# STABLE GROTHENDIECK POLYNOMIALS AND $K$-THEORETIC FACTOR SEQUENCES

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Abstract. We formulate a nonrecursive combinatorial rule for the expansion of the stable Grothendieck polynomials of [Fomin-Kirillov '94] in the basis of stable Grothendieck polynomials for partitions. This gives a common generalization, as well as new proofs of the rule of [Fomin-Greene '98] for the expansion of the stable Schubert polynomials into Schur polynomials, and the $K$-theoretic Grassmannian Littlewood-Richardson rule of [Buch '02]. The proof is based on a generalization of the Robinson-Schensted and Edelman-Greene insertion algorithms. Our results are applied to prove a number of new formulas and properties for $K$-theoretic quiver polynomials, and the Grothendieck polynomials of [Lascoux-Schützenberger '82]. In particular, we provide the first $K$-theoretic analogue of the factor sequence formula of [Buch-Fulton '99] for the cohomological quiver polynomials.

## 1. Introduction and main results

1.1. **Stable Grothendieck polynomials.** For each permutation $\pi$ there is a symmetric power series $G_\pi = G_\pi(x_1, x_2, \dots)$ called the stable Grothendieck polynomial for $\pi$. These power series were defined by Fomin and Kirillov [14, 13] as a limit of the ordinary Grothendieck polynomials of Lascoux and Schützenberger [18]. We recall this definition in Section 2. The term of lowest degree in $G_\pi$ is the Stanley symmetric function (or stable Schubert polynomial) $F_\pi$. The *Stanley coefficients* which appear in the Schur expansion of a Stanley function are interesting combinatorial invariants which generalize the Littlewood-Richardson coefficients.

Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$, the *Grassmannian permutation* $\pi_\lambda$ for $\lambda$ is uniquely defined by the requirement that $\pi_\lambda(i) = i + \lambda_{k+1-i}$ for $1 \leq i \leq k$ and $\pi_\lambda(i) < \pi_\lambda(i+1)$ for $i \neq k$. The power series $G_\lambda := G_{\pi_\lambda}$ play a role in combinatorial $K$-theory similar to the role of Schur functions in cohomology. Buch has shown [3] that any stable Grothendieck polynomial $G_\pi$ can be written as a finite linear combination

$$G_\pi = \sum_\lambda c_{\pi,\lambda} G_\lambda \tag{1}$$

of stable Grothendieck polynomials indexed by partitions, using integer coefficients $c_{\pi,\lambda}$ that generalize the Stanley coefficients [2]. Lascoux gave a recursive formula for stable Grothendieck polynomials which confirms a conjecture that these coefficients have signs that alternate with degree, i.e. $(-1)^{|\lambda|-\ell(\pi)} c_{\pi,\lambda} \geq 0$ [17]. Here $|\lambda| = \lambda_1 + \cdots + \lambda_k$ and $\ell(\pi)$ is the Coxeter length of $\pi$. The central result of this paper

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is a new formula for the coefficients $c_{\pi,\lambda}$ which generalizes Fomin and Greene's combinatorial rule [12] for Stanley coefficients.

To state our formula, we need the 0-*Hecke monoid*, which is the quotient of the free monoid of all finite words in the alphabet $\{1, 2, \dots\}$ by the relations

$$p\,p \equiv p \qquad \text{for all } p \tag{2}$$

$$p\,q\,p \equiv q\,p\,q \qquad \text{for all } p, q \tag{3}$$

$$p\,q \equiv q\,p \qquad \text{for } |p-q| \geq 2. \tag{4}$$

There is a bijection between the 0-Hecke monoid and the infinite symmetric group $S_\infty = \bigcup_{n\geq 1} S_n$. Given any word $a$ there is a unique permutation $\pi \in S_\infty$ such that $a \equiv b$ for some (equivalently every) reduced word $b$ of $\pi$. In this case we write $w(a) = \pi$ and say that $a$ is a *Hecke word* for $\pi$. Notice that the reduced words for $\pi$ are precisely the Hecke words for $\pi$ that are of minimum length. Given an additional permutation $\rho$ with Hecke word $b$, the *Hecke product* of $\pi$ and $\rho$ is defined as the permutation $\pi \cdot \rho = w(ab)$.

We use the English notation for partitions and tableaux. An *increasing* tableau is a Young tableau whose rows increase strictly from left to right, and whose columns increase strictly from top to bottom. A *decreasing* tableau is defined similarly. The (column reading) word of a tableau $T$, denoted $\mathrm{word}(T)$, is obtained by reading the columns of the tableau from bottom to top, starting with the leftmost column, followed by the column to its right, etc. We also define $w(T) := w(\mathrm{word}(T))$.

Our main theorem gives the explicit expansion of the stable Grothendieck polynomial $G_\pi$ in terms of the $G_\lambda$.

**Theorem 1.** *For any permutation $\pi \in S_\infty$, the coefficient $c_{\pi,\lambda}$ in (1) is equal to $(-1)^{|\lambda|-\ell(\pi)}$ times the number of increasing tableaux $T$ of shape $\lambda$ such that* $\mathrm{word}(T)$ *is a Hecke word for $\pi^{-1}$.*

**Example 1.** Consider $\pi = 31524 = s_2 s_1 s_4 s_3$, where each $s_i$ is a simple transposition. The increasing tableaux that provide Hecke words for $\pi^{-1}$ are:

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&4\\\cline{1-2}\end{array}$$

Hence $G_\pi = G_{22} + G_{31} - G_{32}$.

Theorem 1 may be used to give self-contained proofs of a number of known results. For example, the finiteness of the expansion (1) proved in [3] follows immediately from Theorem 1. When the permutation $\pi$ is 321-avoiding, Theorem 1 furthermore generalizes Buch's rule for the coefficients $c_{\pi,\lambda}$ in terms of set-valued tableaux [3], in the sense that there is an explicit bijection between the relevant increasing and set-valued tableaux. As a consequence, we obtain a new proof of the set-valued Littlewood-Richardson rule for the Schubert structure constants in the $K$-theory of Grassmannians, as well as an alternative rule based on increasing tableaux. This is explained in Section 3.5.

1.2. **Hecke insertion.** Fomin and Kirillov proved that the monomial coefficients of (stable) Grothendieck polynomials are counted by combinatorial objects called compatible pairs (also known as resolved wiring diagrams, FK-graphs, pipe dreams, or nonreduced RC-graphs) [14, 13]. This formula was used in [3] to express the monomial coefficients of stable Grothendieck polynomials for partitions in terms of set-valued tableaux (see equation (8)). We prove Theorem 1 by exhibiting an

explicit bijection between the set of compatible pairs for a permutation $\pi$ and the set of pairs $(T, U)$ where $T$ is an increasing tableau with $w(T) = \pi^{-1}$ and $U$ is a set-valued tableau of the same shape as $T$. This bijection is constructed using a new combinatorial algorithm called *Hecke insertion*, which is the technical core of our paper.

Hecke insertion is a generalization of the Edelman-Greene insertion algorithm [10] (also known as Coxeter-Knuth insertion) from the set of reduced words to the set of all (Hecke) words. It specializes to Robinson-Schensted insertion for words of distinct integers [21, 22]. There are two main novelties in our extension. First, we need an operation that "jumps" many columns at once. Second, an accompanying reverse insertion algorithm can pass back different intermediate values than the insertion algorithm generated. Neither of these elements appear in the classical algorithms. We also use Hecke insertion to define products of decreasing tableaux, which enter into our definition of $K$-theoretic factor sequences.

1.3. **Quiver varieties.** Our main application of Theorem 1 concerns the classes of quiver varieties in $K$-theory. Recall that a sequence of vector bundle morphisms $E_0 \to E_1 \to \cdots \to E_n$ over a non-singular variety $\mathfrak{X}$ together with a set of *rank conditions* $r = \{r_{ij}\}$ for $0 \leq i \leq j \leq n$ define a *quiver variety* $\Omega_r \subset \mathfrak{X}$ of points where each composition of bundle maps $E_i \to E_j$ has rank at most $r_{ij}$. We demand that the rank conditions can actually *occur*, and that the bundle maps are generic, so that the quiver variety $\Omega_r$ obtains its expected codimension $d(r) = \sum_{i<j}(r_{i,j-1} - r_{ij})(r_{i+1,j} - r_{ij})$. Buch and Fulton proved a formula for the cohomology class of $\Omega_r$ [6], which was later generalized to $K$-theory by Buch [2]. The $K$-theory version states that the Grothendieck class of $\Omega_r$ is given by

$$[\mathcal{O}_{\Omega_r}] = \sum_{\mu} c_\mu(r)\, G_{\mu_1}(E_1 - E_0) G_{\mu_2}(E_2 - E_1) \cdots G_{\mu_n}(E_n - E_{n-1})\,, \tag{5}$$

where the sum is over sequences $\mu = (\mu_1, \ldots, \mu_n)$ of partitions $\mu_i$ such that $\sum |\mu_i| \geq d(r)$ and each partition $\mu_i$ can be contained in the rectangle $e_i \times e_{i-1}$ with $e_i$ rows and $e_{i-1}$ columns, where $e_i := r_{ii} = \operatorname{rank}(E_i)$. The notation $G_{\mu_i}(E_{i+1} - E_i)$ will be explained in Section 2.

The coefficients $c_\mu(r)$ in formula (5) are integers called *quiver coefficients*. When $\sum |\mu_i| = d(r)$, the coefficient $c_\mu(r)$ also appears in the cohomology formula from [6] and is called a *cohomological quiver coefficient*. A precise conjecture for these cohomological coefficients was posed in [6], which asserts that $c_\mu(r)$ counts the number of *factor sequences* of tableaux with shapes given by the sequence of partitions $\mu$. A factor sequence is a sequence of semistandard Young tableaux that can be obtained by performing a series of plactic factorizations and multiplications of chosen tableaux arranged in a *tableau diagram*. For a specific choice of tableau diagram, this conjecture was proved by Knutson, Miller and Shimozono [16]. However, the original definition of factor sequences from [6] as sequences of tableaux generated using the plactic product has no known generalization to $K$-theory.

In this paper, we prove that $K$-theoretic quiver coefficients are counted by a new type of factor sequence, generalizing the cohomological factor sequences defined by Buch in [4] using the Coxeter-Knuth product of tableaux. The new $K$-theoretic factor sequences are constructed from a tableau diagram of decreasing[1] tableaux

[1]The combinatorics of these factor sequences naturally requires decreasing rather than increasing tableaux.

using the same algorithm that defines the original factor sequences, except that the plactic product is replaced with a product $(U,T) \mapsto U \cdot T$ of decreasing tableaux which is compatible with Hecke products of permutations (see Section 3.7).

For each $0 \leq i < j \leq n$ let $R_{ij}$ be a rectangle with $r_{i+1,j} - r_{ij}$ rows and $r_{i,j-1} - r_{ij}$ columns. Let $U_{ij}$ be the unique decreasing tableau of shape $R_{ij}$ such that the lower left box contains the number $r_{i,j-1}$, and the number in each box is one larger than the number below it and one smaller than the number to the left of it. For example, if $r_{i,j-1} = 6$, $r_{i+1,j} = 5$, and $r_{ij} = 2$ then

$$U_{ij} \;=\; \begin{array}{|c|c|c|c|}\hline 8&7&6&5\\\hline 7&6&5&4\\\hline 6&5&4&3\\\hline\end{array}\,.$$

These tableaux $U_{ij}$ can be arranged in a triangular tableau diagram as in [6, §4]:

$$\begin{array}{ccccccc} U_{01} & & U_{12} & & \cdots & & U_{n-1,n} \\ & U_{02} & & \cdots & & U_{n-2,n} & \\ & & \ddots & & & & \\ & & & U_{0n} & & & \end{array}$$

We define a *$K$-theoretic factor sequence* for the rank conditions $r$ by induction on $n$. If $n = 1$ then the only factor sequence is the sequence $(U_{01})$ consisting of the only tableau in the tableau diagram. If $n \geq 2$ then the numbers $\overline{r} = \{\overline{r}_{ij} : 0 \leq i \leq j \leq n-1\}$ defined by $\overline{r}_{ij} = r_{i,j+1}$ form a valid set of rank conditions corresponding to a sequence of $n-1$ bundle maps. In this case, a factor sequence for $r$ is any sequence of the form $(U_{01} \cdot A_1, \ldots, B_{i-1} \cdot U_{i-1,i} \cdot A_i, \cdots, B_{n-1} \cdot U_{n-1,n})$, for a choice of decreasing tableaux $A_i$ and $B_i$ such that $(A_1 \cdot B_1, \ldots, A_{n-1} \cdot B_{n-1})$ is a factor sequence for $\overline{r}$.

**Theorem 2.** *The $K$-theoretic quiver coefficient $c_\mu(r)$ is equal to $(-1)^{\sum |\mu_i| - d(r)}$ times the number of $K$-theoretic factor sequences $(T_1, \ldots, T_n)$ for the rank conditions $r$, such that $T_i$ has shape $\mu_i$ for each $i$.*

Central to the proof of the nonnegativity of cohomological quiver coefficients given in [16] is the *stable component formula*, which writes the cohomology class of a quiver variety as a sum of products of Stanley functions. This sum is over all *lace diagrams* representing the rank conditions $r$, which have the smallest possible number of crossings. The $K$-theoretic version of the component formula from [4, 20] states that

$$[\mathcal{O}_{\Omega_r}] = \sum_{(\pi_1,\ldots,\pi_n)} (-1)^{\sum \ell(\pi_i) - d(r)} G_{\pi_1}(E_1 - E_0) \cdots G_{\pi_n}(E_n - E_{n-1}) \tag{6}$$

where the sum is over a generalization of minimal lace diagrams, which was named *KMS-factorizations* in [4]. We recall this definition in Section 4. Our proof of Theorem 2 is based on Theorem 1, equation (6), and the following characterization of the $K$-theoretic factor sequences:

**Theorem 3.** *A sequence of decreasing tableaux $(T_1, \ldots, T_n)$ is a $K$-theoretic factor sequence for the rank conditions $r$ if and only if $(w(T_1), \ldots, w(T_n))$ is a KMS-factorization for $r$.*

1.4. **Outline of the paper.** We give a brief overview of the contents of this article. In Section 2 we recall the original definitions of Grothendieck polynomials and stable Grothendieck polynomials, and record two useful monomial expansions for the latter. In Section 3 we define the Hecke insertion algorithm, establish its basic properties, give the proof of Theorem 1, and apply it to reprove the set-valued Littlewood-Richardson rule from [3]. We also give an algorithm for generating all increasing tableaux which represent a given permutation. Finally, our applications to quiver coefficients and factor sequences are contained in Section 4. These include a proof that the $K$-theoretic quiver coefficients are special cases of the coefficients $c_{z(r),\lambda}$ in the expansion (1) of the stable Grothendieck polynomial for a Zelevinsky permutation $z(r)$ (Theorem 6), and a new formula for the decomposition coefficients of universal Grothendieck polynomials (Theorem 7).

## 2. Grothendieck polynomials

Grothendieck polynomials were introduced by Lascoux and Schützenberger [19] as polynomial representatives for the classes of structure sheaves of Schubert varieties in the $K$-theory of the flag variety for $GL_n$. Let $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ be two sequences of commuting independent variables and $\pi \in S_n$. If $\pi = \pi_0$ is the longest permutation in $S_n$, then we set

$$\mathfrak{G}_{\pi_0}(X;Y) = \prod_{i+j\leq n} (x_i + y_j - x_i y_j)\,.$$

If $\pi \neq \pi_0$, we can find a simple transposition $s_i = (i, i+1) \in S_n$ such that $\ell(\pi s_i) = \ell(\pi) + 1$. We then define

$$\mathfrak{G}_{\pi}(X;Y) = \frac{(1-x_{i+1})\mathfrak{G}_{\pi s_i}(X;Y) - (1-x_i)\mathfrak{G}_{\pi s_i}(x_1,\ldots,x_{i+1},x_i,\ldots,x_n;Y)}{x_i - x_{i+1}}\,.$$

For $\pi \in S_\infty$ and $r \geq 0$, let $1^r \times \pi \in S_\infty$ denote the permutation obtained by putting $r$ fixed points in front of $\pi$, that is, $1^r \times \pi = \rho$ where $\rho(i) = i$ for $1 \leq i \leq r$ and $\rho(i) = \pi(i-r) + r$ for $i > r$. The *stable Grothendieck polynomial* $G_\pi(X;Y)$ is the formal power series, symmetric in the $X$ and $Y$ variables separately, defined by

$$G_\pi(X;Y) = \lim_{r\to\infty} \mathfrak{G}_{1^r\times\pi}(X;Y)\,.$$

Given vector bundles $E = L_1 \oplus \cdots \oplus L_p$ and $F = M_1 \oplus \cdots \oplus M_q$ over a variety $\mathfrak{X}$ which are direct sums of line bundles, we write

$$G_\lambda(E-F) = G_\lambda(1-L_1^{-1},\ldots,1-L_p^{-1};1-M_1,\ldots,1-M_q) \in K(\mathfrak{X})\,.$$

The symmetry of $G_\lambda(X;Y)$ implies that this is a polynomial in the exterior powers of $E^\vee$ and $F$. Therefore $G_\lambda(E-F)$ makes sense even for bundles that are not split into direct sums of line bundles. This explains the notation used in (5).

We will be mostly interested in the specialization $G_\pi = G_\pi(X;0)$. We recall Fomin and Kirillov's combinatorial construction of these polynomials [13], using notation which generalizes Billey, Jockusch, and Stanley's formula for Schubert polynomials [1]. Define a *compatible pair* to be a pair $(a,i)$ of words $a = a_1 a_2 \cdots a_p$ and $i = i_1 i_2 \cdots i_p$ of positive integers, such that $i_1 \leq i_2 \leq \cdots \leq i_p$, and so that $a_j > a_{j+1}$ whenever $i_j = i_{j+1}$. The stable Grothendieck polynomial for $\pi \in S_\infty$ is then given by [13]

$$G_\pi = \sum_{(a,i)} (-1)^{\ell(i)-\ell(\pi)}\, x^i \tag{7}$$

where the sum is over all compatible pairs $(a, i)$ such that $w(a) = \pi$. Here $\ell(i)$ is the common length of $a$ and $i$, and $x^i = x_{i_1} x_{i_2} \cdots x_{i_{\ell(i)}}$.

A *set-valued tableau* of shape $\lambda$ is a filling of the boxes of the Young diagram of $\lambda$ with finite nonempty sets of positive integers, such that these sets are weakly increasing along rows and strictly increasing down columns. In other words, all integers in a box must be smaller than or equal to the integers in the box to the right of it, and strictly smaller than the integers in the box below it. For a set-valued tableau $S$, let $x^S$ denote the monomial where the exponent of $x_i$ is equal to the number of boxes containing the integer $i$, and let $|S|$ be the degree of this monomial. Buch's formula for the monomial expansion of $G_\lambda$ is given by [3]

$$G_\lambda = \sum_S (-1)^{|S|-|\lambda|} x^S \tag{8}$$

where $S$ runs over all set-valued tableaux of shape $\lambda$.

## 3. Hecke Insertion and the proof of Theorem 1

In view of (7) and (8), to prove Theorem 1 it suffices to establish a bijection $(a, i) \mapsto (T, U)$ between all compatible pairs $(a, i)$ such that $w(a) = \pi$, and all pairs of tableaux $(T, U)$ of the same shape, such that $T$ is increasing with $w(T) = \pi^{-1}$ and $U$ is set-valued. In addition, this bijection must satisfy $x^U = x^i$. To construct this bijection, we need a new algorithm called *Hecke insertion*.

3.1. **Hecke insertion.** We shall define the Hecke (column) insertion of a non-negative integer $x$ into the increasing tableau $Y$, resulting in the increasing tableau $Z$. The shape of $Z$ always contains the shape of $Y$, and contains at most one extra box $c$. Unlike ordinary Robinson-Schensted insertion, it is possible that $Z$ has the same shape as $Y$, but even in this case it will contain a special corner $c$ where the insertion algorithm terminated. To keep track of these cases, we will use a parameter $\alpha \in \{0, 1\}$, which is set to 1 if and only if the corner $c$ is outside the shape of $Y$. Thus the complete output of the insertion algorithm is the triple $(Z, c, \alpha)$. We will use the notation $Z = (x \xrightarrow{H} Y)$.

The algorithm proceeds by inserting the integer $x$ into the first column of $Y$. This may modify this column, and possibly produce an *output integer*, which is then inserted into the second column of $Y$, etc. This process is repeated until an insertion does not produce an output integer. The procedure for inserting an integer $x$ into a column $C$ is as follows.

If $x$ is larger than or equal to all boxes of $C$, then no new output value is produced and the algorithm terminates. If adjoining $x$ as a new box below $C$ results in an increasing tableau, then $Z$ is the resulting tableau, $\alpha = 1$, and $c$ is the new corner where $x$ was added. If $x$ cannot be added, then no further modifications are carried out to produce $Z$, $\alpha = 0$, and $c$ is the corner of the row of $Z$ containing the bottom box of the column $C$.

Otherwise $C$ contains boxes strictly larger than $x$, and we let $y$ be the smallest such box. If replacing $y$ with $x$ results in an increasing tableau, then this is done. In either case, $y$ is the output integer, which is inserted into the next column.

**Example 2.**

$$3 \xrightarrow{H} \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&5&6\\\cline{1-3} 3\\\cline{1-1} 5\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&5&6\\\cline{1-3} 3\\\cline{1-1} 5\\\cline{1-1}\end{array}$$

The integer 3 is inserted into the first column, which contains 3. So 5 is inserted into the second column, whose largest value is 5. The algorithm terminates with $\alpha = 0$, and $c = (2, 3)$ is the corner in the second row and third column.

**Example 3.**

$$2 \xrightarrow{H} \begin{array}{|c|c|}\hline 1&2\\\hline 2&5\\\cline{1-2} 4\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 2&4\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

The integer 2 is inserted into the first column, which contains 2. So 4 is inserted into the second column, displacing the 5. The 5 is inserted into the third column, where it comes to rest. We get $\alpha = 1$ and $c = (1, 3)$.

**Example 4.**

$$2 \xrightarrow{H} \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 2&3&4\\\cline{1-3} 4\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 2&3&4\\\cline{1-3} 4\\\cline{1-1}\end{array}$$

The integer 2 is inserted into the first column, which contains a 2. So 4 is inserted into the second column, which has largest entry 3. Since the first column still contains the value 4 in its bottom box, it is not possible to add a box with 4 to the second column. We obtain $\alpha = 0$, and $c = (2, 3)$ is the corner of the second row.

**Example 5.**

$$1 \xrightarrow{H} \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 3&4&5\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 3&4&5\\\hline\end{array}$$

The integer 1 is inserted into the first column, which already contains a 1. So 3 is inserted into the second column. It would have replaced 4, but this replacement would place a 3 directly to the right of another 3, violating the increasing tableau condition. So the second column is unchanged and 4 is inserted into the third column. Similarly 4 cannot replace 5, so 5 is inserted into the fourth column, where it comes to rest in the cell $c = (1, 4)$ with $\alpha = 1$.

3.2. **Reverse Hecke insertion.** Let $Z$ be an increasing tableau, $c$ a corner of $Z$, and $\alpha \in \{0, 1\}$. Reverse Hecke insertion applied to the triple $(Z, c, \alpha)$ produces a pair $(Y, x)$ of an increasing tableau $Y$ and a positive integer $x$ as follows. Let $y$ be the integer in the cell $c$ of $Z$. If $\alpha = 1$ then remove $y$. In any case, reverse insert $y$ into the column to the left of the corner $c$.

Whenever a value $y$ is reverse inserted into a column $C$, let $x$ be the largest entry of $C$ such that $x < y$. If replacing $x$ with $y$ results in an increasing tableau, then this is done. In any case, the integer $x$ is passed along to the left. If $C$ is not the left-most column, this means that $x$ is reverse inserted into the column left of $C$; otherwise $x$ becomes the final output value, along with the modified tableau.

**Example 6.** Let us apply reverse Hecke insertion to the tableau computed in Example 5 at the cell $c = (1, 4)$ with $\alpha = 1$. The integer 5 in this cell is then removed, and 5 is reverse inserted into the third column. Since 5 is already in the third column, it is not changed, and 3 is reverse inserted into the second column. Here 3 cannot replace 2 because this would place a 3 directly to the left of a 3. The

second column is unchanged and 2 is reverse inserted into the first column. The 2 cannot replace 1 for the same reason, so the first column is unchanged and $x = 1$ is the output value. This recovers the initial tableau of Example 5.

Let $\mathcal{I}$ denote the set of pairs $(Y, x)$ where $Y$ is an increasing tableau and $x$ is a positive integer. Let $\mathcal{R}$ be the set of triples $(Z, c, \alpha)$ where $Z$ is an increasing tableau, $c$ a corner cell of $Z$, and $\alpha \in \{0, 1\}$.

**Theorem 4.** *Hecke insertion* $(Y, x) \mapsto (Z, c, \alpha)$ *and reverse Hecke insertion* $(Z, c, \alpha) \mapsto (Y, x)$ *define mutually inverse bijections between the sets* $\mathcal{I}$ *and* $\mathcal{R}$.

*Proof.* Assume at first that $(Z, c, \alpha)$ has been obtained by applying Hecke insertion to $(Y, x)$. We must show that reverse Hecke insertion recovers $(Y, x)$ from $(Z, c, \alpha)$. This is clear if $Y$ is the empty tableau. In general we proceed by induction on the number of columns in $Y$.

If $x$ is strictly larger than all the integers in the first column of $Y$, then $Z$ is obtained by adding a box containing $x$ to the first column of $Y$, $c$ is this box, $\alpha = 1$, and reverse Hecke insertion clearly maps $(Z, c, \alpha)$ back to $(Y, x)$. If $x$ is equal to the largest integer in the first column of $Y$, then $Z = Y$, $\alpha = 0$, and $c$ is the leftmost corner of $Y$. Also in this case it is easy to see that reverse Hecke insertion applied to $(Z, c, \alpha)$ recovers $(Y, x)$.

We can therefore assume that the first column of $Y$ contains at least one integer that is strictly larger than $x$. We let $Y'$ denote the first column of $Y$ and let $Y''$ be the rest of $Y$. We define $Z'$ and $Z''$ similarly, and we regard $c$ as a corner of both $Z$ and of $Z''$. Let $a$ be the smallest box of $Y'$ for which $x < a$. If $(Z'', c, \alpha)$ is the result of applying Hecke insertion to $(Y'', a)$, then we know by induction that reverse Hecke insertion applied to $(Z, c, \alpha)$ first recovers $Y''$ from $Z''$, after which $a$ is reverse inserted into $Z'$. If the box of $Y'$ containing $a$ was replaced with $x$ in the construction of $Z'$, then the reverse insertion puts $a$ back in this box to recover $Y'$, and the output value of the reverse insertion is $x$. If the box of $Y'$ containing $a$ was not changed to $x$ when $Z'$ was formed, then $Z' = Y'$ must contain $x$ in the box immediately over $a$. Reverse inserting $a$ therefore leaves $Z'$ unchanged, and gives $x$ as the output value, as required.

It remains to consider the case when $Z''$ differs from the tableau $(a \xrightarrow{H} Y'')$. This can only happen if the box of $Y'$ containing $a$ was not replaced with $x$, and if the leftmost box in the same row of $(a \xrightarrow{H} Y'')$ contains $a$. Let $b$ be the largest box of the first column of $Y''$ for which $b \leq a$, and observe that $(Z'', c, \alpha)$ must be the result of applying Hecke insertion to $(b, Y'')$. It therefore follows by induction that reverse Hecke insertion applied to $(Z, c, \alpha)$ first recovers $Y''$ from $Z''$, after which $b$ is reverse inserted into $Z' = Y'$. Finally, notice that $Y'$ must contain $x$ in the box above the box containing $a$, since otherwise $a$ would have been replaced with $x$ in the initial Hecke insertion. Furthermore we have $x < b$, since $b$ is in the box of $Y''$ to the right of $x$ in $Y'$. Since $x < b \leq a$, we conclude that reverse insertion of $b$ to $Y'$ leaves this column unchanged, and $x$ is the final output value, as required. This verifies that Hecke insertion followed by reverse Hecke insertion is the identity map.

To finish the proof, assume that $(Y, x)$ has been obtained by applying reverse Hecke insertion to $(Z, c, \alpha)$. We must show that Hecke insertion maps $(Y, x)$ back to $(Z, c, \alpha)$. This is easily checked if $c$ is the corner of the bottom row of $Z$. In general we use induction on the number of columns in $Z$. Assume that $c$ is not in

the bottom row of $Z$, and let $Y'$, $Y''$, $Z'$, and $Z''$ be defined as above. We know that for some integer $a$, the pair $(Y'', a)$ is the result of applying reverse Hecke insertion to $(Z'', c, \alpha)$. We also know that $Y'$ is obtained by reverse inserting $a$ into $Z'$, keeping in mind that $Y''$ resides to the right, and $x$ is the output value resulting from this insertion.

We first prove that $Y'$ contains integers strictly larger than $x$. In fact, if this was not true, then $x$ would be equal to the bottom box of $Y' = Z'$. Since $a$ did not replace $x$ in $Z'$, we deduce that $Z'$ and $Y''$ have equally many rows, and $a$ is the content of the bottom-left box of $Y''$. On the other hand, since $c$ is not in the bottom row of $Z''$, it follows by induction on the number of columns of $Z$ that the first column of $Y''$ contains integers strictly larger than $a$, a contradiction.

If the box of $Z'$ containing $x$ is replaced by $a$ in $Y'$, then Hecke insertion applied to $(Y, x)$ first restores $Z'$ in the first column, after which $a$ is inserted into the first column of $Y''$. Since we know by induction that $(Z'', c, \alpha)$ is the result of applying Hecke insertion to $(a, Y'')$, we deduce that Hecke insertion maps $(Y, a)$ to $(Z, c, \alpha)$, as required.

Finally assume that $Y' = Z'$, i.e. $x$ is not replaced by $a$. Since $Y'$ contains integers strictly larger than $x$, we know that $x$ is not in the bottom box of $Y'$. Let $y$ be the integer in the box of $Y'$ just under $x$. When $x$ is Hecke inserted into $Y' = Z'$, this column is not changed, and $y$ is inserted into $Y''$. If $y = a$, then we know by induction that this recovers $(Z'', c, \alpha)$. Otherwise we must have $y > a$, and $a$ must be contained in the box in the first column of $Y''$ which is in the same row as the box containing $x$ in $Y'$. In this case Hecke insertion of $y$ into $Y''$ with $Z'$ to the left will produce the same result as Hecke inserting $a$ into $Y''$ with nothing to the left, namely $(Z'', c, \alpha)$. This verifies that reverse Hecke insertion followed by Hecke insertion is the identity map, which completes the proof. □

3.3. **Properties of Hecke insertion.** We will need two additional properties of Hecke insertion. The first property says that Hecke insertion respects Hecke words.

**Lemma 1.** *Let $Y$ be an increasing tableau, $x$ a positive integer, and set $Z = (x \xrightarrow{H} Y)$. Then* $\mathrm{word}(Z) \equiv x\ \mathrm{word}(Y)$.

*Proof.* It is easiest to check that reverse Hecke insertion preserves Hecke words. It is enough to consider the following situation. Let $C$ be a column, $y$ a number that is reverse inserted into $C$, $U$ the modified tableau to the right of $C$ from which $y$ comes, $C'$ the modification of $C$, and $x$ the output value. We must show that

$$\mathrm{word}(C)\, y\ \mathrm{word}(U) \equiv x\ \mathrm{word}(C')\ \mathrm{word}(U)\,.$$

If $x$ is replaced by $y$ in $C$ then we have $\mathrm{word}(C)\, y \equiv x\ \mathrm{word}(C')$ by the relations (4). If $x$ is not replaced because $y$ is immediately below $x$ in $C$, then $\mathrm{word}(C)\, y \equiv x\ \mathrm{word}(C')$ holds by the relations (3) and (4).

Finally, assume that $x$ is not replaced by $y$ in $C$ because the box of $C$ containing $x$ is just left of a box in $U$ containing $y$. In this case we show that $\mathrm{word}(C) = \mathrm{word}(C') \equiv x\ \mathrm{word}(C')$ and $y\ \mathrm{word}(U) \equiv \mathrm{word}(U)$. If $x$ is contained in the bottom box of $C$, then the first relation follows from (2). Otherwise let $z$ be the box just under $x$ in $C$. Since we must have $x < y < z$, the relation $\mathrm{word}(C') \equiv x\ \mathrm{word}(C')$ follows from (2) and (4). Similarly, if the first column of $U$ contains an integer $w$ in a box just below $y$, then $y < z < w$, and (2) and (4) imply that $y\ \mathrm{word}(U) \equiv \mathrm{word}(U)$. □

We also need the following "Pieri property" of Hecke insertion.

**Lemma 2.** *Let $Y$ be an increasing tableau, and $x_1, x_2$ two positive integers. Suppose that Hecke insertion of $x_1$ into $Y$ results in $(Z, c_1, \alpha_1)$ and that Hecke insertion of $x_2$ into $Z$ results in $(T, c_2, \alpha_2)$. Then $c_2$ is strictly to the right of $c_1$ if and only if $x_1 > x_2$.*

*Proof.* As in the proof of Lemma 1, it is easier to work with reverse Hecke insertion. We consider $x_2$ as the output value obtained from applying reverse Hecke insertion to $T$ starting at the corner $c_2$, and $x_1$ as the output value obtained by applying reverse insertion to the corner $c_1$ of the result. We first consider the case that $c_2$ is strictly to the right of $c_1$.

Suppose $c_1$ is in the first column of $T$. If the path of the first reverse insertion went through $c_1$, then the number in $c_1$ became larger, so the lemma holds. If the first insertion path went above $c_1$, then clearly the lemma also holds.

Consider $T$ as split vertically into the subtableau of columns weakly to the right of $c_1$ and the subtableau of columns strictly to the left. The above observations imply that it is enough to prove the following: Let $y_2 < y_1$ and assume $y_2$ is first reverse inserted into a column $C$, and then $y_1$ is reverse inserted into the modification of $C$, then the first output value is strictly smaller than the second.

The first output value $x$ is the bottom-most entry in $C$ which is strictly smaller than $y_2$. If the first reverse insertion replaces $x$ by $y_2$ or if the box below $x$ contains $y_2$, then the second reverse insertion produces an output value which is greater than or equal to $y_2$.

Finally assume that when the first reverse insertion occurs, $y_2$ is contained in a box just to the right of the box of $C$ containing $x$. In this case $y_1$ must reside in a box below $y_2$ in the column to the right of $C$, which implies that the second output value will come from a box below $x$ in $C$, and thus be strictly larger than $x$.

Now we consider the case when $c_2$ is further to the left or in the same column as $c_1$. We must show that the first output value is larger than or equal to the second. If $c_2$ is in the first column then this is clear. The first output value is the largest entry in the first column. After the reverse insertion from $c_2$, the second output value must come from the first column, all of whose entries are smaller than or equal to the first output value.

For reasons similar to those of the previous case, we need to show the following: suppose $y_2 \geq y_1$ and that $y_2$ is reverse inserted into a column $C$, and then $y_1$ is reverse inserted into the modification of $C$, then the first output value is larger than or equal to the second.

The first output value $x$ is the bottom-most entry of $C$ that is strictly smaller than $y_2$. If $x$ is replaced by $y_2$, then the bottom-most entry of the modified column which is strictly smaller than $y_1$ must be located above the current location of $y_2$. And this entry is still smaller than $x$.

Suppose $x$ is not replaced by $y_2$. In this case, the bottom-most entry that is strictly smaller than $y_1$ is either $x$ or something above $x$. In both cases, the second output value is less than or equal to $x$. □

3.4. **Proof of Theorem 1.** Recall from the discussion at the beginning of Section 3 that to prove Theorem 1, it suffices to exhibit a bijection $(a, i) \mapsto (T, U)$ where $(a, i)$ is a compatible pair with $w(a) = \pi$, $T$ is an increasing tableau with $w(T) = \pi^{-1}$,

and $U$ is a set-valued tableau of the same shape as $T$. Moreover this bijection must satisfy $x^i = x^U$.

Let $(a, i)$ be as above with $a = a_1 \cdots a_p$ and $i = i_1 \cdots i_p$. We start with the empty tableau pair $(T_0, U_0) = (\varnothing, \varnothing)$. If $(T_{j-1}, U_{j-1})$ has been defined for some $j \geq 1$, let $(T_j, c_j, \alpha_j)$ be the result of Hecke inserting $a_j$ into $T_{j-1}$. If $\alpha_j = 1$ then $U_j$ is obtained from $U_{j-1}$ by adding the corner $c_j$ and putting the singleton set $\{i_j\}$ in this box. Otherwise $c_j$ is already a corner of $U_{j-1}$, and $U_j$ is obtained by putting $i_j$ into the existing set in this corner of $U_{j-1}$. We finally set $(T, U) = (T_p, U_p)$.

The map $(a, i) \mapsto (T, U)$ has the desired properties. $U$ is a set-valued tableau by Lemma 2 and $x^i = x^U$ by definition. The fact that $w(T) = \pi^{-1}$ follows from Lemma 1, combined with the fact that the reversal of words gives a bijection between the Hecke words for $\pi$ and those for $\pi^{-1}$.

Finally, for $j \geq 1$ we note that $i_j$ is the largest integer appearing in $U_j$, and $c_j$ is the (unique) rightmost corner of $U_j$ containing this integer. If this corner of $U_j$ contains only a singleton, then $\alpha_j = 1$, and $U_{j-1}$ can be obtained by removing the box $c_j$ from $U_j$. Otherwise we have $\alpha_j = 0$ and $U_{j-1}$ is obtained by removing the integer $i_j$ from the box $c_j$ of $U_j$. Since the pair $(T_{j-1}, a_j)$ is the result of applying reverse Hecke insertion to the triple $(T_j, c_j, \alpha_j)$, this shows that the integers $i_j$ and $a_j$ and the pair $(T_{j-1}, U_{j-1})$ can be recovered from $(T_j, U_j)$. Repetition of this procedure provides an inverse map $(T, U) \mapsto (a, i)$. This completes the proof of Theorem 1.

**Example 7.** Let $(a, i)$ be the compatible pair given by $a = 41443$ and $i = 11244$. Then the above proof constructs the following sequence of tableau pairs $(T_j, U_j)$.

$$(\ydiagram{1}, \ydiagram{1}), (\ydiagram{2}, \ydiagram{2}), \left(\begin{ytableau}1 & 4\\4\end{ytableau}, \begin{ytableau}1 & 1\\2\end{ytableau}\right), \left(\begin{ytableau}1 & 4\\4\end{ytableau}, \begin{ytableau}1 & 1\\24\end{ytableau}\right), \left(\begin{ytableau}1 & 4\\3\end{ytableau}, \begin{ytableau}1 & 14\\24\end{ytableau}\right)$$

3.5. **Increasing tableaux and set-valued tableaux.** In this section we sketch how to recover the set-valued Littlewood-Richardson rule of [3] from Theorem 1.

A compatible pair $(a, i)$ can be identified with a diagram of columns of boxes containing positive integers, which increase strictly from top to bottom. The boxes of column $p$ contain the integers $a_j$ for which $i_j = p$. Some of the columns may be empty. For example, the compatible pair from Example 7 is represented by the diagram:

$$\begin{ytableau}1 & 4 & \none & 3\\4 & \none & \none & 4\end{ytableau}$$

Let $\lambda/\mu$ be the skew Young diagram between two partitions $\lambda$ and $\mu$. If we let $\pi_\lambda$ and $\pi_\mu$ be the corresponding Grassmannian permutations with descent at the same position $k$, then the 321-avoiding permutation corresponding to the skew shape $\lambda/\mu$ is given by $\pi_{\lambda/\mu} = \pi_\lambda \, \pi_\mu^{-1}$ (see [1]). It was shown in [3, Thm. 3.1] that the stable Grothendieck polynomial $G_{\lambda/\mu} := G_{\pi_{\lambda/\mu}}$ can be written as the sum

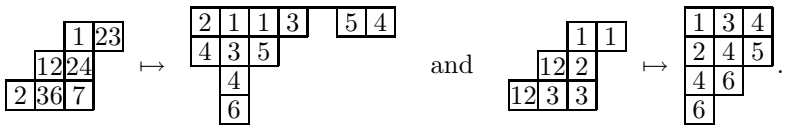
$$G_{\lambda/\mu} = \sum_S (-1)^{|S|-|\lambda/\mu|} x^S$$

over all set-valued tableaux $S$ of shape $\lambda/\mu$. The proof is based on a bijection between set-valued tableaux and monomials equivalent to compatible pairs.

Given an integer $n$ such that $\pi_{\lambda/\mu} \in S_n$, we can formulate this as a bijection between set-valued tableaux of shape $\lambda/\mu$ and (diagrams of) compatible pairs $(a, i)$ with $w(a) = \pi_0 \pi_{\lambda/\mu}^{-1} \pi_0$, where $\pi_0$ is the longest permutation in $S_n$. Number the

north-west to south-east diagonals of $\lambda$ (and $\mu$) consecutively from right to left, so that the upper-left box of $\lambda$ is in diagonal number $n-k$. Then the set-valued tableau $S$ of shape $\lambda/\mu$ is mapped to the diagram in which column $p$ consists of the diagonal numbers of the boxes of $S$ that contain the integer $p$.

**Example 8.** If $n-k=4$, then the bijection makes the assignments




Recall that the column reading word of a set-valued tableau is obtained by reading its boxes from bottom to top and then left to right. The integers of each box are arranged in increasing order. The set-valued tableaux in Example 8 have column reading words 2 3 6 1 2 7 2 4 1 2 3 and 1 2 3 1 2 3 2 1 1. Recall also that a word is a *reverse lattice word* if every occurrence of an integer $i$ with $i>1$ is followed by more $i-1$'s than $i$'s. The *content* of a word is the integer sequence $(\nu_1,\nu_2,\dots)$ where $\nu_i$ is the number of occurrences of $i$ in the word. The following lemma says that the condition that the diagram of a compatible pair is an increasing tableau naturally generalizes the condition that the reading word of a tableau is a reverse lattice word.

**Lemma 3.** *If the set-valued tableau $S$ is mapped to the diagram $T$, then the column reading word of $S$ is a reverse lattice word if and only if $T$ is an increasing tableau.*

*Proof.* Consider an integer $i>1$ contained in some box $B$ of $S$. The lemma follows from the observation that all occurrences of $i$ and $i-1$ that follow the integers of $B$ in the column reading word of $S$ have diagonal numbers that are strictly smaller than the diagonal number of $B$, and all other occurrences of $i$ and $i-1$ have diagonal numbers that are larger than or equal to the diagonal number of $B$. □

**Corollary 1** (Theorem 6.9 of [3])**.** *The coefficient $c_{\pi_{\lambda/\mu},\nu}$ is equal to the number of set-valued tableaux $S$ of shape $\lambda/\mu$ such that the column reading word of $S$ is a reverse lattice word with content $\nu$.*

*Proof.* Notice that if the set-valued tableau $S$ is mapped to the diagram $T$, then the content of $S$ is the list of column lengths of $T$. The number of set-valued tableau $S$ of the corollary is therefore equal to the number of increasing tableaux $T$ of shape conjugate to $\nu$ such that $w(T)=\pi_0\pi_{\lambda/\mu}^{-1}\pi_0$, which by Theorem 1 equals $c_{\pi_0\pi_{\lambda/\mu}\pi_0,\nu'}=c_{\pi_{\lambda/\mu},\nu}$ (see [3, Lemma 3.4]). □

3.6. **Generating increasing tableaux.** For practical applications of Theorem 1, it is desirable to have an efficient algorithm for generating all increasing tableaux which represent a given permutation. We will address the following slightly more general problem.

**Problem.** *Given a column $C_0$ of boxes containing integers that increase from top to bottom and a permutation $\pi\in S_\infty$, find all increasing tableaux $T$ such that $w(T)=\pi$ and so that $T$ can be attached to the right hand side of $C_0$ to form a larger increasing tableau.*

We can generate the solutions to this problem as follows. First we find all pairs $(C,\sigma)$ consisting of an increasing column $C$ and a permutation $\sigma$, such that

$\pi = w(C) \cdot \sigma$ and $C$ can be attached to the right hand side of $C_0$. For each such pair we recursively find all increasing tableaux $T'$ for which $w(T') = \sigma$ and $T'$ can be attached to the right side of $C$, thus forming one of the solutions $T$. Notice that it is sufficient to consider pairs $(C, \sigma)$ such that $\sigma(i) = i$ for the smallest integer $i$ for which $\pi(i) > i$. Furthermore, these pairs can be generated very quickly.

However, the algorithm in its present form is not efficient, because in many cases there are no increasing tableaux $T$ satisfying the stated conditions, and it may require many recursive applications of the algorithm to discover this. We will fix this problem by describing an easy way to decide up front if at least one solution $T$ exists.

Stanley has proved [23] that the Schur expansion of his symmetric function $F_{\pi^{-1}}$ contains two special terms, each with coefficient one, which are indexed by partitions that are minimal and maximal in the dominance order among partitions occurring in $F_{\pi^{-1}}$. Accordingly, there is exactly one increasing tableau representing $\pi$ on each of these shapes. Let $M_\pi$ be the unique increasing tableau of the maximal shape. Then the integers $i_1 < i_2 < \cdots < i_p$ in the top row of $M_\pi$ satisfy that each $i_k$ is the largest descent position smaller than $i_{k+1}$ of the permutation $\pi s_{i_p} s_{i_{p-1}} \cdots s_{i_{k+1}}$ (we set $i_{p+1} = \infty$), and the permutation $\pi s_{i_p} \cdots s_{i_1}$ has no descent positions smaller than $i_1$. This characterizes the top row of $M_\pi$, and the part of $M_\pi$ below this row is equal to $M_{\pi s_{i_p} \cdots s_{i_1}}$.

We leave it as an exercise for the reader to show that the integers of the first column of $M_\pi$ are larger than the integers in the first column of any other increasing tableau representing $\pi$. In other words, if $w(T) = \pi$ and the leftmost box of row $r$ in $T$ contains the integer $x$, then either $M_\pi$ has fewer than $r$ rows, or the first integer in its $r$th row is larger than or equal to $x$. This property of $M_\pi$ implies that the set of solutions $T$ to our problem is nonempty if and only if $M_\pi$ is a solution. When this criterion is incorporated, our algorithm is fairly efficient.

3.7. **Products of increasing tableaux.** Given two increasing tableau $T_1$ and $T_2$, we let $T_1 \cdot T_2$ denote the increasing tableau obtained by Hecke inserting the word of $T_1$ into $T_2$. More precisely, if $a_1 a_2 \cdots a_p$ is the word of $T_1$ then we define $T_1 \cdot T_2 = (a_1 \xrightarrow{H} (a_2 \xrightarrow{H} (\cdots (a_p \xrightarrow{H} T_2) \cdots)))$. When the concatenation of the words of $T_1$ and $T_2$ is a reduced word of a permutation, this product agrees with the Coxeter-Knuth product, which is known to be associative. Unfortunately our more general product of increasing tableaux is not associative. We make the convention that $T_1 \cdot T_2 \cdot T_3$ means $T_1 \cdot (T_2 \cdot T_3)$.

**Example 9.** Let $T_1 = \begin{array}{|c|}\hline 1 \\ \hline \end{array}$, $T_2 = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}$, and $T_3 = \begin{array}{|c|}\hline 2 \\ \hline \end{array}$. Then

$$(T_1 \cdot T_2) \cdot T_3 \;=\; \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \cdot \begin{array}{|c|}\hline 2 \\ \hline \end{array} \;=\; \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}$$

whereas

$$T_1 \cdot (T_2 \cdot T_3) \;=\; \begin{array}{|c|}\hline 1 \\ \hline \end{array} \cdot \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline \end{array} \;=\; \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 & 5 \\ \hline \end{array}\,.$$

The product of increasing tableaux has the following properties, whose proofs are straightforward from the definitions.

**Lemma 4.** *Let $T$ and $T'$ be increasing tableaux. Then we have*

(1) $w(T \cdot T') = w(T) \cdot w(T')$.

(2) *Suppose $T$ is cut along a vertical line into $T_{left}$ and $T_{right}$. Then $T = T_{left} \cdot T_{right}$.*
(3) *Suppose $T$ is cut along a horizontal line into tableaux $T_{bottom}$ and $T_{top}$. Then $T = T_{bottom} \cdot T_{top}$.*

3.8. **Decreasing tableaux.** For our applications to $K$-theoretic factor sequences in the next section, it is more natural to work with decreasing tableaux, which by definition are Young tableaux with strictly decreasing rows and columns. By regarding decreasing tableaux as increasing tableaux with the order of the natural numbers inverted, we obtain well defined operations of Hecke insertion and products of decreasing tableaux as in Sections 3.1 and 3.7. For example we have

$$\begin{array}{|c|c|}\hline 5&3\\\hline 4\\\cline{1-1}\end{array} \cdot \begin{array}{|c|c|c|}\hline 6&3&1\\\hline 4&2\\\cline{1-2} 3\\\cline{1-1} 2\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 6&4&3&1\\\hline 5&3&2\\\cline{1-3} 4\\\cline{1-1} 2\\\cline{1-1}\end{array}$$

If $T$ is a decreasing tableau, we let $w(T)$ be the unique permutation that has the column word of $T$ as a Hecke word. Lemma 4 then remains true for products of decreasing tableaux. In addition we have the following decreasing version of Theorem 1.

**Theorem** 1′. *For any permutation $\pi$, the coefficient $c_{\pi,\lambda}$ of (1) equals $(-1)^{|\lambda|-\ell(\pi)}$ times the number of decreasing tableaux $T$ of shape $\lambda$ such that $w(T) = \pi$.*

*Proof.* Let $\pi \in S_n$ and let $\pi_0 \in S_n$ be the longest permutation. By replacing each entry $x$ in a tableau with $n - x$, we obtain a bijection between the decreasing tableaux representing $\pi$ and the increasing tableaux representing $\pi_0\pi\pi_0$. The theorem therefore follows from the identity $G_{\pi_0\pi^{-1}\pi_0} = G_\pi$, which is a consequence of Fomin and Kirillov's construction of Grothendieck polynomials. □

## 4. Quiver coefficients

4.1. **$K$-theoretic factor sequences.** Let $r = \{r_{ij}\}$ be a set of rank conditions for $0 \le i \le j \le n$, and set $N = e_0 + \cdots + e_n$ where $e_i = r_{ii}$. A result of Zelevinsky shows that when the base variety $\mathfrak{X}$ is a product of matrix spaces, the quiver variety $\Omega_r \subset \mathfrak{X}$ is isomorphic to a dense open subset of a Schubert variety [24]. The *Zelevinsky permutation* corresponding to this Schubert variety was used in [16] to prove the ratio formula for quiver varieties.

With the notation from [4], the Zelevinsky permutation can be constructed as a product of permutations as follows (see [16, Prop. 1.6] for a different construction). Extend the rank conditions $r = \{r_{ij}\}$ by setting $r_{ij} = e_j + \cdots + e_i$ for $0 \le j < i \le n$. Then define decreasing tableaux $U_{ij}$ as in the introduction, but for all $0 \le i < n$ and $0 < j \le n$. The corresponding permutations $W_{ij} = w(U_{ij})$ are given by

$$W_{ij}(p) = \begin{cases} p + r_{i,j-1} - r_{ij} & \text{if } r_{ij} < p \le r_{i+1,j} \\ p - r_{i+1,j} + r_{ij} & \text{if } r_{i+1,j} < p \le r_{i+1,j} + r_{i,j-1} - r_{ij} \\ p & \text{otherwise.} \end{cases}$$

The Zelevinsky permutation can now be defined by $z(r) = \prod_{j=1}^{n} \prod_{i=0}^{n-1} W_{ij}$. The descent positions of $z(r)$ are contained in the set $\{r_{nj} : 0 < j \le n\}$, and the descent positions of $z(r)^{-1}$ are contained in $\{r_{i0} : 0 \le i < n\}$.

For each $1 \leq j \leq n-1$ we set $\delta_j = W_{jj}W_{j+1,j}\cdots W_{n-1,j} \in S_N$. A *KMS-factorization* for the rank conditions $r$ is any sequence $(\pi_1, \ldots, \pi_n)$ of permutations with $\pi_i \in S_{e_{i-1}+e_i}$, such that the Zelevinsky permutation $z(r)$ is equal to the Hecke product

$$\pi_1 \cdot \delta_1 \cdot \pi_2 \cdot \delta_2 \cdots \delta_{n-1} \cdot \pi_n \,.$$

These sequences of permutations generalize the notion of a minimal lace diagram from [16] and give the index set in the $K$-theoretic stable component formula (6) from [4, 20].

We begin by defining a *$K$-theoretic factor sequence* for the rank conditions $r$ to be any sequence $(T_1, \ldots, T_n)$ of decreasing tableaux, such that $(w(T_1), \ldots, w(T_n))$ is a KMS-factorization for $r$. With this definition, Theorem 2 is an immediate consequence of Theorem 1 combined with the $K$-theoretic stable component formula (6). To obtain the inductive definition of factor sequences given before Theorem 2 we need the following result, proved in [4, Thm. 7], which shows that KMS-factorizations can themselves be defined as 'factor sequences'. Recall the definition of $\overline{r}$ from Section 1.3.

**Theorem 5.** *(a) If $(\pi_1, \ldots, \pi_n)$ is a KMS-factorization for $r$, then each permutation $\pi_i$ has a reduced factorization $\pi_i = \rho_{i-1} \cdot W_{i-1,i} \cdot \sigma_i$ with $\rho_{i-1} \in S_{e_{i-1}}$ and $\sigma_i \in S_{e_i}$, such that $\rho_0 = \sigma_n = 1$.*

*(b) Let $\sigma_1, \rho_1, \ldots, \sigma_{n-1}, \rho_{n-1}$ be permutations with $\sigma_i, \rho_i \in S_{e_i}$. Then the sequence $(W_{01} \cdot \sigma_1, \rho_1 \cdot W_{12} \cdot \sigma_2, \ldots, \rho_{n-1} \cdot W_{n-1,n})$ is a KMS-factorization for $r$ if and only if $(\sigma_1 \cdot \rho_1, \sigma_2 \cdot \rho_2, \ldots, \sigma_{n-1} \cdot \rho_{n-1})$ is a KMS-factorization for $\overline{r}$.*

We also need the following statement.

**Lemma 5.** *Let $T$ be any decreasing tableau such that $w(T) \in S_m$, and for some integers $a, b < m$ we have $w(T)(p) \leq b$ for all $a < p \leq m$. Then $T$ contains the rectangle $R = (m-b) \times (m-a)$ in its upper left corner. The upper-left box of $R$ equals $m-1$, and the boxes of $R$ decrease by one for each step down or to the right.*

*Proof.* After deleting the contents of some boxes of $T$, the permutation $w(T)$ becomes equal to the south-west to north-east (permutation) product of the simple transpositions corresponding to the non-empty boxes in $T$. Since the integers in all these boxes are smaller than $m$, the assumption that $w(T)(m) \leq b$ implies that the top of the first column of $T$ must contain the integers $m-1, m-2, \ldots, b$. The assumption that $w(T)(m-1) \leq b$ then implies that the second column of $T$ starts with $m-2, m-3, \ldots, b-1$, etc. □

Let $(T_1, T_2) \mapsto T_1 \cdot T_2$ be the product of decreasing tableaux from Section 3.8.

**Corollary 2.** *A sequence of decreasing tableaux $(T_1, \ldots, T_n)$ is a $K$-theoretic factor sequence for the rank conditions $r$ if and only if there exist decreasing tableaux $A_i, B_i$ for $1 \leq i \leq n-1$, such that $T_i = B_{i-1} \cdot U_{i-1,i} \cdot A_i$ for each $i$ (with $B_0 = A_n = \emptyset$) and $(A_1 \cdot B_1, \ldots, A_{n-1} \cdot B_{n-1})$ is a $K$-theoretic factor sequence for $\overline{r}$.*

*Proof.* Let $(T_1, \ldots, T_n)$ be a factor sequence for $r$ and $(\pi_1, \ldots, \pi_n)$ the corresponding KMS-factorization. It follows from Theorem 5 (a) that $\pi_i \in S_{e_{i-1}+e_i-r_{i-1,i}}$ and that $\pi_i(p) \leq e_{i-1}$ for all $e_i < p \leq e_{i-1} + e_i - r_{i-1,i}$. Since $T_i$ represents $\pi_i$, Lemma 5 implies that $T_i$ contains the tableau $U_{i-1,i}$ in its upper-left corner. Now write $T_i = B_{i-1} \cdot U_{i-1,i} \cdot A_i$ where $A_i$ is the part of $T_i$ to the right of $U_{i-1,i}$ and $B_{i-1}$ is

the part below $U_{i-1,i}$ and $A_i$.

$$T_i \;=\; \begin{array}{|c|c|}\hline U_{i-1,i} & A_i \\ \hline B_{i-1} \\ \cline{1-1}\end{array}$$

Then we have $\pi_i = w(B_{i-1}) \cdot W_{i-1,i} \cdot w(A_i)$ by Lemma 4, and all entries of $A_i$ and $B_i$ are smaller than $e_i$. Since all descent positions of $z(r) = \pi_1 \cdot \delta_1 \cdot \pi_2 \cdots \delta_{n-1} \cdot \pi_n$ are greater than or equal to $e_n$, the same must be true for $\pi_n$, so $A_n$ must be empty. Similarly, since the descent positions of $\pi_1^{-1}$ are greater than or equal to $e_0$, $B_0$ is empty. Now it follows from Theorem 5 (b) that $(w(A_1)\cdot w(B_1), \ldots, w(A_{n-1}) \cdot w(B_{n-1}))$ is a KMS-factorization for $\overline{r}$, or equivalently that $(A_1 \cdot B_1, \ldots, A_{n-1} \cdot B_{n-1})$ is a factor sequence.

On the other hand, if we are given decreasing tableaux $A_1, B_1, \ldots, A_{n-1}, B_{n-1}$ such that $(A_1 \cdot B_1, \ldots, A_{n-1} \cdot B_{n-1})$ is a factor sequence for $\overline{r}$ then Theorem 5 (a) implies that the entries of $A_i$ and $B_i$ are smaller than $r_{i-1,i} + r_{i,i+1} - r_{i-1,i+1} \leq e_i$, so it follows from Theorem 5 (b) that $(U_{01} \cdot A_1, B_1 \cdot U_{12} \cdot A_2, \ldots, B_{n-1} \cdot U_{n-1,n})$ is a factor sequence for $r$. □

This completes the proof of Theorems 2 and 3.

**Example 10.** Consider a sequence of vector bundles $E_0 \to E_1 \to E_2 \to E_3$ of ranks $1, 4, 3, 3$ together with the rank conditions

$$r \;=\; \begin{Bmatrix} r_{00} & & r_{11} & & r_{22} & & r_{33} \\ & r_{01} & & r_{12} & & r_{23} & \\ & & r_{02} & & r_{13} & & \\ & & & r_{03} & & & \end{Bmatrix} \;=\; \begin{Bmatrix} 1 & & 4 & & 3 & & 3 \\ & 1 & & 2 & & 2 & \\ & & 1 & & 1 & & \\ & & & 0 & & & \end{Bmatrix}.$$

These rank conditions result in the following diagram of decreasing tableaux $U_{ij}$:

$$\begin{array}{ccccc} \emptyset & & \boxed{4}\boxed{3} & & \boxed{3} \\ & \emptyset & & \boxed{2} & \\ & & \boxed{1} & & \end{array}$$

The two bottom rows of this diagram produce the following three factor sequences for the inductive rank conditions $\overline{r}$:

$$(\boxed{1}, \boxed{2})\,,\; \left(\emptyset, \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}\right),\; \left(\boxed{1}, \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}\right).$$

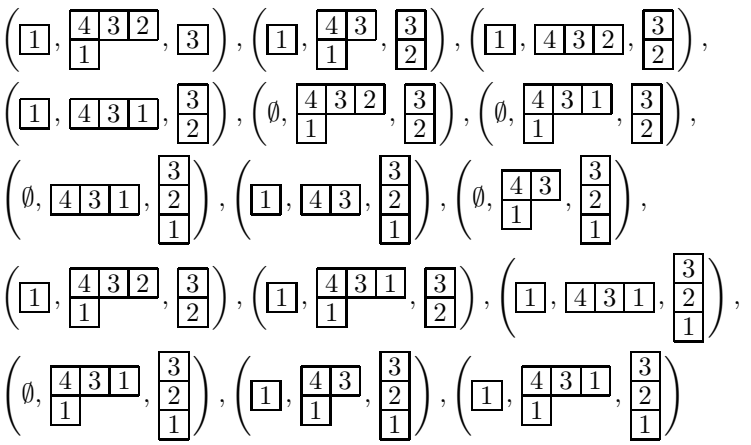
Since the decreasing tableau $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$ has the factorizations

$$\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \;=\; \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \cdot \emptyset \;=\; \boxed{1} \cdot \boxed{2} \;=\; \emptyset \cdot \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \;=\; \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \cdot \boxed{2} \;=\; \boxed{1} \cdot \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$$

and a single box has the factorizations $\boxed{x} = \boxed{x} \cdot \emptyset = \emptyset \cdot \boxed{x} = \boxed{x} \cdot \boxed{x}$, the factor sequences for $\overline{r}$ produce the following list of factor sequences for $r$:

$$(\boxed{1}, \boxed{4}\boxed{3}\boxed{2}, \boxed{3})\,, \left(\boxed{1}, \boxed{4}\boxed{3}, \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array}\right), \left(\emptyset, \begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}, \boxed{3}\right),$$

$$\left(\emptyset, \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array}\right), \left(\emptyset, \boxed{4}\boxed{3}\boxed{1}, \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array}\right), \left(\emptyset, \boxed{4}\boxed{3}, \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}\right),$$



These factor sequences can also be obtained by first working out the 13 possible KMS-factorizations for $r$, for example by using Theorem 5 or the transformations on KMS-factorizations given in [5, §5]. We conclude that the Grothendieck class of the quiver variety $\Omega_r(E_\bullet)$ is obtained by replacing each tensor $G_{\mu_1} \otimes G_{\mu_2} \otimes G_{\mu_3}$ in the following expression with the class $G_{\mu_1}(E_1 - E_0) \cdot G_{\mu_2}(E_2 - E_1) \cdot G_{\mu_3}(E_3 - E_2)$:

$$\begin{aligned}
&G_1 \otimes G_3 \otimes G_1 + G_1 \otimes G_2 \otimes G_{11} + 1 \otimes G_{31} \otimes G_1 + 1 \otimes G_{21} \otimes G_{11} \\
&+ 1 \otimes G_3 \otimes G_{11} + 1 \otimes G_2 \otimes G_{111} - G_1 \otimes G_{31} \otimes G_1 - G_1 \otimes G_{21} \otimes G_{11} \\
&- 2 \cdot G_1 \otimes G_3 \otimes G_{11} - 2 \cdot 1 \otimes G_{31} \otimes G_{11} - 1 \otimes G_3 \otimes G_{111} - G_1 \otimes G_2 \otimes G_{111} \\
&- 1 \otimes G_{21} \otimes G_{111} + 2 \cdot G_1 \otimes G_{31} \otimes G_{11} + G_1 \otimes G_3 \otimes G_{111} + 1 \otimes G_{31} \otimes G_{111} \\
&+ G_1 \otimes G_{21} \otimes G_{111} - G_1 \otimes G_{31} \otimes G_{111} \,.
\end{aligned}$$

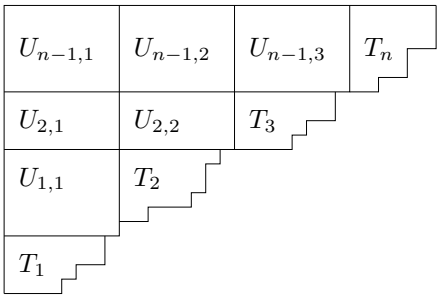
4.2. **Grothendieck polynomials for Zelevinsky permutations.** Using results about Demazure characters it was proved in [16] that cohomological quiver coefficients are special cases of the Stanley coefficients associated to the Zelevinsky permutation $z(r)$. As an application of our results, we will prove more generally that the $K$-theoretic quiver coefficients are special cases of the coefficients $c_{z(r),\lambda}$ in the expansion (1) of the stable Grothendieck polynomial for $z(r)$. This result also sharpens the fact from [4, 9] that quiver coefficients are special cases of the decomposition coefficients of Grothendieck polynomials studied in [7] (see Section 4.3). Given a sequence of partitions $\mu = (\mu_1, \ldots, \mu_n)$ such that $\mu_i$ is contained in the rectangle $e_i \times e_{i-1}$, let $\lambda(\mu)$ be the partition obtained by concatenating the partitions $(e_0 + e_1 + \cdots + e_{i-2})^{e_i} + \mu_i$ for $i = n, n-1, \ldots, 1$.

**Theorem 6.** *For any set of rank conditions $r$ and sequence of partitions $\mu$ we have $c_\mu(r) = c_{z(r),\lambda(\mu)}$.*

Our proof of the above identity is based on a bijection between the $K$-theoretic factor sequences for $r$ and the decreasing tableaux representing $z(r)$. Given a sequence $(T_1, \ldots, T_n)$ of decreasing tableaux, such that each tableau $T_i$ can be contained in the rectangle $e_i \times e_{i-1}$ and all entries of $T_i$ are smaller than $e_{i-1} + e_i$, we let $\Phi(T_1, \ldots, T_n)$ denote the decreasing tableau constructed from this sequence

as well as the tableaux $U_{ij}$ for $i \geq j$ as follows.

$\Phi(T_1, \ldots, T_n) \quad =$




Notice that the upper-left box of $U_{n-1,1}$ is equal to $N-1$, and the boxes in the union of tableaux $U_{ij}$ decrease by one for each step down or to the right. Theorem 6 follows from the following proposition combined with Theorems 1 and 2.

**Proposition 1.** *The map $(T_1, \ldots, T_n) \mapsto \Phi(T_1, \ldots, T_n)$ gives a bijection of the set of all $K$-theoretic factor sequences for $r$ with the set of all decreasing tableaux representing $z(r)$.*

*Proof.* Since the permutation of a decreasing tableau can be defined as the south-west to north-east Hecke product of the simple reflections given by the boxes of the tableau, it follows from the definition of KMS-factorizations that $(T_1, \ldots, T_n)$ is a factor sequence if and only if $\Phi(T_1, \ldots, T_n)$ represents the Zelevinsky permutation $z(r)$. It remains to show that any decreasing tableau $T$ representing $z(r)$ contains the arrangement of rectangular tableaux $U_{ij}$ in its upper-left corner, and has no boxes strictly south-east of the tableaux $U_{ii}$ for $1 \leq i \leq n-1$. The inclusion of the tableaux $U_{ij}$ in $T$ follows from Lemma 5 because $z(r) \in S_N$ and for each $0 < i < n$ and $p > r_{ni}$ we have $z(r)(p) \leq r_{i0}$, see [16, Prop 1.6] or [4, Lemma 3.1].

To see that $T$ contains no boxes strictly south-east of $U_{ii}$, we use that the Grothendieck polynomial $\mathfrak{G}_{\widehat{z}(r)}(x_1, \ldots, x_N)$ is separately symmetric in each group of variables $\{x_p \mid r_{n,i} < p \leq r_{n,i-1}\}$, where $\widehat{z}(r) = \pi_0^{(N)} z(r)^{-1} \pi_0^{(N)}$ and $\pi_0^{(N)}$ is the longest permutation in $S_N$. This is true because the descent positions of $\widehat{z}(r)$ are contained in the set $\{r_{nj} \mid 0 < j \leq n\}$. It follows that the exponent of $x_{r_{ni}+1}$ in any monomial of $\mathfrak{G}_{\widehat{z}(r)}(x_1, \ldots, x_N)$ is less than or equal to $N - r_{n,i-1} = r_{i-2,0}$. Now $T$ can be used to construct a unique compatible pair $(a, k)$ for $\widehat{z}(r)$, such that $T$ contains the integer $p$ in some box of row $q$ if and only if $(a_l, k_l) = (N-p, q)$ for some $l$. Since this pair contributes the monomial $x^k$ to $\mathfrak{G}_{\widehat{z}(r)}(x_1, \ldots, x_N)$, it follows that row $r_{ni}+1$ of $T$ has at most $r_{i-2,0}$ boxes. This means exactly that $T$ contains no boxes south-east of $U_{i-1,i-1}$, as required. □

**Example 11.** The Zelevinsky permutation for the rank conditions $r$ of Example 10 is given by $z(r) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 6 & 9 & 1 & 10 & 11 & 3 & 4 & 7 & 8 & 5 \end{pmatrix} \in S_{11}$. The decreasing tableaux representing this permutation are obtained by attaching the factor sequences for $r$ to the bottom side, the middle corner, and the right side of the

tableau

| | | | | |
|---|---|---|---|---|
| 10 | 9 | 8 | 7 | 6 |
| 9 | 8 | 7 | 6 | 5 |
| 8 | 7 | 6 | 5 | 4 |
| 7 | | | | |
| 6 | | | | |
| 5 | | | | |

.

4.3. **Universal Grothendieck polynomials.** Fulton's *universal Schubert polynomials* [15] describe certain quiver varieties associated to a sequence of vector bundles $E_1 \to \cdots \to E_{n-1} \to E_n \to F_n \to F_{n-1} \to \cdots \to F_1$ over $X$, such that $\mathrm{rank}(E_i) = \mathrm{rank}(F_i) = i$ for each $i$. They are also known to specialize to, e.g., the quantum Schubert polynomials [11], where a nonrecursive combinatorial formula was given in [8]. We now describe an extension of this result to $K$-theory.

Given a permutation $\pi \in S_{n+1}$, we let $\Omega_\pi \subset X$ be the degeneracy locus of points where the rank of each composed map $E_q \to F_p$ is at most equal to the number of integers $i \leq p$ such that $\pi(i) \leq q$. The quiver formula (5) can be applied to give a formula

$$[\mathcal{O}_{\Omega_\pi}] = \sum_\mu c^{(n)}_{\pi,\mu}\, G_{\mu_1}(E_2 - E_1) \cdots G_{\mu_n}(F_n - E_n) \cdots G_{\mu_{2n-1}}(F_1 - F_2) \tag{9}$$

for the Grothendieck class of $\Omega_\pi$, where the coefficients $c^{(n)}_{\pi,\mu}$ are special cases of quiver coefficients. It was shown in [2] that the coefficients $c_{\pi,\lambda}$ of the expansion (1) of the stable Grothendieck polynomial for $\pi$ can be obtained as the specializations $c^{(n)}_{\pi,(\emptyset^{n-1},\lambda,\emptyset^{n-1})}$, where $\emptyset^{n-1}$ denotes a sequence of $n-1$ empty partitions. More generally, it was proved in [7, Thm. 4] that the coefficients $c^{(n)}_{\pi,\lambda}$ can be used to expand a double Grothendieck polynomial as a linear combination of products of stable Grothendieck polynomials applied to disjoint intervals of variables. In [7], the formula (9) was also used to prove that

$$[\mathcal{O}_{\Omega_\pi}] = \sum (-1)^{\ell(\sigma_1 \cdots \sigma_{2n-1}\pi)} G_{\sigma_1}(E_2 - E_1) \cdots G_{\sigma_n}(F_n - E_n) \cdots G_{\sigma_{2n-1}}(F_1 - F_2)$$

where this sum is over all sequences of permutations $(\sigma_1, \ldots, \sigma_{2n-1})$ such that $\sigma_i \in S_{\min(i,2n-i)+1}$ and $\pi$ is equal to the Hecke product $\sigma_1 \cdot \sigma_2 \cdots \sigma_{2n-1}$. Combining this with Theorem 1, we obtain the following generalization of [8, Thm. 1].

**Theorem 7.** *The coefficient $c^{(n)}_{\pi,\mu}$ of (9) is equal to $(-1)^{\sum |\mu_i| - \ell(\pi)}$ times the number of sequences $(T_1, \ldots, T_{2n-1})$ of increasing tableaux of shapes $(\mu_1, \ldots, \mu_{2n-1})$, such that the entries of $T_i$ are at most $\min(i, 2n-i)$ and $w(T_{2n-1} \cdots T_2 \cdot T_1) = \pi^{-1}$.*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