variables x^i. This argument can also be turned around to show that a linear combination of products of Schubert polynomials over minimal lace diagrams is symmetric if and only if all coefficients are equal. This makes it possible to prove the component formula based on the Gröbner degeneration of [13] combined with some ideas from Thom polynomial theory, at least up to a constant, which can then be determined by applying the original quiver formula from [4]. We explain this alternative proof in section 4.

We remark that the component formula can also be proved combinatorially [18, 3] from the ratio formula of [13]. In fact, among the four geometric approaches to quiver formulas currently known to us (which are resolution of singularities [4], equations coming from restrictions [9], the Zelevinsky map, and degeneration [13]), only the original approach of [4] offers no easy path to positivity of quiver coefficients. On the other hand, given the central role played by quiver coefficients in [4], this approach arguably makes the question of positivity more natural to ask.

The component formula also has a K-theory variant [3, 17], which implies that the K-theoretic quiver coefficients defined in [2] have alternating signs. This formula expresses the structure sheaf of a quiver variety as an alternating sum of products of Grothendieck polynomials indexed by *KMS-factorizations*, which generalize minimal lace diagrams. In the last section we apply the methods of this paper to give a new description of KMS-factorizations based on transformations of lace diagrams.

This paper is organized as follows. In section 2 we explain basic notions like minimal lace diagrams and Schubert polynomials, and we prove that the component formula is symmetric and equivalent to the stable component formula. In section 3 we prove the component formula using Thom polynomial theory, while section 4 contains the alternative proof based on the Gröbner degeneration of [13]. Section 5 finally contains the classification of KMS-factorizations.

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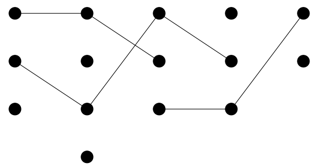
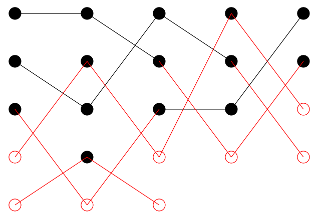
2. The component formula

A *lace diagram* for the dimension vector $(e_0, \dots, e_n)$ is a diagram of dots arranged in columns, with e_i dots in column i, together with line segments connecting dots of consecutive columns. Each dot may be connected to at most one dot in the column to the left of it, and to at most one dot in the column to the right of it. The corresponding orbit r satisfies that the rank condition r_{ij} is the number of connections from column i to column j [1]. For example, the following lace diagram represents an orbit r of quiver representations through 5 vector spaces of dimensions $(e_0, \dots, e_4) = (3, 4, 3, 3, 2)$, and we have $r_{01} = r_{02} = 2$, $r_{03} = 1$, etc.



A lace diagram may be identified with a sequence $(w_1, \dots, w_n)$ of permutations [13] (see also [11]). Here we let w_i be the permutation of minimal length such that $w_i(q) = p$ whenever dot q of column i is connected to dot p of column $i-1$. Equivalently, this permutation describes the connections from the i'th to the $i-1$'st column of an *extension* of the lace diagram. This extended diagram is constructed by adding extra dots to the columns, so that the original dots without connections to both sides can be connected to the new dots. For example, the above lace diagram is extended as follows; in particular we have $w_2 = 31524$.



Notice that a sequence $(w_1, \dots, w_n)$ of permutations represent a lace diagram for the dimension vector $(e_0, \dots, e_n)$ if and only if each permutation w_i is a *partial permutation* from e_i elements to e_{i-1} elements, which means that all descent positions of w_i are smaller than or equal to e_i, and all descent positions of w_i^{-1} are smaller than or equal to e_{i-1}.

A *strand* of a lace diagram is a maximal sequence of connected dots and line segments, and the *extension* of a strand is obtained by also including the extra line segments that it is directly connected to in the extended lace diagram.

The *length* of a lace diagram is the sum of the lengths of the permutations w_i, or equivalently the total number of crossings in the extended lace diagram. The smallest possible length of a lace diagram representing an orbit r is equal to the codimension $d(r)$ of the orbit. In fact, the codimension is given by $d(r) = \sum_{i<j}(r_{i,j-1} - r_{ij})(r_{i+1,j} - r_{ij})$, so this follows because (the extensions of) all of the

$r_{i+1,j} - r_{ij}$ strands starting in column $i+1$ and passing through column j must intersect all of the $r_{i,j-1} - r_{ij}$ strands passing through column i and terminating in column $j-1$. The lace diagram is called *minimal* if its length is equal to $d(r)$. This is equivalent to demanding that (the extensions of) any two strands can cross at most once, and not at all if they start or end at the same column [13, Thm. 3.8].

To state the component formula, we also need the Schubert polynomials of Lascoux and Schützenberger [16]. The *divided difference operator* $\partial_{a,b}$ with respect to two variables a and b is defined by

$$\partial_{a,b}(f) = \frac{f(a,b) - f(b,a)}{a-b}\,,$$

where f is any polynomial in these (and possibly other) variables. The *double Schubert polynomials* $\mathfrak{S}_w(x;y) = \mathfrak{S}_w(x_1,\dots,x_m;y_1,\dots,y_m)$ given by permutations $w \in S_m$ are uniquely determined by the identity

$$\partial_{x_i,x_{i+1}}(\mathfrak{S}_w(x;y)) = \begin{cases} \mathfrak{S}_{ws_i}(x;y) & \text{if } w(i) > w(i+1) \\ 0 & \text{if } w(i) < w(i+1) \end{cases} \tag{2}$$

together with the expression

$$\mathfrak{S}_{w_0}(x;y) = \prod_{i+j\leq m} (x_i - y_j)$$

for the longest permutation w_0 in S_m. Using that $\mathfrak{S}_w(y;x) = (-1)^{\ell(w)}\mathfrak{S}_{w^{-1}}(x;y)$ we similarly have that $\partial_{y_i,y_{i+1}}(\mathfrak{S}_w(x;y))$ is equal to $-\mathfrak{S}_{s_i w}(x;y)$ if $\ell(s_i w) < \ell(w)$, and is zero otherwise. If k and l are the last descent positions of w and w^{-1}, respectively, then only the variables $x_1,\dots,x_k,y_1,\dots,y_l$ occur in $\mathfrak{S}_w(x;y)$.

The component formula can now be stated as follows. Recall that the variables x^i_j are the Chern roots of the group G of the introduction.

Theorem 1. *The Thom polynomial of a G-orbit r is given by*

$$\mathrm{Tp}_r = \sum_{(w_1,\dots,w_n)} \mathfrak{S}_{w_1}(x^1;x^0)\,\mathfrak{S}_{w_2}(x^2;x^1)\cdots\mathfrak{S}_{w_n}(x^n;x^{n-1})$$

where the sum is over all minimal lace diagrams representing the orbit.

Since the Thom polynomial is a G-equivariant class, it follows from this theorem that the sum on its right hand side is symmetric in each set of variables x^i. This can also be proved directly. We let

$$Q_r = \sum_{(w_1,\dots,w_n)} \mathfrak{S}_{w_1}(x^1;x^0)\,\mathfrak{S}_{w_2}(x^2;x^1)\cdots\mathfrak{S}_{w_n}(x^n;x^{n-1}) \tag{3}$$

denote the polynomial of the component formula.

Lemma 1. *The polynomial Q_r is symmetric in each set of variables x^i.*

Proof. We must show that for any $0 \leq i \leq n$ and $1 \leq j < e_i$, the divided difference operator $\partial^i_j = \partial_{x^i_j,x^i_{j+1}}$ maps Q_r to zero. Notice at first that any minimal lace diagram $(w_1,\dots,w_n)$ must satisfy that $\ell(s_k w_1) > \ell(w_1)$ for $k < e_0$ and $\ell(w_n s_k) > \ell(w_n)$ for $k < e_n$. Using (2) this implies that $\partial^i_j(Q_r) = 0$ for $i = 0$ or $i = n$.

Given any sequence of permutations $(w_1,\dots,w_n)$ we write $\mathfrak{S}(w_1,\dots,w_n) = \prod \mathfrak{S}_{w_i}(x^i;x^{i-1})$ for the corresponding product of Schubert polynomials. Now suppose that $1 \leq i \leq n-1$ and let $(w_1,\dots,w_n)$ be a minimal lace diagram for r. There are four cases to consider:

(i) $w_i(j) < w_i(j+1)$ and $w_{i+1}^{-1}(j) < w_{i+1}^{-1}(j+1)$. We get $\partial_j^i(\mathfrak{S}(w_1, \dots, w_n)) = 0$.

(ii) $w_i(j) < w_i(j+1)$ and $w_{i+1}^{-1}(j) > w_{i+1}^{-1}(j+1)$. We get $\partial_j^i(\mathfrak{S}(w_1, \dots, w_n)) = -\mathfrak{S}(w_1, \dots, w_i, s_j w_{i+1}, \dots, w_n)$.

(iii) $w_i(j) > w_i(j+1)$ and $w_{i+1}^{-1}(j) < w_{i+1}^{-1}(j+1)$. We get $\partial_j^i(\mathfrak{S}(w_1, \dots, w_n)) = \mathfrak{S}(w_1, \dots, w_i s_j, w_{i+1}, \dots, w_n)$.

(iv) $w_i(j) > w_i(j+1)$ and $w_{i+1}^{-1}(j) > w_{i+1}^{-1}(j+1)$. This is impossible since $(w_1, \dots, w_i s_j, s_j w_{i+1}, \dots, w_n)$ would be a shorter lace diagram for the orbit r.

Notice that if our minimal lace diagram $\mathbf{w} = (w_1, \dots, w_n)$ falls in one of the cases (ii) or (iii), then the sequence $\mathbf{w}' = (w_1, \dots, w_i s_j, s_j w_{i+1}, \dots, w_n)$ is also a minimal lace diagram for r. For example, if $w_i(j) > w_i(j+1)$ then since two crossing strands cannot both terminate at column i, we must have $w_{i+1}^{-1}(j) \leq e_{i+1}$, which implies that $\mathbf{w}'$ is also a lace diagram. Since $\partial_j^i(\mathfrak{S}(\mathbf{w}) + \mathfrak{S}(\mathbf{w}')) = 0$, we conclude that $\partial_j^i(Q_r) = 0$ as required. □

The *double Stanley symmetric function* F_w for a permutation w is defined by

$$F_w(x_1, \dots, x_p; y_1, \dots, y_q) = \mathfrak{S}_{1^k \times w}(x_1, \dots, x_p, 0, \dots, 0; y_1, \dots, y_q, 0, \dots, 0)\,,$$

where k is any integer larger than p and q, and the shifted permutation $1^k \times w$ acts as the identity on the set $\{1, \dots, k\}$, and maps $k+j$ to $k+w(j)$ for $j \geq 1$. We also need the identity

$$\mathfrak{S}_{1^k \times w}(0^k, x_1, \dots, x_m; 0^k, y_1, \dots, y_m) = \mathfrak{S}_w(x_1, \dots, x_m; y_1, \dots, y_m) \tag{4}$$

where 0^k denotes k zeros. This identity is proved in [5, Cor. 4].

The following consequence of Lemma 1 shows that Theorem 1 is equivalent to the stable component formula, which states that the Thom polynomial Tp_r equals the sum of products of Stanley symmetric functions in the corollary. By the Schur positivity of Stanley symmetric functions [6, 16], this formula implies that quiver coefficients are non-negative. The statement of the corollary was first proved in [13] using a combination of geometry and combinatorics.

Corollary (Knutson, Miller, Shimozono)**.** *For any orbit r we have*

$$Q_r = \sum_{(w_1, \dots, w_n)} F_{w_1}(x^1; x^0)\, F_{w_2}(x^2; x^1) \cdots F_{w_n}(x^n; x^{n-1})$$

where the sum is over all minimal lace diagrams for r.

Proof. Let $r+k$ be denote the orbit corresponding to the dimension vector $(e_0 + k, \dots, e_n + k)$ and rank conditions $\{r_{ij} + k\}$. The above discussion of lace diagrams implies that the minimal lace diagrams for $r+k$ are exactly those obtained by adding k strands of length n to the top of a minimal lace diagram for r [13, Cor. 4.12]. Equivalently, such a diagram is given by a sequence of permutations $(1^k \times w_1, \dots, 1^k \times w_n)$, for which $(w_1, \dots, w_n)$ is a minimal lace diagram for r. For $k \geq \max(e_0, \dots, e_n)$, the symmetry of the polynomial Q_{r+k} therefore implies that

$$\begin{aligned}\sum_{\mathbf{w}} \prod_{i=1}^n \mathfrak{S}_{w_i}(x^i; x^{i-1}) &= \sum_{\mathbf{w}} \prod_{i=1}^n \mathfrak{S}_{1^k \times w_i}(0^k, x^i\,;\, 0^k, x^{i-1}) \\ &= \sum_{\mathbf{w}} \prod_{i=1}^n \mathfrak{S}_{1^k \times w_i}(x^i, 0^k\,;\, x^{i-1}, 0^k) = \sum_{\mathbf{w}} \prod_{i=1}^n F_{w_i}(x^i; x^{i-1})\end{aligned}$$

where the sums are over all minimal lace diagrams $\mathbf{w} = (w_1, \dots, w_n)$ for r. □

We remark that the first equality in the above proof can also be deduced from Theorem 1 together with the property $c_\lambda(r) = c_\lambda(r+k)$ of quiver coefficients. Since the proof of Theorem 1 using Thom polynomial theory in section 3 does not rely on the corollary, one can therefore prove that quiver coefficients are non-negative without relying on (4). However, the alternative proof of the component formula in section 4 does rely on the corollary, which makes the given combinatorial proof preferable.

3. Proof using Thom polynomials

Let G be a complex Lie group acting on a vector space V with finitely many orbits. An orbit μ of complex codimension d defines a G-equivariant cohomology class $\mathrm{Tp}_\mu = [\overline{\mu}] \in H^{2d}_G(V) = H^{2d}(BG;\mathbb{Z})$ called its *Thom polynomial*. When μ is an orbit of a space of quiver representations as in the introduction, this Thom polynomial is equivalent to the quiver formula (1).

We let G_μ denote the stabilizer subgroup of a point p_μ in μ. The inclusion of G_μ into G induces a map $BG_\mu \to BG$ between the classifying spaces, which gives a ring homomorphism $\phi_\mu : H^*(BG) \to H^*(BG_\mu)$ on cohomology. The Euler class $e(\mu) \in H^*_G(\mu) = H^*(BG_\mu)$ is the top G-equivariant Chern class of the normal bundle to μ in V.

In [7] a general theory for computing Thom polynomials is developed. The special case of this theory that is needed here is summarized in the following theorem.

Theorem 2. *Let μ and η be orbits of a G-representation with finitely many orbits.*
(i) *If $\mu \not\subset \overline{\eta}$ then $\phi_\mu(\mathrm{Tp}_\eta) = 0$;*
(ii) $\phi_\eta(\mathrm{Tp}_\eta) = e(\eta)$.
Furthermore, if for every orbit μ the Euler class $e(\mu)$ is not a zero-divisor in $H^*(BG_\mu)$, then Tp_η is uniquely determined by these conditions.

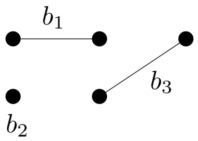
For the application to quiver formulas that concerns us here, we use the group $G = \prod_{i=0}^n \mathrm{GL}(e_i)$ with its usual quiver action on $V = \bigoplus_{i=1}^n \mathrm{Hom}(\mathbb{C}^{e_{i-1}}, \mathbb{C}^{e_i})$. In this case the cohomology ring $H^*(BG)$ is the ring of polynomials in the Chern roots x^i_j, that are symmetric in each group of variables $x^i = \{x^i_1, \dots, x^i_{e_i}\}$:

$$H^*(BG) = \mathbb{Z}[x^i_j \mid 0 \le i \le n, 1 \le j \le e_i]^{\prod S_{e_i}}$$

In [9] a combinatorial description of the cohomology ring $H^*(BG_\mu)$, the restriction map ϕ_μ, and the Euler class $e(\mu)$ was given, which works for representations of any quiver that is shaped like a Dynkin diagram. In our case of equioriented quivers of type A, this works as follows (see [9, §4–5]).

Let $r \subset V$ be an orbit, and fix a lace diagram $\mathbf{w}$ representing r. Choose variables $b_1, \dots, b_k$ corresponding to the strands of $\mathbf{w}$. Then $H^*(BG_r)$ can be identified with a subring of the polynomial ring $\mathbb{Z}[b_1, \dots, b_k]$, and $\phi_r : H^*(BG) \to H^*(BG_r)$ maps each variable x^i_j to the variable of the strand passing through dot j of column i in $\mathbf{w}$. We notice that this description makes it possible to extend ϕ_r to a map on all polynomials in the Chern roots x^i_j. This extended map depends on the chosen lace diagram, and is denoted by $\phi_{\mathbf{w}}$. Finally, if $\mathbf{w}$ is a minimal lace diagram, then the Euler class $e(r) \in H^*(BG_r)$ is the product of all differences $(b_p - b_q)$ of variables for which the extensions of the corresponding strands cross in $\mathbf{w}$; here the strand of b_p should have the higher slope at the crossing point. In particular, the Euler class $e(r)$ is not a zero-divisor, so Theorem 2 applies.

Example 1. In the following minimal lace diagram, the strands have been labeled with the associated variables.



If r denotes the corresponding orbit, then $H^*(BG_r) = \mathbb{Z}[b_1, b_2, b_3]$, and the map $\phi_r : H^*(BG) \to H^*(BG_r)$ is given by $\phi_r(x_1^0) = \phi_r(x_1^1) = b_1$, $\phi_r(x_2^0) = b_2$, and $\phi_r(x_2^1) = \phi_r(x_1^2) = b_3$. Finally we have $e(r) = (b_3 - b_1)(b_3 - b_2) \in H^*(BG_r)$.

Let $u, w \in S_m$ be permutations. Our proof of the component formula uses that the specialization $\mathfrak{S}_w(b_u; b) = \mathfrak{S}_w(b_{u(1)}, \dots, b_{u(m)}; b_1, \dots, b_m)$ is zero unless $w \leq u$ in the Bruhat order on S_m, and for $u = w$ we have

$$\mathfrak{S}_u(b_{u(1)}, \dots, b_{u(m)}; b_1, \dots, b_m) = \prod_{i<j\,:\,u(i)>u(j)} (b_{u(i)} - b_{u(j)})\,. \tag{5}$$

These statements follow by descending induction on $\ell(w)$ from the identity

$$(b_{u(i+1)} - b_{u(i)})\,\mathfrak{S}_w(b_u; b) = \mathfrak{S}_{ws_i}(b_{us_i}; b) - \mathfrak{S}_{ws_i}(b_u; b)$$

which holds whenever $w(i) < w(i+1)$. The vanishing statement is part of Goldin's characterization of the Bruhat order [12], and both statements can also be deduced from Theorem 2 applied to the representation studied in [8, §4]. More general formulas for specializations of Schubert polynomials are proved in [5].

Proof of Theorem 1 using Thom polynomial theory. Lemma 1 shows that the polynomial Q_r of (3) is an element of $H^*(BG)$. We must show that Q_r satisfies the requirements (i) and (ii) of Theorem 2.

As in the proof of Lemma 1 we set $\mathfrak{S}(w_1, \dots, w_n) = \prod \mathfrak{S}_{w_i}(x^i; x^{i-1})$. Notice that if $\mathbf{u} = (u_1, \dots, u_n)$ is any lace diagram, then $\phi_{\mathbf{u}}(\mathfrak{S}(w_1, \dots, w_n))$ is zero unless $w_i \leq u_i$ in the Bruhat order for all i. In fact, if $b_1, \dots, b_m$ are the variables of the strands through column $i-1$ in the extended lace diagram for $\mathbf{u}$, ordered from top to bottom, then $\phi_{\mathbf{u}}$ maps the i'th factor of $\mathfrak{S}(w_1, \dots, w_n)$ to $\mathfrak{S}_{w_i}(b_{u_i}; b)$.

Now suppose that $s \subset V$ is an orbit which is not contained in the closure of r. This implies that $s_{ij} > r_{ij}$ for some $0 \leq i < j \leq n$, see [1, §3]. Choose a lace diagram $\mathbf{u} = (u_1, \dots, u_n)$ for s such that $u_k(p) = p$ for all $i < k \leq j$ and $1 \leq p \leq s_{ij}$. Since no lace diagram $\mathbf{w} = (w_1, \dots, w_n)$ for r can satisfy these requirements, some w_k is not dominated by u_k in the Bruhat order, which implies that $\phi_{\mathbf{u}}(\mathfrak{S}(\mathbf{w})) = 0$. We therefore get $\phi_s(Q_r) = \phi_{\mathbf{u}}(Q_r) = 0$ which proves (i).

For (ii), let $\mathbf{u}$ be a fixed minimal lace diagram for r. If $\mathbf{w}$ is any minimal lace diagram for this orbit such that $\phi_{\mathbf{u}}(\mathfrak{S}(\mathbf{w})) \neq 0$, then since $w_i \leq u_i$ for all i we must have $\mathbf{w} = \mathbf{u}$. It therefore suffices to show that $\phi_{\mathbf{u}}(\mathfrak{S}(\mathbf{u})) = e(r)$, which follows from (5) because $\phi_{\mathbf{u}}$ maps each factor $\mathfrak{S}_{u_i}(x^i; x^{i-1})$ to the product of the differences $(b_p - b_q)$ corresponding to strands of $\mathbf{u}$ that cross between column $i-1$ and column i. This finishes the proof. $\square$

4. Proof using Gröbner degeneration

In [13] the closure of an orbit r in the space of quiver representations V was degenerated into a union of products of matrix Schubert varieties. As a consequence of this, it was proved [13, Cor. 4.9] that the Thom polynomial can be written as a non-negative integral linear combination

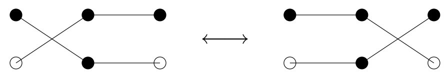
$$\mathrm{Tp}_r = \sum_{\mathbf{w}} c_{\mathbf{w}}\, \mathfrak{S}_{w_1}(x^1; x^0)\, \mathfrak{S}_{w_2}(x^2; x^1) \cdots \mathfrak{S}_{w_n}(x^n; x^{n-1}) \tag{6}$$

of products of Schubert polynomials for lace diagrams $\mathbf{w} = (w_1, \dots, w_n)$ of length equal to the codimension $d(r)$. The degeneration argument also shows that $c_{\mathbf{w}}$ is non-zero when $\mathbf{w}$ is a minimal lace diagram for r, but we will not need this fact. If $\mathbf{w}$ is not a minimal lace diagram for r, then $\mathbf{w}$ represents an orbit that is disjoint from the closure of r, and this implies that $c_{\mathbf{w}} \phi_{\mathbf{w}}(\prod \mathfrak{S}_{w_i}(x^i; x^{i-1})) = \phi_{\mathbf{w}}(\mathrm{Tp}_r) = 0$ by Theorem 2 (i). The first equality holds because all other terms of Tp_r are mapped to zero by $\phi_{\mathbf{w}}$, as explained in part (ii) of the Thom polynomial proof. We conclude by (5) that $c_{\mathbf{w}} = 0$, so (6) writes the Thom polynomial as a linear combination over the minimal lace diagrams for r.

In this section we give a new proof of the component formula based on this fact. The crucial combinatorial observation is that the linear combination (6) can be symmetric only if all coefficients are equal. Notice that our argument showing that only minimal lace diagrams for r occur in (6) does not use the uniqueness statement of Theorem 2, which is the non-trivial part of this theorem. However, it would be interesting to find an argument based purely on degeneration.

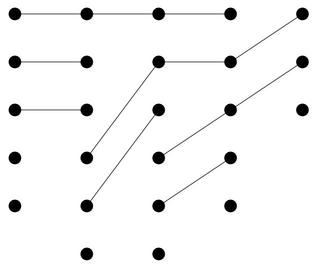
Let $P = \sum c_{\mathbf{w}} \mathfrak{S}(\mathbf{w})$ be an arbitrary linear combination of products $\mathfrak{S}(\mathbf{w}) = \prod \mathfrak{S}_{w_i}(x^i; x^{i-1})$ for the minimal lace diagrams $\mathbf{w}$ representing r. Recall from the proof of Lemma 1 that if a divided difference operator ∂^i_j is evaluated on P, with $1 \leq i \leq n-1$ and $1 \leq j < e_i$, then the result is a linear combination of products $\mathfrak{S}(\mathbf{u})$ for lace diagrams $\mathbf{u} = (u_1, \dots, u_n)$, such that $u_i(j) < u_i(j+1)$ and $u_{i+1}^{-1}(j) < u_{i+1}^{-1}(j+1)$. Furthermore, the coefficient of $\mathfrak{S}(\mathbf{u})$ is equal to $c_{\mathbf{u}'} - c_{\mathbf{u}''}$, where $\mathbf{u}' = (u_1, \dots, u_i s_j, u_{i+1}, \dots, u_n)$ and $\mathbf{u}'' = (u_1, \dots, u_i, s_j u_{i+1}, \dots, u_n)$. It follows that if P is symmetric in all groups of variables x^i, then for any minimal lace diagram $\mathbf{w} = (w_1, \dots, w_n)$ such that $w_i(j) > w_i(j+1)$ or $w_{i+1}^{-1}(j) > w_{i+1}^{-1}(j+1)$ we have $c_{\mathbf{w}} = c_{\mathbf{w}'}$ where $\mathbf{w}' = (w_1, \dots, w_i s_j, s_j w_{i+1}, \dots, w_n)$. The transformation between $\mathbf{w}$ and $\mathbf{w}'$ is illustrated by the following picture (of parts of the extended lace diagrams):

(7)



Notice that this transformation can be applied to any lace diagram, as long as the upper and middle dots are not in the extended part of the diagram.

We define the *left-most lace diagram* for the orbit r as follows. Start with an empty diagram (with zero dots in each column). Then for each $i = 0, 1, \dots, n$, and each $j = n, n-1, \dots, i$ (in this order) we add $r_{ij} - r_{i-1,j} - r_{i,j+1} + r_{i-1,j+1}$ strands starting at column i and terminating at column j to the bottom of the diagram. Notice that any left-most diagram is also minimal. The following picture shows an example of a left-most lace diagram.



Proposition 1. (i) *Two minimal lace diagrams are connected via the transformations (7) if and only if they represent the same orbit.*

(ii) *A linear combination $P = \sum c_{\mathbf{w}} \mathfrak{S}(\mathbf{w})$ over minimal lace diagrams $\mathbf{w}$ for an orbit r is symmetric in each group of variables x^i if and only if all coefficients $c_{\mathbf{w}}$ are equal.*

Proof. It is enough to show that any minimal lace diagram can be converted to a left-most lace diagram using the transformations (7). We give an explicit algorithm for doing this.

Consider the strand which starts at the top dot of column 0 in the lace diagram. If this strand is not entirely in the top row of the diagram, we let i be the first column where the strand contains a dot below the top row, and $k > 1$ the row number of this dot. The line segment to this dot from the top dot of column $i-1$ must then cross the strand going through dot $k-1$ of column i, so these strands do not cross between column i and column $i+1$. Furthermore, since these strands cannot terminate in the same column, the strand through dot $k-1$ of column i continues to a dot of column $i+1$ which is not in the extended part of the diagram. We can therefore use a transformation (7) to move the crossing one step to the right; in the resulting diagram, the strand starting at the top dot of the first column will now contain the $k-1$'st dot of column i. By continuing to apply this method, we eventually reach a lace diagram in which the strand starting at the top dot of column 0 is entirely in the top row. The same procedure is now carried out for the remaining strands that start at the first column, from top to bottom, then the strands starting at the second column, and so on; for each of these strands one ignores the previous strands which have already been moved to the correct position. Finally, notice that since strands starting in the same column cannot cross each other, this algorithm will deal with the strands in the same order as they are added when a left-most lace diagram is constructed. We conclude that the resulting lace diagram is left-most. □

Proof of Theorem 1 using Gröbner degeneration. By (6) and part (ii) of Proposition 1, the Thom polynomial Tp_r is equal to a non-negative integer c times the polynomial Q_r. By the corollary to Lemma 1 this says that

$$\mathrm{Tp}_r = c \sum_{\mathbf{w}} \prod_{i=1}^{n} F_{w_i}(x^i; x^{i-1})$$

where the sum is over all minimal lace diagrams for r. Since each Stanley symmetric function F_{w_i} is an integral linear combination of Schur polynomials, it follows that c must divide all the quiver coefficients $c_\lambda(r)$ for the orbit r. To show that $c = 1$ it is therefore enough to find a quiver coefficient equal to 1.

This can be done explicitly as follows. For all $0 \leq i < j \leq n$ we let R_{ij} be a rectangular partition with $r_{i+1,j} - r_{ij}$ rows and $r_{i,j-1} - r_{ij}$ columns, and we let λ_i be the Young diagram obtained by arranging the rectangles $R_{i-1,j}$ for $i \leq j \leq n$ side by side from left to right.

$$\lambda_i \;=\; \boxed{R_{i-1,i}}\,\boxed{R_{i-1,i+1}}\;\cdots\;\boxed{R_{i-1,n}}$$

It then follows from the algorithm of [4, §2.1] that $c_\lambda(r) = 1$ for the sequence of partitions $\lambda = (\lambda_1, \dots, \lambda_n)$. □

Remark. (a) With slightly more care, one can use the transformations (7) to prove that a linear combination $\sum c_{\mathbf{w}} \mathfrak{S}(\mathbf{w})$ over all lace diagrams $\mathbf{w}$ for a given dimension vector is symmetric in each set of variables x^i if and only if all coefficients corresponding to non-minimal lace diagrams are zero, and coefficients for minimal lace diagrams representing the same orbit are equal.

(b) M. Shimozono reports that the moves of (7) can also be used to prove that the components of the Gröbner degeneration of an orbit closure intersect in codimension two or higher.

5. Grothendieck classes of quiver varieties

In [2] a formula for the Grothendieck class of a quiver variety was proved, which generalizes (1). This formula can be interpreted as an expression for the structure sheaf $\mathcal{O}_{\overline{r}}$ of an orbit closure $\overline{r}$ in the torus-equivariant Grothendieck ring of the representation space V [13, 10]. It has the form

$$[\mathcal{O}_{\overline{r}}] = \sum_\lambda c_\lambda(r)\, G_{\lambda_1}(x^1; x^0)\, G_{\lambda_2}(x^2; x^1) \cdots G_{\lambda_n}(x^n; x^{n-1}) \tag{8}$$

where G_{λ_i} denotes the double stable (Laurent) Grothendieck polynomial for the partition λ_i (see [3, §2] for notation). The sequences λ of partitions in this formula all satisfy that the sum $\sum |\lambda_i|$ of the weights is greater than or equal to the expected codimension $d(r)$. The cohomological quiver coefficients of (1) are the subset of the coefficients $c_\lambda(r)$ in (8) for which $\sum |\lambda_i| = d(r)$. It was conjectured in [2] that the K-theoretic quiver coefficients have signs which alternate with codimension, that is $(-1)^{\sum |\lambda_i| - d(r)}\, c_\lambda(r) \geq 0$.

This conjecture was proved in [3] by giving K-theoretic generalizations of the component formulas. E. Miller has found a different proof of the stable variant [17]. The K-theoretic component formula has the form

$$[\mathcal{O}_{\overline{r}}] = \sum_{\mathbf{w}} (-1)^{\sum \ell(w_i) - d(r)}\, \mathfrak{G}_{w_1}(x^1; x^0)\, \mathfrak{G}_{w_2}(x^2; x^1) \cdots \mathfrak{G}_{w_n}(x^n; x^{n-1}) \tag{9}$$

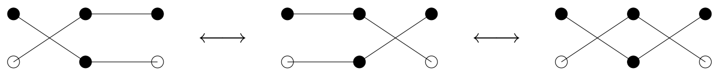
where $\mathfrak{G}_{w_i}$ is the (Laurent) Grothendieck polynomial of Lascoux and Schützenberger [16, 15], and the sum is over certain lace diagrams $\mathbf{w} = (w_1, \dots, w_n)$ called *KMS-factorizations* for the orbit r. These lace diagrams can be defined as certain factorizations of the Zelevinsky permutation of [13]. In this final section we explain how the methods of the present paper can be used to give a concrete description of the KMS-factorizations associated to a given orbit r.

Let $P = \sum c_{\mathbf{w}} \mathfrak{G}(\mathbf{w})$ be a linear combination of products of Grothendieck polynomials $\mathfrak{G}(\mathbf{w}) = \prod \mathfrak{G}_{w_i}(x^i; x^{i-1})$ for all lace diagrams $\mathbf{w}$ for the dimension vector $(e_0, \dots, e_n)$. The arguments of section 4 can be generalized to show that P is symmetric in each set of variables x^i if and only if the following conditions are satisfied:

(I) The coefficient $c_{\mathbf{w}}$ of a lace diagram $\mathbf{w} = (w_1, \dots, w_n)$ is non-zero only if $w_1^{-1}(j) < w_1^{-1}(j+1)$ for all $1 \le j < e_0$ and $w_n(j) < w_n(j+1)$ for all $1 \le j < e_n$.

(II) For every lace diagram $\mathbf{u} = (u_1, \dots, u_n)$ and integers $1 \le i < n$ and $1 \le j < e_i$ such that $u_i(j) < u_i(j+1)$ and $u_{i+1}^{-1}(j) < u_{i+1}^{-1}(j+1)$, we have $c_{\mathbf{u}'} = c_{\mathbf{u}''} = -c_{\mathbf{u}'''}$, where $\mathbf{u}' = (u_1, \dots, u_i s_j, u_{i+1}, \dots, u_n)$, $\mathbf{u}'' = (u_1, \dots, u_i, s_j u_{i+1}, \dots, u_n)$, and $\mathbf{u}''' = (u_1, \dots, u_i s_j, s_j u_{i+1}, \dots, u_n)$.

It follows easily from the definition of KMS-factorizations given in [3] that any KMS-factorization $\mathbf{w}$ satisfies the requirement of (I), and that each of the lace diagrams $\mathbf{u}'$, $\mathbf{u}''$, and $\mathbf{u}'''$ of (II) are KMS-factorizations for r if and only if all three are KMS-factorizations for r (see the remark at the end of [3, §6]). Since this is sufficient to prove the description of KMS-factorizations presented here, we will skip the proof of the above classification of symmetric linear combinations of products of Grothendieck polynomials. Notice that the transformation on lace diagrams corresponding to (II) can be pictured as follows.

(10) 

We also need to know that a minimal lace diagram for an orbit r is a KMS-factorization for this orbit and no other orbits. Again, this fact is immediate from the definition of KMS-factorizations [3].

Theorem 3. *The KMS-factorizations for an orbit r are exactly the lace diagrams that can be obtained by applying a series of transformations (10) to the left-most lace diagram for r.*

Proof. Let $\mathbf{w} = (w_1, \dots, w_n)$ be any KMS-factorization for r. By applying a series of transformations (10) to $\mathbf{w}$, each replacing an occurrence of the first or the third diagram of (10) with the middle diagram, one arrives at a KMS-factorization $\mathbf{w}' = (w_1', \dots, w_n')$ in which only the middle situation of (10) can be found. It is enough to prove that $\mathbf{w}'$ is a minimal lace diagram. In fact, if this is true then $\mathbf{w}'$ must be the left-most diagram for r, since the algorithm in the proof of Proposition 1 will not change this diagram.

If $\mathbf{w}'$ has a crossing outside the extended part of the diagram, say between column $i-1$ and column i, then one can find $1 \le j < e_i$ such that $w_i(j) > w_i(j+1)$. By (I) this implies that $i < n$. Since the first and third situations of (10) cannot occur, the two crossing strands must both terminate at column i, which implies that $\mathbf{w}'' = (w_1', \dots, w_i' s_j, s_j w_{i+1}', \dots, w_n')$ is not a lace diagram. On the other hand, (II) requires $\mathbf{w}''$ to be a KMS-factorization, a contradiction.

We conclude from this that every crossing of $\mathbf{w}'$ must involve a line segment in the extended part of the diagram, which extends the right end of a strand. In particular, two strands can cross at most once, and not at all if they terminate at the same column.

It remains to show that no two crossing strands of $\mathbf{w}'$ can start at the same column. Assume for contradiction that a strand starting at dot j of column i crosses another starting at dot k of column i, where $j < k$. Assume also that $k - j$ is minimal with these properties. Since all crossings involve line segments extending the right end of a strand, it follows that the strand starting at dot j is shorter than the strand starting at dot k. Furthermore, if $j+1 < k$ then the strand containing dot $j+1$ of column i must start at this dot; otherwise it would cross the left side extension of the strand starting at dot j. Since the strand starting at dot $j+1$ is either longer than the strand starting at dot j or shorter than the strand starting at dot k, the minimality of $k-j$ forces $k=j+1$. Now a series of the moves (7), from right to left, will move the crossing of the two strands so that it occurs between columns i and $i+1$. But this is again impossible: (I) implies that $i>0$, after which (II) can be used to produce a KMS-factorization which is not a lace diagram. This contradiction shows that $\mathbf{w}'$ is a minimal lace diagram, which concludes the proof. $\square$

References

[1] S. Abeasis and A. Del Fra, *Degenerations for the representations of an equioriented quiver of type A_m*, Boll. Un. Mat. Ital. Suppl. **1980**, no. 2, 157–171. MR **84e:**16019

[2] A. S. Buch, *Grothendieck classes of quiver varieties*, Duke Math. J. **115** (2002), no. 1, 75–103. MR 1 932 326

[3] ______, *Alternating signs of quiver coefficients*, J. Amer. Math. Soc., to appear.

[4] A. S. Buch and W. Fulton, *Chern class formulas for quiver varieties*, Inv. Math. **135** (1999), 665–687.

[5] A. S. Buch and R. Rimányi, *Specializations of Grothendieck polynomials*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 1, 1–4. MR 2 075 223

[6] P. Edelman and C. Greene, *Balanced tableaux*, Adv. in Math. **63** (1987), no. 1, 42–99. MR **88b:**05012

[7] L. M. Fehér and R. Rimányi, *Calculation of Thom polynomials and other cohomological obstructions for group actions*, Real and Complex Singularities (Sao Carlos, 2002), Ed. T. Gaffney and M. Ruas, Contemp. Math., #354, Amer. Math. Soc., Providence, RI, 2004.

[8] ______, *Schur and Schubert polynomials as Thom polynomials—cohomology of moduli spaces*, Cent. European J. Math. **4** (2003), 418–434.

[9] ______, *Classes of degeneracy loci for quivers: the Thom polynomial point of view*, Duke Math. J. **114** (August 2002), no. 2, 193–213.

[10] W. Fulton, *Notes from a course on equivariant cohomology*, Winter, 2003.

[11] ______, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. **65** (1992), no. 3, 381–420.

[12] R. F. Goldin, *The cohomology ring of weight varieties and polygon spaces*, Adv. in Math. **160** (2001), 175–204.

[13] A. Knutson, E. Miller, and M. Shimozono, *Four positive formulas for type A quiver polynomials*, preprint, 2003.

[14] V. Lakshmibai and P. Magyar, *Degeneracy schemes, quiver schemes, and Schubert varieties*, Internat. Math. Res. Notices (1998), no. 12, 627–640. MR **99g:**14065

[15] A. Lascoux, *Anneau de Grothendieck de la variété de drapeaux*, The Grothendieck Festschrift, Vol. III, Birkhäuser Boston, Boston, MA, 1990, pp. 1–34. MR **92j:**14064

[16] A. Lascoux and M.-P. Schützenberger, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 11, 629–633. MR **84b:**14030

[17] E. Miller, *Alternating formulae for K-theoretic quiver polynomials*, Duke Math. J., to appear.

[18] A. Yong, *On combinatorics of quiver component formulas*, J. Algebraic Combin., to appear.

[19] A. V. Zelevinskiĭ, *Two remarks on graded nilpotent classes*, Uspekhi Mat. Nauk **40** (1985), no. 1(241), 199–200. MR **86e:**14027

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