Combinatoire/Combinatorics (Geometrie algébrique/Algebraic Geometry)

# SPECIALIZATIONS OF GROTHENDIECK POLYNOMIALS

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ABSTRACT. We prove a formula for double Schubert and Grothendieck polynomials specialized to two rearrangements of the same set of variables. Our formula generalizes the usual formulas for Schubert and Grothendieck polynomials in terms of RC-graphs, and it gives immediate proofs of many other important properties of these polynomials.

# SPÉCIALIZATION DE POLYNÔMES DE GROTHENDIECK

RÉSUMÉ. On démontre une formule pour les polynômes de Schubert et de Grothendieck dans le cas de réarrangements du même ensemble de variables. Cette formule généralise les formules usuelles pour ces polynômes en termes de RC-graphs et donne des démonstrations immédiates de plusieurs propriétés importantes de ces polynomes.

## 1. Introduction

Let $v, w \in S_n$ be permutations and let $\mathfrak{S}_w(x;y)$ and $\mathfrak{G}_w(a;b)$ denote the double Schubert and Grothendieck polynomials of Lascoux and Schützenberger [15]. The goal of this note is to prove a formula for the specializations of these polynomials to different rearrangements of the same set of variables. For example

$$\mathfrak{S}_w(y_v;y) = \mathfrak{S}_w(y_{v(1)},\dots,y_{v(n)}\,;\,y_1,\dots,y_n)\,.$$

The double Schubert polynomial $\mathfrak{S}_w(x;y)$ represents the class of the Schubert variety for $w$ in the torus-equivariant cohomology of $\mathrm{SL}_n(\mathbb{C})/B$. The specialization $\mathfrak{S}_w(y_v;y)$ gives the restriction of this class to the fixed point corresponding to $v$ [9, Thm. 2.3]. Equivalently, $\mathfrak{S}_w(x;y)$ represents the class of an orbit in the $B \times B$-equivariant cohomology of $\mathbb{C}^{n\times n}$ [8, 4, 10], and $\mathfrak{S}_w(y_v;y)$ is the restriction of this class to another orbit, i.e. an 'incidence class' in the sense of [19]. Specialized Grothendieck polynomials $\mathfrak{G}_w(b_v;b)$ have similar interpretations in equivariant $K$-theory. Both kinds of polynomials also show up in transition matrices between natural bases of Hecke algebras [12, 16, 17, 14]. For example, the operator on polynomials that maps $y_i$ to $y_{v(i)}$ for all $i$ is equal to $\sum_w \mathfrak{S}_w(y_v;y)\,\partial_w$, where $\partial_w$ is a divided difference operator [17, Prop. 3.2]. Some ways to compute specialized Schubert and Grothendieck polynomials using operators on polynomials are furthermore suggested in [14].

The formula proved in this paper generalizes the usual formulas for Schubert and Grothendieck polynomials in terms of RC-graphs [6, 5, 1, 11], and it furthermore gives immediate proofs of several important properties of these polynomials, some of which have natural proofs in the above interpretations. We include a characterization of the Bruhat order [13, 17, 9], the existence and supersymmetry of stable

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Schubert and Grothendieck polynomials [6], and the statements about Schubert and Grothendieck polynomials needed in [3] and [2]. The proof of our formula relies on Fomin and Kirillov's construction of Grothendieck polynomials [5].

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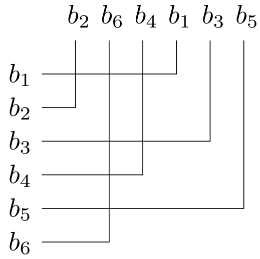
## 2. The main theorem

Consider the diagram $\mathfrak{D}_v$ consisting of hooks of lines going due north and due west from the points $(v(j), j)$, and with each such hook labeled by $b_{v(j)}$. For example, when $v = 264135$ we get:

$\mathfrak{D}_v$ =




We let $C(\mathfrak{D}_v)$ denote the crossing positions of this diagram, i.e. the points $(i, j)$ such that $v(j) > i$ and $v^{-1}(i) > j$. Notice that $C(\mathfrak{D}_v) = D(v^{-1})$ with the notation of [18, p. 8]. For $(i, j) \in C(\mathfrak{D}_v)$ we let $\nu(i, j)$ be one plus the number of hooks going north-west of $(i, j)$ in the diagram $\mathfrak{D}_v$, i.e.

$$\nu(i, j) = j + \#\{k > j : v(k) < i\} .$$

We need the *degenerate Hecke algebra*, which is the free $\mathbb{Z}$-algebra $\mathcal{H}$ generated by symbols $s_1, s_2, \ldots$, modulo the relations (i) $s_i s_j = s_j s_i$ if $|i - j| \geq 2$, (ii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, and (iii) $s_i^2 = -s_i$. This algebra has a basis of permutations.

For a subset $D \subset C(\mathfrak{D}_v)$, consider the product in $\mathcal{H}$ of the simple reflections $s_{\nu(i,j)}$ for $(i, j) \in D$, in south-west to north-east order, i.e. $s_{\nu(i,j)}$ must come before $s_{\nu(i',j')}$ if $i \geq i'$ and $j \leq j'$. This product is equal to plus or minus a single permutation $w(D)$. We say that $D$ is a *Fomin-Kirillov graph* (or *FK-graph*) for this permutation w.r.t. the diagram $\mathfrak{D}_v$, and that $D$ is *reduced* if $|D|$ equals the length of $w(D)$.

An FK-graph $D$ can be pictured by replacing the crossing positions of $\mathfrak{D}_v$ which belong to $D$ with the symbol "┼", while the remaining crossing positions are replaced with the symbol "╯╭". If $D$ is reduced then the string entering the resulting diagram at column $j$ at the top will exit at row $w(D)(j)$ at the left hand side.

Our main result is the following theorem, which is proved combinatorially in the next section. It is natural to ask for a geometric proof as well.

**Theorem.** *For permutations $v, w \in S_n$ and variables $b_1, \ldots, b_n$ we have*

$$\mathfrak{G}_w(b_v; b) \;=\; \sum_D (-1)^{|D|-\ell(w)} \prod_{(i,j)\in D} \left(1 - \frac{b_i}{b_{v(j)}}\right)$$

*where the sum is over all FK-graphs $D$ for $w$ w.r.t. $\mathfrak{D}_v$.*

**Corollary 1.** *For permutations $v, w \in S_n$ and variables $y_1, \ldots, y_n$ we have*

$$\mathfrak{S}_w(y_v; y) \;=\; \sum_D \prod_{(i,j)\in D} (y_{v(j)} - y_i)$$

*where the sum is over all reduced FK-graphs $D$ for $w$ w.r.t. $\mathfrak{D}_v$.*

**Corollary 2.** *The usual formulas for double Schubert and Grothendieck polynomials in terms of RC-graphs are true (see* [6, 5, 1, 11]*).*

*Proof.* Apply the theorem to $\mathfrak{G}_w(a_1,\dots,a_n,b_1,\dots,b_n\,;\,b_1,\dots,b_n,a_1,\dots,a_n)$. $\square$

The next corollary recovers the characterization of the Bruhat order proved in [13, Cor. 3.2] (cf. [17, p. 171]) and [9, Thm. 2.4].

**Corollary 3.** *Let $v, w \in S_n$. The following are equivalent:*

(1) *$w \le v$ in the Bruhat order.*
(2) $\mathfrak{S}_w(y_v;y) \neq 0$.
(3) $\mathfrak{G}_w(b_v;b) \neq 0$.

*Proof.* The product defining $w(C(\mathfrak{D}_v))$ is a reduced expression for $v$. There exists a reduced FK-graph $D \subset C(\mathfrak{D}_v)$ for $w$ if and only if $w$ equals a reduced subexpression of this product. The later is equivalent to $w \le v$. This shows that each of (2) and (3) imply (1) (these implications are clear from geometry, too.) It is also clear that (2) implies (3). To see that (1) implies (2), notice that if $(i,j) \in C(\mathfrak{D}_v)$ then $i < v(j)$. Therefore each reduced FK-graph $D$ for $w$ in Corollary 1 contributes a positive polynomial in the variables $z_i = y_{i+1} - y_i$. $\square$

The following corollary implies that stable double Schubert and Grothendieck polynomials exist and are supersymmetric [20, 15, 6, 5].

**Corollary 4.** *Let $w \in S_n$ and $m \le n$. Then we have*

$$\mathfrak{G}_w(c_1,\dots,c_m,a_{m+1},\dots,a_n\,;\,c_1,\dots,c_m,b_{m+1},\dots,b_n) = \begin{cases} \mathfrak{G}_u(a_{m+1},\dots,a_n;b_{m+1},\dots,b_n) & \text{if } w = 1^m \times u \text{ for some } u \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Apply the theorem to $\mathfrak{G}_w(c,a,b\,;\,c,b,a)$. $\square$

It was proved in [4, Prop. 4.1] that the Schubert polynomial $\mathfrak{S}_v(y_v;y)$ is a product of linear factors. This also follows from the viewpoint of [12, 16, 17], or from our theorem.

**Corollary 5.** *For $v \in S_n$ we have*

$$\mathfrak{G}_v(b_v;b) = \prod_{(i,j)\in C(\mathfrak{D}_v)} \left(1 - \frac{b_i}{b_{v(j)}}\right).$$

Corollaries 3 through 5 above include all the facts about Schubert polynomials required in [3]. We remark that Cor. 5.1 and Prop. 9.2 of [2] are also special cases of our theorem.

## 3. Proof of the main theorem

Let $R$ be the ring of Laurent polynomials in the variables $a_i$ and $b_i$, $1 \le i \le n$. For $c \in R$ we set $h_i(c) = (1 + (1-c)\,s_i) \in \mathcal{H} \otimes R$. As observed in [5], these elements satisfy the Yang-Baxter identities $h_i(c)\,h_j(d) = h_j(d)\,h_i(c)$ for $|i-j| \ge 2$; $h_i(c)\,h_i(d) = h_i(cd)$; and $h_i(c)\,h_{i+1}(cd)\,h_i(d) = h_{i+1}(d)\,h_i(cd)\,h_{i+1}(c)$.

For $p \ge q$ we furthermore set

$$A_p^q(c;k) \;=\; h_{k-1+p}(b_p/c)\,h_{k-1+p-1}(b_{p-1}/c)\cdots h_{k-1+q}(b_q/c)$$

and define, following [7] and [5, (2.1)], the product

$$\mathfrak{G}^{(n)}(a;b) = A^1_{n-1}(a_1;1)\,A^1_{n-2}(a_2;2)\cdots A^1_1(a_{n-1};n-1) \;\in\; \mathcal{H}\otimes R\,.$$

Fomin and Kirillov have proved that the coefficient of each permutation $w \in S_n$ in $\mathfrak{G}^{(n)}(a;b)$ is equal to the Grothendieck polynomial $\mathfrak{G}_w(a;b)$ (see Thm. 2.3 and the remark on page 7 of [5], and use the change of variables $x_i = 1 - a_i^{-1}$ and $y_i = 1 - b_i$.) We claim that the specialization $\mathfrak{G}^{(n)}(b_v;b)$ is equal to the south-west to north-east product of the factors $h_{\nu(i,j)}(b_i/b_{v(j)})$ for all $(i,j) \in C(\mathfrak{D}_v)$.

By descending induction on $q$, the above Yang-Baxter identities imply that

$$A^q_p(c;k-1)\,A^q_{p-1}(d;k)\,h_{q+k-2}(c/d) = A^q_p(d;k-1)\,A^q_{p-1}(c;k)\,,$$

from which we deduce that, for $2 \le k \le n-p$, we have

$$A^{p+1}_{n-k+1}(b_p;k-1)\,A^1_{n-k}(a_k;k) \;=\; A^{p+1}_{n-k+1}(a_k;k-1)\,A^1_{p-1}(a_k;k)\,A^{p+1}_{n-k}(b_p;k)\,.$$

By using this identity repeatedly, and setting $\widetilde{a} = (a_2,\dots,a_n)$ and $\widetilde{b} = (b_1,\dots,b_{p-1}, b_{p+1},\dots,b_n)$, we obtain that

$$\begin{aligned}
\mathfrak{G}^{(n)}(b_p,\widetilde{a}\,;\,b) \;&=\; A^1_{p-1}(b_p;1)\,A^{p+1}_{n-1}(b_p;1)\prod_{k=2}^{n-1} A^1_{n-k}(a_k;k) \\
&= A^1_{p-1}(b_p;1)\left(\prod_{k=2}^{n-p} A^{p+1}_{n-k+1}(a_k;k-1)\,A^1_{p-1}(a_k;k)\right)\left(\prod_{k=n-p+1}^{n-1} A^1_{n-k}(a_k;k)\right) \\
&= A^1_{p-1}(b_p;1)\left(1\times\mathfrak{G}^{(n-1)}(\widetilde{a};\widetilde{b})\right)\,.
\end{aligned}$$

Here "1×" is the operator on $\mathcal{H}\otimes R$ which maps $s_i$ to $s_{i+1}$ for all $i$. By setting $p = v(1)$ and $\widetilde{a} = (b_{v(2)},\dots,b_{v(n)})$, the above claim follows by induction, and our theorem is an immediate consequence of the claim.

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