SCHUBERT POLYNOMIALS AND QUIVER FORMULAS

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Abstract. Fulton's universal Schubert polynomials [F3] represent degeneracy loci for morphisms of vector bundles with rank conditions coming from a permutation. The quiver formula of Buch-Fulton [BF] expresses these polynomials as an integer linear combination of products of Schur determinants. We present a positive, nonrecursive combinatorial formula for the coefficients. Our result is applied to obtain new expansions for the Schubert polynomials of Lascoux and Schützenberger [LS1] and explicit Giambelli formulas in the classical and quantum cohomology ring of any partial flag variety.

1. Introduction

The work of Buch and Fulton [BF] established a formula for a general kind of degeneracy locus associated to an oriented quiver of type A. The main ingredients in this formula are Schur determinants and certain integers, the *quiver coefficients*, which generalize the classical Littlewood-Richardson coefficients. Our aim in this paper is to prove a positive combinatorial formula for the quiver coefficients when the rank conditions defining the degeneracy locus are given by a permutation. In particular, this gives new expansions for Fulton's universal Schubert polynomials [F3] and the Schubert polynomials of Lascoux and Schützenberger [LS1].

Let $\mathfrak{X}$ be a smooth complex algebraic variety and let

$$G_1 \to \cdots \to G_{n-1} \to G_n \to F_n \to F_{n-1} \to \cdots \to F_1 \tag{1}$$

be a sequence of vector bundles and morphisms over $\mathfrak{X}$, such that G_i and F_i have rank i for each i. For every permutation w in the symmetric group S_{n+1} there is a degeneracy locus

$$\Omega_w(G_\bullet \to F_\bullet) = \{x \in \mathfrak{X} \mid \operatorname{rank}(G_q(x) \to F_p(x)) \leqslant r_w(p,q) \text{ for all } 1 \leqslant p,q \leqslant n\}\,,$$

where $r_w(p,q)$ is the number of $i \leqslant p$ such that $w(i) \leqslant q$. The universal double Schubert polynomial $\mathfrak{S}_w(c;d)$ of Fulton gives a formula for this locus; this is a polynomial in the variables $c_i(j)$ and $d_i(j)$ for $1 \leqslant i \leqslant j \leqslant n$. When the codimension of $\Omega_w(G_\bullet \to F_\bullet)$ is equal to the length of w, its class $[\Omega_w]$ in the cohomology (or Chow ring) of $\mathfrak{X}$ is obtained by evaluating $\mathfrak{S}_w(c;d)$ at the Chern classes $c_i(p)$ and $d_i(q)$ of the bundles F_p and G_q, respectively.

The quiver formula given in [BF] specializes to a formula for universal double Schubert polynomials:

$$\mathfrak{S}_w(c;d) = \sum_\lambda c_{w,\lambda}^{(n)}\, s_{\lambda^1}(d(2)-d(1)) \cdots s_{\lambda^n}(c(n)-d(n)) \cdots s_{\lambda^{2n-1}}(c(1)-c(2))\,.$$

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Here the sum is over sequences of $2n-1$ partitions $\lambda = (\lambda^1, \dots, \lambda^{2n-1})$ and each s_{λ^i} is a Schur determinant in the difference of the two alphabets in its argument. The quiver coefficients $c_{w,\lambda}^{(n)}$ can be computed by an inductive algorithm, and are conjectured to be nonnegative [BF].

Let $\mathrm{col}(T)$ denote the column word of a semistandard Young tableau T, the word obtained by reading the entries of the columns of the tableau from bottom to top and left to right. The following theorem is our main result.

Theorem 1. *Suppose that* $w \in \mathcal{S}_{n+1}$ *and* $\lambda = (\lambda^1, \dots, \lambda^{2n-1})$ *is a sequence of partitions. Then* $c_{w,\lambda}^{(n)}$ *equals the number of sequences of semistandard tableaux* $(T_1, \dots, T_{2n-1})$ *such that the shape of* T_i *is conjugate to* λ^i*, the entries of* T_i *are at most* $\min(i, 2n-i)$*, and* $\mathrm{col}(T_1) \cdots \mathrm{col}(T_{2n-1})$ *is a reduced word for* w*.*

As a consequence of Theorem 1, we obtain formulas for the Schubert polynomials of Lascoux and Schützenberger, expressing them as linear combinations of products of Schur polynomials in disjoint sets of variables. The coefficients in these expansions are all quiver coefficients $c_{w,\lambda}^{(n)}$; this is true in particular for the expansion of a Schubert polynomial as a linear combination of monomials (see, e.g., [BJS, FS]).

The *Stanley symmetric functions* or stable Schubert polynomials [St] play a central role in this paper. Our main result generalizes the formula of Fomin and Greene [FG] for the expansion of these symmetric functions in the Schur basis, as well as the connection between quiver coefficients and Stanley symmetric functions obtained in [B]. In fact, we show that the universal double Schubert polynomials can be expressed as a multiplicity-free sum of products of Stanley symmetric functions (Theorem 3).

It should be noted that the formula for Schubert polynomials suggested in this paper is different from the one given in [BF]. For example, the formula from [BF, §2.3] does not make it clear that the monomial coefficients of Schubert polynomials are quiver coefficients, or even that these monomial coefficients are nonnegative. We remark however that the arguments used in this article do not imply the quiver formula of [BF] in the case of universal Schubert polynomials, but rather rely on some results of loc. cit.

Knutson, Miller, and Shimozono have recently announced that they can prove that the general quiver coefficients defined in [BF] are non-negative, using different methods. On the other hand, the techniques of the present paper may also be used to obtain an analogous treatment of Grothendieck polynomials and quiver formulas for the structure sheaves of degeneracy loci in K-theory. This application is presented in [BKTY].

We review the universal Schubert polynomials and Stanley symmetric functions in Section 2; in addition, we prove some required properties. In Section 3 we introduce quiver varieties and we prove Theorem 1. In the following section we apply our results to the case of ordinary Schubert polynomials. Finally, in Section 5 we use our expressions for single Schubert polynomials to obtain explicit Giambelli formulas for the classical and quantum cohomology rings of partial flag varieties.

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2. Preliminaries

2.1. **Universal Schubert polynomials.** We begin by recalling the definition of the double Schubert polynomials of Lascoux and Schützenberger [LS1, L1]. Let $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ be two sequences of commuting independent variables. Given a permutation $w \in S_n$, the double Schubert polynomial $\mathfrak{S}_w(X;Y)$ is defined recursively as follows. If $w = w_0$ is the longest permutation in S_n then we set

$$\mathfrak{S}_{w_0}(X;Y) = \prod_{i+j \leqslant n} (x_i - y_j)\,.$$

Otherwise we can find a simple transposition $s_i = (i, i+1) \in S_n$ such that $\ell(ws_i) = \ell(w) + 1$. Here $\ell(w)$ denotes the length of w, which is the smallest number ℓ for which w can be written as a product of ℓ simple transpositions. We then define

$$\mathfrak{S}_w(X;Y) = \partial_i(\mathfrak{S}_{ws_i}(X;Y))$$

where ∂_i is the divided difference operator given by

$$\partial_i(f) = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}}\,.$$

The (single) Schubert polynomial is defined by $\mathfrak{S}_w(X) = \mathfrak{S}_w(X;0)$.

Suppose now that w is a permutation in S_{n+1}. If $u_1, \ldots, u_r$ are permutations, we will write $u_1 \cdots u_r = w$ if $\ell(u_1) + \cdots + \ell(u_r) = \ell(w)$ and the product of $u_1, \ldots, u_r$ is equal to w. In this case we say $u_1 \cdots u_r$ is a *reduced factorization* of w.

The *double universal Schubert polynomial* $\mathfrak{S}_w(c;d)$ of [F3] is a polynomial in the variables $c_i(j)$ and $d_i(j)$ for $1 \leqslant i \leqslant j \leqslant n$. For convenience we set $c_0(j) = d_0(j) = 1$ for all j and $c_i(j) = d_i(j) = 0$ if $i < 0$ or $i > j$. The classical Schubert polynomial $\mathfrak{S}_w(X)$ can be written uniquely in the form

$$\mathfrak{S}_w(X) = \sum a_{i_1,\ldots,i_n}\, e_{i_1}(x_1)\, e_{i_2}(x_1, x_2) \cdots e_{i_n}(x_1, \ldots, x_n)$$

where the sum is over all sequences $(i_1, \ldots, i_n)$ with each $i_\alpha \leqslant \alpha$ and $\sum i_\alpha = \ell(w)$ [LS3]. The coefficients $a_{i_1,\ldots,i_n}$ are uniquely determined integers depending on w. Define the single universal Schubert polynomial for w by

$$\mathfrak{S}_w(c) = \sum a_{i_1,\ldots,i_n}\, c_{i_1}(1)\, c_{i_2}(2) \cdots c_{i_n}(n)$$

and the double universal Schubert polynomial by

$$\mathfrak{S}_w(c;d) = \sum_{u \cdot v = w} (-1)^{\ell(u)} \mathfrak{S}_{u^{-1}}(d) \mathfrak{S}_v(c)\,.$$

Since the ordinary Schubert polynomial $\mathfrak{S}_w(X)$ does not depend on which symmetric group w belongs to, the same is true for $\mathfrak{S}_w(c;d)$.

As explained in the introduction, the double universal Schubert polynomials describe the degeneracy loci $\Omega_w(G_\bullet \to F_\bullet)$ of morphisms between vector bundles on a smooth algebraic variety $\mathfrak{X}$. The precise statement is the following result.

Theorem 2 (Fulton [F3])**.** *If the codimension of $\Omega_w(G_\bullet \to F_\bullet)$ is equal to $\ell(w)$ (or if this locus is empty) then the class of $\Omega_w(G_\bullet \to F_\bullet)$ in the cohomology ring of $\mathfrak{X}$ is obtained from $\mathfrak{S}_w(c;d)$ by evaluating at the Chern classes of the bundles, i.e., setting $c_i(j) = c_i(F_j)$ and $d_i(j) = c_i(G_j)$.*

We will need the following consequence of Theorem 2, which generalizes a result of Kirillov [K, (5.5)], see also [F3, (26)].

Proposition 1. *Choose $m \geqslant 0$ and substitute $b(j)$ for $c(j)$ and for $d(j)$ in $\mathfrak{S}_w(c;d)$ for all $j \geqslant m+1$. We then have*

$$\mathfrak{S}_w(c(1),\dots,c(m),b(m+1),\dots\,;\,d(1),\dots,d(m),b(m+1),\dots) = \begin{cases} \mathfrak{S}_w(c;d) & \text{if } w \in S_{m+1} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $w \in S_{m+1}$ then it follows from the definition that $\mathfrak{S}_w(c;d)$ is independent of $c_i(j)$ and $d_i(j)$ for $j \geqslant m+1$.

Assume that $w \in S_{n+1} \smallsetminus S_n$ where $n > m$. We claim that $\mathfrak{S}_w(c;d)$ vanishes as soon as we set $c_i(n) = d_i(n)$. To see this, consider a variety $\mathfrak{X}$ with bundles F_j for $1 \leqslant j \leqslant n$ and G_j for $1 \leqslant j \leqslant n-1$, such that $\operatorname{rank}(F_j) = \operatorname{rank}(G_j) = j$, and such that all monomials of degree $\ell(w)$ in the Chern classes of these bundles are linearly independent. For example we can take a product of Grassmann varieties $\mathfrak{X} = \mathrm{Gr}(1,N) \times \cdots \times \mathrm{Gr}(n,N) \times \cdots \times \mathrm{Gr}(1,N)$ for N large and use the tautological bundles.

If we set $G_n = F_n$ then we have a sequence of bundles (1) for which the map $G_n \to F_n$ is the identity and all other maps are zero. Since $r_w(n,n) = n-1$ it follows that the locus $\Omega_w(G_\bullet \to F_\bullet)$ is empty. Theorem 2 therefore implies that $\mathfrak{S}_w(c;d)$ is zero when $c_i(n) = d_i(n) = c_i(F_n)$ for $1 \leqslant i \leqslant n$. □

The next identity is due to Kirillov [K]; we include a proof for completeness.

Corollary 1 (Kirillov). *If $b_i(j)$, $c_i(j)$, and $d_i(j)$ are three sets of variables, $1 \leqslant i \leqslant j \leqslant n$, then we have*

$$\mathfrak{S}_w(c;d) = \sum_{u \cdot v = w} \mathfrak{S}_u(b;d)\mathfrak{S}_v(c;b)\,.$$

Proof. Let $\mathfrak{S}(c;d)$ denote the function from permutations to polynomials which maps w to $\mathfrak{S}_w(c;d)$. Following [M1, §6] we define the product of two such functions f and g by the formula

$$(fg)(w) = \sum_{u \cdot v = w} f(u)g(v)\,.$$

Fulton's definition of universal Schubert polynomials then says that $\mathfrak{S}(c;d) = \mathfrak{S}(0;d)\mathfrak{S}(c;0)$, and Proposition 1 shows that $\mathfrak{S}(0;c)\mathfrak{S}(c;0) = \mathfrak{S}(c;c) = \underline{\underline{1}}$, where $\underline{\underline{1}}(w) = \delta_{1,w}$. Now (6.6) of [M1] implies that also $\mathfrak{S}(c;0)\mathfrak{S}(0;c) = \underline{\underline{1}}$. We therefore obtain

$$\mathfrak{S}(c;d) = \mathfrak{S}(0;d)\mathfrak{S}(c;0) = \mathfrak{S}(0;d)\mathfrak{S}(b;0)\mathfrak{S}(0;b)\mathfrak{S}(c;0) = \mathfrak{S}(b;d)\mathfrak{S}(c;b)\,,$$

as required. □

Later, we will need the following observation: Proposition 1 and Corollary 1 together imply that

$$\mathfrak{S}_w(c;d) = \sum_{u \cdot v = w} \mathfrak{S}_u(0,\dots,0,c(r+1),c(r+2),\dots\,;\,d) \cdot \mathfrak{S}_v(c(1),\dots,c(r);0) \tag{2}$$

and similarly,

$$\mathfrak{S}_w(c;d) = \sum_{u \cdot v = w} \mathfrak{S}_u(0;d(1),\dots,d(r)) \cdot \mathfrak{S}_v(c;0,\dots,0,d(r+1),d(r+2),\dots)\,. \tag{3}$$

2.2. **Symmetric functions.** For each integer partition $\alpha = (\alpha_1 \geqslant \cdots \geqslant \alpha_p \geqslant 0)$, let $|\alpha| = \sum \alpha_i$ and let α' denote the conjugate of α. Let $c = \{c_1, c_2, \ldots\}$ and $d = \{d_1, d_2, \ldots\}$ be ordered sets of independent variables. Define the Schur determinant

$$s_\alpha(c - d) = \det(h_{\alpha_i + j - i})_{p \times p}$$

where the elements h_k are determined by the identity of formal power series

$$\sum_{k \in \mathbb{Z}} h_k t^k = \frac{1 - d_1 t + d_2 t^2 - \cdots}{1 - c_1 t + c_2 t^2 - \cdots}.$$

In particular, $h_0 = 1$ and $h_k = 0$ for $k < 0$. The *supersymmetric Schur functions* $s_\alpha(X/Y)$ are obtained by setting $c_i = e_i(X)$ and $d_i = e_i(Y)$ for all i, and the usual Schur polynomials are given by the specializations

$$s_\alpha(X/Y)|_{Y=0} = s_\alpha(X) \qquad \text{and} \qquad s_\alpha(X/Y)|_{X=0} = (-1)^{|\alpha|} s_{\alpha'}(Y).$$

If E and F are two vector bundles with total Chern classes $c(E)$ and $c(F)$, respectively, we will denote $s_\alpha(c(E) - c(F))$ by $s_\alpha(E - F)$. For any three bundles E, F, and G, there is a basic combinatorial identity [M2, §1.5]:

$$s_\alpha(G - E) = \sum N^\alpha_{\beta\gamma}\, s_\beta(F - E)\, s_\gamma(G - F), \tag{4}$$

where the sum is over partitions β and γ with $|\beta| + |\gamma| = |\alpha|$, and $N^\alpha_{\beta\gamma}$ is a Littlewood-Richardson coefficient.

Let Λ denote the ring of symmetric functions (as in [M2]). For each permutation $w \in S_n$ there is a stable Schubert polynomial or Stanley symmetric function $F_w \in \Lambda$ which is uniquely determined by the property that

$$F_w(x_1, \ldots, x_k) = \mathfrak{S}_{1^m \times w}(x_1, \ldots, x_k)$$

for all $m \geqslant k$. [1] Here $1^m \times w \in S_{n+m}$ is the permutation which is the identity on $\{1, \ldots, m\}$ and which maps j to $w(j - m) + m$ for $j > m$ (see [M1, (7.18)]). When F_w is written in the basis of Schur functions, one has

$$F_w = \sum_{\alpha\,:\,|\alpha| = \ell(w)} d_{w\alpha}\, s_\alpha$$

for some nonnegative integers $d_{w\alpha}$ [EG, LS2]. Fomin and Greene show that the coefficient $d_{w\alpha}$ equals the number of semistandard tableaux T of shape α' such that the column word of T is a reduced word for w [FG, Thm. 1.2]. On the other hand, Buch [B, Cor. 4.1] proved that the $d_{w\alpha}$ are special cases of quiver coefficients, and we will use this connection in the sequel.

3. Quiver varieties

3.1. **Definitions.** Let

$$E_\bullet \;:\; E_1 \to E_2 \to \cdots \to E_n$$

be a sequence of vector bundles and bundle maps over a non-singular variety $\mathfrak{X}$. Given rank conditions $r = \{r_{ij}\}$ for $1 \leqslant i < j \leqslant n$ there is a *quiver variety* given by

$$\Omega_r(E_\bullet) = \{x \in \mathfrak{X} \mid \operatorname{rank}(E_i(x) \to E_j(x)) \leqslant r_{ij} \ \forall i < j\}.$$

For convenience, we set $r_{ii} = \operatorname{rank} E_i$ for all i, and we demand that the rank conditions satisfy $r_{ij} \geqslant \max(r_{i-1,j}, r_{i,j+1})$ and $r_{ij} + r_{i-1,j+1} \geqslant r_{i-1,j} + r_{i,j+1}$ for

[1] In Stanley's notation, the function $F_{w^{-1}}$ is assigned to w.

all $i \le j$. In this case, the expected codimension of $\Omega_r(E_\bullet)$ is the number $d(r) = \sum_{i<j}(r_{i,j-1} - r_{ij})(r_{i+1,j} - r_{ij})$. The main result of [BF] states that when the quiver variety $\Omega_r(E_\bullet)$ has this codimension, its cohomology class is given by

$$[\Omega_r(E_\bullet)] = \sum_\lambda c_\lambda(r)\, s_{\lambda^1}(E_2 - E_1) \cdots s_{\lambda^{n-1}}(E_n - E_{n-1})\,. \tag{5}$$

Here the sum is over all sequences of partitions $\lambda = (\lambda^1, \ldots, \lambda^{n-1})$ such that $\sum |\lambda^i| = d(r)$, and the coefficients $c_\lambda(r)$ are integers computed by a combinatorial algorithm which we will not reproduce here. These coefficients are uniquely determined by the condition that (5) is true for all varieties $\mathfrak{X}$ and sequences $E_\bullet$, as well as the condition that $c_\lambda(r) = c_\lambda(r')$, where $r' = \{r'_{ij}\}$ is the set of rank conditions given by $r'_{ij} = r_{ij} + 1$ for all $i \leqslant j$.

Suppose the index p is such that all rank conditions $\operatorname{rank}(E_i(x) \to E_p(x)) \leqslant r_{ip}$ and $\operatorname{rank}(E_p(x) \to E_j(x)) \leqslant r_{pj}$ may be deduced from other rank conditions. Following [BF, §4], we will then say that the bundle E_p is *inessential*. Omitting an inessential bundle E_p from $E_\bullet$ produces a sequence

$$E'_\bullet \;:\; E_1 \to \cdots \to E_{p-1} \to E_{p+1} \to \cdots \to E_n,$$

where the map from E_{p-1} to E_{p+1} is the composition $E_{p-1} \to E_p \to E_{p+1}$. If r' denotes the restriction of the rank conditions to $E'_\bullet$, we have that $\Omega_{r'}(E'_\bullet) = \Omega_r(E_\bullet)$. We can use (4) to expand any factor $s_\alpha(E_{p+1} - E_{p-1})$ occurring in the quiver formula for $\Omega_{r'}(E'_\bullet)$ into a sum of products of the form $s_\beta(E_p - E_{p-1}) s_\gamma(E_{p+1} - E_p)$, and thus arrive at the quiver formula (5) for $\Omega_r(E_\bullet)$.

The loci associated with universal Schubert polynomials are special cases of quiver varieties. Given $w \in S_{n+1}$ we define rank conditions $r^{(n)} = \{r^{(n)}_{ij}\}$ for $1 \leqslant i \leqslant j \leqslant 2n$ by

$$r^{(n)}_{ij} = \begin{cases} r_w(2n+1-j, i) & \text{if } i \leqslant n < j \\ i & \text{if } j \leqslant n \\ 2n+1-j & \text{if } i \geqslant n+1. \end{cases}$$

Then $\Omega_w(G_\bullet \to F_\bullet)$ is identical to the quiver variety $\Omega_{r^{(n)}}(G_\bullet \to F_\bullet)$, and furthermore we have $\ell(w) = d(r^{(n)})$. If we let $c^{(n)}_{w,\lambda} = c_\lambda(r^{(n)})$ denote the quiver coefficients corresponding to this locus, it follows that

$$\mathfrak{S}_w(c; d) = \sum_\lambda c^{(n)}_{w,\lambda}\, s_{\lambda^1}(d(2) - d(1)) \cdots s_{\lambda^n}(c(n) - d(n)) \cdots s_{\lambda^{2n-1}}(c(1) - c(2))\,. \tag{6}$$

3.2. **Proof of Theorem 1.** It will be convenient to work with the element $P^{(n)}_w \in \Lambda^{\otimes 2n-1}$ defined by

$$P^{(n)}_w = \sum_\lambda c^{(n)}_{w,\lambda}\, s_{\lambda^1} \otimes \cdots \otimes s_{\lambda^{2n-1}}\,.$$

Theorem 1 is a consequence of Fomin and Greene's formula for stable Schubert polynomials combined with the following result.

Theorem 3. *For $w \in S_{n+1}$ we have*

$$P^{(n)}_w = \sum_{u_1 \ldots u_{2n-1} = w} F_{u_1} \otimes \cdots \otimes F_{u_{2n-1}}$$

where the sum is over all reduced factorizations $w = u_1 \cdots u_{2n-1}$ such that $u_i \in S_{\min(i,2n-i)+1}$ for each i.

Proof. Since $r_w(p,q)+m = r_{1^m\times w}(p+m,q+m)$ for $m \geq 0$, it follows that the coefficients $c^{(n)}_{w,\lambda}$ are uniquely defined by the condition that

$$(7)\quad \mathfrak{S}_{1^m\times w}(c;d) = \sum_\lambda c^{(n)}_{w,\lambda}\, s_{\lambda^1}(d(2+m)-d(1+m))\cdots s_{\lambda^n}(c(n+m)-d(n+m))\cdots s_{\lambda^{2n-1}}(c(1+m)-c(2+m))$$

for all $m \geqslant 0$ (see also [B, §4]).

Given any two integers $p \leqslant q$ we let $P^{(n)}_w[p,q]$ denote the sum of the terms of $P^{(n)}_w$ for which λ^i is empty when $i<p$ or $i>q$:

$$P^{(n)}_w[p,q] = \sum_{\lambda:\lambda^i=\emptyset \text{ for } i\notin[p,q]} c^{(n)}_{w,\lambda}\, s_{\lambda^1}\otimes\cdots\otimes s_{\lambda^{2n-1}}\,.$$

Lemma 1. *For any $1 < i \leqslant 2n-1$ we have*

$$(8)\qquad P^{(n)}_w = \sum_{u\cdot v=w} P^{(n)}_u[1,i-1]\cdot P^{(n)}_v[i,2n-1]\,.$$

Proof. We will do the case $i \leqslant n$; the other one is similar. For $f=\sum c_\lambda s_{\lambda^1}\otimes\cdots\otimes s_{\lambda^{2N-1}} \in \Lambda^{\otimes 2N-1}$, we set

$$f(c;d) = \sum c_\lambda\, s_{\lambda^1}(d(2)-d(1))\cdots s_{\lambda^N}(c(N)-d(N))\cdots s_{\lambda^{2N-1}}(c(1)-c(2))\,.$$

Equation (7) implies that $P^{(n)}_w \in \Lambda^{\otimes 2n-1}$ is the unique element satisfying that $(1^{\otimes m}\otimes P^{(n)}_w \otimes 1^{\otimes m})(c;d) = \mathfrak{S}_{1^m\times w}(c;d)$ for all m. This uniqueness is preserved even if we set $d(i+m)=0$. The right hand side of the identity (8) satisfies this by equation (3) applied to $1^m\times w$. □

Lemma 2. *For $1 \leqslant i \leqslant 2n-1$ we have*

$$P^{(n)}_w[i,i] = \begin{cases} 1^{\otimes i-1}\otimes F_w\otimes 1^{\otimes 2n-1-i} & \text{if } w\in S_{m+1},\ m=\min(i,2n-i)\\ 0 & \text{otherwise.}\end{cases}$$

Proof. If $w\notin S_{m+1}$ then this follows from Proposition 1, so assume $w\in S_{m+1}$. For simplicity we will furthermore assume that $m=i$. We have that

$$P^{(m)}_w[m,m] = \sum_\alpha c^{(m)}_{w,\alpha}\, 1^{\otimes m-1}\otimes s_\alpha\otimes 1^{\otimes m-1}\,,$$

the sum over partitions α (which are identified with sequences where α is surrounded by empty partitions). According to [B, Cor. 4.1], the quiver coefficients in the above formula satisfy $c^{(m)}_{w,\alpha} = d_{w\alpha}$, where $d_{w\alpha}$ is the integer defined in §2.2. It follows that $P^{(m)}_w[m,m] = 1^{\otimes m-1}\otimes F_w\otimes 1^{\otimes m-1}$.

Let $\Phi : \Lambda^{\otimes 2m-1}\to\Lambda^{\otimes 2n-1}$ be the linear map given by

$$\Phi(s_{\lambda^1}\otimes\cdots\otimes s_{\lambda^{2m-1}}) = s_{\lambda^1}\otimes\cdots\otimes s_{\lambda^{m-1}}\otimes\Delta^{2n-2m}(s_{\lambda^m})\otimes s_{\lambda^{m+1}}\otimes\cdots\otimes s_{\lambda^{2m-1}}$$

where $\Delta^{2n-2m}:\Lambda\to\Lambda^{\otimes 2n-2m+1}$ denotes the $2n-2m$ fold coproduct, that is

$$\Delta^{2n-2m}(s_{\lambda^m}) = \sum_{\tau_1,\dots,\tau_{2n-2m+1}} N^{\lambda^m}_{\tau_1,\dots,\tau_{2n-2m+1}}\, s_{\tau_1}\otimes\cdots\otimes s_{\tau_{2n-2m+1}}$$

where $N^{\lambda^m}_{\tau_1,\dots,\tau_{2n-2m+1}}$ is the coefficient of s_{λ^m} in the product $s_{\tau_1}s_{\tau_2}\cdots s_{\tau_{2n-2m+1}}$. In the definition of the locus $\Omega_w(G_\bullet\to F_\bullet)$, the bundles F_i and G_i for $i\geqslant m+1$ are inessential in the sense of §3.1, and we deduce from the remarks there that

$\Phi(P_w^{(m)}) = P_w^{(n)}$. Now the result follows from the identity $P_w^{(n)}[m, 2n-m] = \Phi(P_w^{(m)}[m,m]) = 1^{\otimes m-1} \otimes \Delta^{2n-2m}(F_w) \otimes 1^{\otimes m-1}$. □

Theorem 3 follows immediately from lemmas 1 and 2. □

Example 1. For the permutation $w = s_2 s_1 = 312$ in S_3, the sequences of tableaux which satisfy the conditions of Theorem 1 are

$$\left(\ \emptyset\ ,\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\ ,\ \emptyset\ \right) \quad \text{and} \quad \left(\ \emptyset\ ,\ \boxed{2}\ ,\ \boxed{1}\ \right).$$

It follows that

$$\begin{aligned} \mathfrak{S}_{312}(c;d) &= s_2(c(2)-d(2)) + s_1(c(2)-d(2))s_1(c(1)-c(2)) \\ &= c_1(1)c_1(2) - c_1(1)d_1(2) - c_2(2) + d_2(2)\,. \end{aligned}$$

In [BF], a conjectural combinatorial rule for general quiver coefficients $c_\lambda(r)$ was given. Although this rule was also stated in terms of sequences of semistandard tableaux satisfying certain conditions, it is different from Theorem 1 in the case of universal Schubert polynomials. It would be interesting to find a bijection that establishes the equivalence of these two rules.

3.3. **Skipping bundles.** A permutation w has a descent position at i if $w(i) > w(i+1)$. We say that a sequence $\{a_k\} : a_1 < \cdots < a_p$ of integers is *compatible with* w if all descent positions of w are contained in $\{a_k\}$. Suppose that $w \in S_{n+1}$ and let $1 \leqslant a_1 < a_2 < \cdots < a_p \leqslant n$ and $1 \leqslant b_1 < b_2 < \cdots < b_q \leqslant n$ be two sequences compatible with w and w^{-1}, respectively.

We let $E_\bullet$ denote the subsequence

$$G_{b_1} \to G_{b_2} \to \cdots \to G_{b_q} \to F_{a_p} \to \cdots \to F_{a_2} \to F_{a_1}\,,$$

and define rank conditions $\tilde{r}^{(n)} = \{\tilde{r}_{ij}^{(n)}\}$ for $1 \leqslant i \leqslant j \leqslant p+q$ by

$$\tilde{r}_{ij}^{(n)} = \begin{cases} r_w(a_{p+q+1-j}, b_i) & \text{if } i \leqslant q < j \\ b_i & \text{if } j \leqslant q \\ a_{p+q+1-j} & \text{if } i \geqslant q+1 \end{cases}$$

Then the expected codimension of the locus $\Omega_{\tilde{r}^{(n)}}(E_\bullet)$ is equal to $\ell(w)$. However, in general this locus may contain $\Omega_w(G_\bullet \to F_\bullet)$ as a proper closed subset. We will need the following criterion for equality (see also the remarks in [F3, §3] and Exercise 10 of [F2, §10].)

Lemma 3. *Suppose that the map $G_{i-1} \to G_i$ is injective for $i \notin \{b_k\}$ and the map $F_i \to F_{i-1}$ is surjective for $i \notin \{a_k\}$. Then $\Omega_{\tilde{r}^{(n)}}(E_\bullet) = \Omega_w(G_\bullet \to F_\bullet)$ as subschemes of $\mathfrak{X}$.*

Proof. Let $1 \leqslant i, j \leqslant n$ be given. If i is not a descent position for w then either $w(i) \leqslant j$ or $w(i+1) > j$. In the first case this implies that $r_w(i,j) = r_w(i-1,j)+1$ so the condition $\operatorname{rank}(G_j \to F_i) \leqslant r_w(i,j)$ follows from $\operatorname{rank}(G_j \to F_{i-1}) \leqslant r_w(i-1,j)$ because the map $F_i \to F_{i-1}$ is surjective. In the second case we have $r_w(i,j) = r_w(i+1,j)$ so the rank condition on $G_j \to F_i$ follows from the one on $G_j \to F_{i+1}$. A similar argument works if w^{-1} does not have a descent at position j. We conclude that the locus $\Omega_w(G_\bullet \to F_\bullet)$ does not change if the bundles G_j for $j \notin \{b_k\}$ and F_i for $i \notin \{a_k\}$ are disregarded. □

Corollary 2. *Let $w \in S_{n+1}$ and $\{a_k\}$ and $\{b_k\}$ be as above. Then we have*

$$[\Omega_{\tilde{r}^{(n)}}(E_\bullet)] = \sum_\mu \tilde{c}^{(n)}_{w,\mu}\, s_{\mu^1}(G_{b_2} - G_{b_1}) \cdots s_{\mu^q}(F_{a_p} - G_{b_q}) \cdots s_{\mu^{p+q-1}}(F_{a_1} - F_{a_2})$$

with coefficients $\tilde{c}^{(n)}_{w,\mu} = c^{(n)}_{w,\lambda}$ where the sequence $\lambda = (\lambda^1, \dots, \lambda^{2n-1})$ is given by

$$\lambda^i = \begin{cases} \mu^k & \text{if } i = b_{k+1} - 1 \\ \mu^q & \text{if } i = n \\ \mu^{p+q-k} & \text{if } i = 2n+1-a_{k+1} \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. We may assume that $F_i = F_{i-1} \oplus \mathbb{C}$ when $i \notin \{a_k\}$ and $G_j = G_{j-1} \oplus \mathbb{C}$ when $j \notin \{b_k\}$. In this case, notice that $s_\alpha(F_i - F_{i-1})$ is non-zero only when α is the empty partition or when $i = a_k$ for some k. Similarly $s_\alpha(G_{i-1} - G_i)$ is zero unless α is empty or $i = b_k$ for some k. The result therefore follows from Lemma 3 and equation (6). ☐

4. Schubert polynomials

4.1. **Degeneracy loci.** In this section, we will interpret the previous results for ordinary double Schubert polynomials. Let V be a vector bundle of rank n and let

$$G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset V \twoheadrightarrow F_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow F_2 \twoheadrightarrow F_1$$

be a complete flag followed by a dual complete flag of V. If $w \in S_n$ then Fulton has proved [F1] that

$$[\Omega_w(G_\bullet \to F_\bullet)] = \mathfrak{S}_w(x_1, \dots, x_n; y_1, \dots, y_n)$$

where $x_i = c_1(\ker(F_i \to F_{i-1}))$, $y_i = c_1(G_i/G_{i-1})$, and $\mathfrak{S}_w(X;Y)$ is the double Schubert polynomial of Lascoux and Schützenberger.

Set $G'_i = V/G_i$ and $F'_i = \ker(V \to F_i)$. Then we have a sequence

$$F'_{n-1} \subset \cdots \subset F'_1 \subset V \twoheadrightarrow G'_1 \twoheadrightarrow \cdots \twoheadrightarrow G'_{n-1}$$

and it is easy to check that $\Omega_w(G_\bullet \to F_\bullet) = \Omega_{w_0 w^{-1} w_0}(F'_\bullet \to G'_\bullet)$ as subschemes of $\mathfrak{X}$, where w_0 is the longest permutation in S_n.

Let $1 \leqslant a_1 < \cdots < a_p \leqslant n-1$ and $0 \leqslant b_1 < \cdots < b_q \leqslant n-1$ be two sequences compatible with w and w^{-1}, respectively. Then by applying section 3.3 to the subsequence $F'_{a_p} \to \cdots \to F'_{a_1} \to G'_{b_1} \to \cdots \to G'_{b_q}$ we obtain

$$[\Omega_w(G_\bullet \to F_\bullet)] = \sum_\mu \tilde{c}^{(n-1)}_{w_0 w^{-1} w_0, \mu}\, s_{\mu^1}(F'_{a_{p-1}} - F'_{a_p}) \cdots s_{\mu^p}(G'_{b_1} - F'_{a_1}) \cdots s_{\mu^{p+q-1}}(G'_{b_q} - G'_{b_{q-1}})\,.$$

Set $a_0 = b_0 = 0$. If we let $X_i = \{x_{a_{i-1}+1}, \dots, x_{a_i}\}$ denote the Chern roots of $\ker(F_{a_i} \to F_{a_{i-1}})$ and $Y_i = \{y_{b_{i-1}+1}, \dots, y_{b_i}\}$ be the Chern roots of $G_{b_i}/G_{b_{i-1}}$ then the previous equality can be written as

$$\mathfrak{S}_w(X;Y) = \sum_\mu \tilde{c}^{(n-1)}_{w_0 w^{-1} w_0, \mu}\, s_{\mu^1}(X_p) \cdots s_{\mu^p}(X_1/Y_1) \cdots s_{\mu^{p+q-1}}(0/Y_q)\,. \tag{9}$$

This equation is true in the cohomology ring $H^*(\mathfrak{X};\mathbb{Z})$, in which there are relations between the variables x_i and y_i (including e.g. the relations $e_j(x_1, \dots, x_n) = c_j(V) = e_j(y_1, \dots, y_n)$ for $1 \leqslant j \leqslant n$). We claim, however, that (9) holds as an identity of polynomials in independent variables. For this, notice that the identity is independent of n, i.e. the coefficient $\tilde{c}^{(n-1)}_{w_0 w^{-1} w_0,\mu}$ does not change when n is replaced with $n+1$ and w_0 with the longest element in S_{n+1}. If we choose n sufficiently large, we can construct a variety $\mathfrak{X}$ on which (9) is true, and where all monomials in the variables x_i and y_i of total degree at most $\ell(w)$ are linearly independent, which establishes the claim.

4.2. **Splitting Schubert polynomials.** We continue by reformulating equation (9) to obtain a more natural expression for double Schubert polynomials.

It follows from (9) together with the main result of [B] that $F_{w_0 w^{-1} w_0} = F_w$. Therefore, the coefficient $d_{w\alpha}$ of the Schur expansion of F_w is equal to the number of semistandard tableaux of shape α' such that the column word is a reduced word for $w_0 w^{-1} w_0$. Notice that if $e = (e_1, e_2, \dots, e_\ell)$ is a reduced word for $w_0 w^{-1} w_0$ then $\widetilde{e} = (n+1-e_\ell, \dots, n+1-e_1)$ is a reduced word for w. Furthermore, if e is the column word of a tableau of shape α', then $\widetilde{e}$ is the column word of a skew tableau whose shape is the 180 degree rotation of α'. If we denote this rotated shape by $\widetilde{\alpha}$ then we conclude that $d_{w\alpha}$ is also equal to the number of skew tableaux of shape $\widetilde{\alpha}$ such that the column word is a reduced word for w. Theorem 3 now implies the following variation of our main theorem:

Theorem 1′. *If $w \in S_{n+1}$ then the coefficient $c^{(n)}_{w,\lambda}$ equals the number of sequences of semistandard skew tableaux $(T_1, \dots, T_{2n-1})$ such that T_i has shape $\widetilde{\lambda}^i$, the entries of T_i are at most $\min(i, 2n-i)$, and $\mathrm{col}(T_1)\cdots\mathrm{col}(T_{2n-1})$ is a reduced word for w.*

We say that a sequence of tableaux $(T_1, \dots, T_r)$ is *strictly bounded below* by an integer sequence $(a_1, \dots, a_r)$ if the entries of T_i are strictly greater than a_i, for each i.

Theorem 4. *Let $w \in S_n$ and let $1 \leqslant a_1 < \cdots < a_p$ and $0 \leqslant b_1 < \cdots < b_q$ be two sequences compatible with w and w^{-1}, respectively. Then we have*

$$\mathfrak{S}_w(X;Y) = \sum_\lambda c_\lambda \, s_{\lambda^1}(0/Y_q) \cdots s_{\lambda^q}(X_1/Y_1) \cdots s_{\lambda^{p+q-1}}(X_p) \tag{10}$$

where $X_i = \{x_{a_{i-1}+1}, \dots, x_{a_i}\}$ and $Y_i = \{y_{b_{i-1}+1}, \dots, y_{b_i}\}$ and the sum is over all sequences of partitions $\lambda = (\lambda^1, \dots, \lambda^{p+q-1})$. Each c_λ is a quiver coefficient, equal to the number of sequences of semistandard tableaux $(T_1, \dots, T_{p+q-1})$ strictly bounded below by $(b_{q-1}, \dots, b_1, 0, a_1, a_2, \dots, a_{p-1})$, such that the shape of T_i is conjugate to λ^i and $\mathrm{col}(T_1)\cdots\mathrm{col}(T_{p+q-1})$ is a reduced word for w.

Proof. This follows from equation (9) together with Theorem 1′ applied to $w_0 w^{-1} w_0$. To translate between the sequences of skew tableaux in Theorem 1′ and the sequences in the present theorem, simply rotate a whole sequence of skew tableaux by 180 degrees (this means invert the order of the sequence, and turn each skew tableau on its head). Then replace each entry e with $n+1-e$.

A purely algebraic proof of the splitting formula (10) is also possible. This requires versions of the results of Section 2.1 for ordinary Schubert polynomials, available e.g. from [L2] and [FK], together with [FG, Thm. 1.2]. However, some geometric reasoning is needed to interpret the c_λ as quiver coefficients. □

Notice that if one takes $b_1 = 0$ in Theorem 4 then the set of variables Y_1 is empty, so equation (10) contains only signed products of single Schur polynomials. Observe also that a factor $s_{\lambda^i}(0/Y_k)$ in (10) will vanish if λ^i has more than $b_k - b_{k-1}$

columns and that $s_{\lambda^i}(X_k)$ vanishes if λ^i has more than $a_k - a_{k-1}$ rows. Therefore equation (10) uses only a subset of the quiver coefficients of Theorem 4.

Example 2. Let $w = 321$ be the longest element in S_3, and choose the sequences $\{a_i\} = \{b_i\} = \{1 < 2\}$. Then the four sequences of tableaux satisfying the conditions of Theorem 4 are

$$\left(\,\boxed{2}\,,\,\boxed{1}\,,\,\boxed{2}\,\right), \quad \left(\,\boxed{2}\,,\,\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\,,\,\emptyset\,\right), \quad \left(\,\emptyset\,,\,\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\,,\,\boxed{2}\,\right), \quad \text{and} \quad \left(\,\emptyset\,,\,\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\,,\,\emptyset\,\right),$$

all of which give nonvanishing terms and correspond to the reduced word $s_2s_1s_2$. We thus have

$$\begin{gathered}\mathfrak{S}_{321}(X;Y) = \\ s_1(0/y_2)s_1(x_1/y_1)s_1(x_2) + s_1(0/y_2)s_{1,1}(x_1/y_1) + s_2(x_1/y_1)s_1(x_2) + s_{2,1}(x_1/y_1) \\ = -y_2(x_1-y_1)x_2 - y_2(y_1^2 - x_1y_1) + (x_1^2 - x_1y_1)x_2 + (x_1y_1^2 - x_1^2y_1) \\ = (x_1-y_1)(x_1-y_2)(x_2-y_1)\,.\end{gathered}$$

Corollary 3. *Suppose that $w \in S_n$ is a permutation compatible with the sequence $a_1 < \cdots < a_p$. Then we have*

$$\mathfrak{S}_w(X) = \sum_{\lambda} c_\lambda\, s_{\lambda^1}(X_1)\cdots s_{\lambda^p}(X_p)$$

where $X_i = \{x_{a_{i-1}+1}, \dots, x_{a_i}\}$ and the sum is over all sequences of partitions $\lambda = (\lambda^1, \dots, \lambda^p)$. Each c_λ is a quiver coefficient, equal to the number of sequences of semistandard tableaux $(T_1, \dots, T_p)$ strictly bounded below by $(0, a_1, a_2, \dots, a_{p-1})$, such that the shape of T_i is conjugate to λ^i and $\mathrm{col}(T_1)\cdots\mathrm{col}(T_p)$ is a reduced word for w.

Example 3. Consider the permutation $w = s_1s_2s_1s_3s_4s_3 = 32541$ in S_5, with descent positions at 1, 3, and 4. The two sequences of tableaux satisfying the conditions of Corollary 3 which give nonvanishing terms are

$$\left(\,\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline\end{array}\,,\,\begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}\,,\,\boxed{4}\,\right) \quad \text{and} \quad \left(\,\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\,,\,\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}\,,\,\boxed{4}\,\right).$$

The reduced words for w corresponding to these sequences are $s_4s_2s_1s_2s_3s_4$ and $s_2s_1s_4s_2s_3s_4$, respectively. It follows that

$$\begin{aligned}\mathfrak{S}_{32541}(X) &= s_3(x_1)s_{1,1}(x_2,x_3)s_1(x_4) + s_2(x_1)s_{2,1}(x_2,x_3)s_1(x_4) \\ &= x_1^3 \cdot x_2x_3 \cdot x_4 + x_1^2(x_2^2x_3 + x_3^2x_2)x_4\,.\end{aligned}$$

The special case of Corollary 3 with $a_k = k$ gives a formula for the coefficient of each monomial in $\mathfrak{S}_w(X)$, which is equivalent to that of [BJS, Thm. 1.1]. We deduce that these monomial coefficients are quiver coefficients. The same conclusion holds for double Schubert polynomials:

Corollary 4. *Let $w \in S_n$ and let $x^u y^v = x_1^{u_1}\cdots x_{n-1}^{u_{n-1}} y_1^{v_1}\cdots y_{n-1}^{v_{n-1}}$ be a monomial of total degree $\ell(w)$. Set $g_i = \sum_{k=n-i}^{n-1} v_k$ and $f_i = g_{n-1} + \sum_{k=1}^{i} u_k$. Then the coefficient of $x^u y^v$ in the double Schubert polynomial $\mathfrak{S}_w(X;Y)$ is equal to $(-1)^{g_{n-1}}$ times the number of reduced words $(e_1, \dots, e_{\ell(w)})$ for w such that $n-i \leqslant e_{g_{i-1}+1} < \cdots < e_{g_i}$ and $e_{f_{i-1}+1} > \cdots > e_{f_i} \geqslant i$ for all $1 \leqslant i \leqslant n-1$.*

Proof. Take $\{a_k\} = \{1, 2, \ldots, n-1\}$ and $\{b_k\} = \{0, 1, 2, \ldots, n-1\}$ in Theorem 4 and notice that a Schur polynomial $s_\alpha(0/y_i)$ is non-zero only if $\alpha = (1^{v_i})$ is a single column with v_i boxes, in which case $s_\alpha(0/y_i) = (-y_i)^{v_i}$. Similarly $s_\alpha(x_i)$ is equal to $x_i^{u_i}$ if $\alpha = (u_i)$ is a single row with u_i boxes, and is zero otherwise. The reduced words of the corollary are exactly those that form a sequence of tableaux on the conjugates of these shapes and strictly bounded below by $(n-2, n-3, \ldots, 1, 0, 0, 1, \ldots, n-2)$. □

Remark. Corollary 3 implies that the single Schubert polynomial $\mathfrak{S}_w(X)$ is the character of a $\mathrm{GL}(a_1, \mathbb{C}) \times \mathrm{GL}(a_2 - a_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(a_p - a_{p-1}, \mathbb{C})$ module. Our expression for the coefficients c_λ gives an isotypic decomposition of this module. This is closely related to a theorem of Kraskiewicz and Pragacz [KP], which shows that $\mathfrak{S}_w(X)$ is a character of a Borel module. P. Magyar reports that the former property may be deduced from the latter (private communication). We note that alternative positive expressions for the c_λ may be deduced from [LS2, 1.5] and [BS, Cor. 1.2].

5. Giambelli formulas

In this final section, we observe that Corollaries 2 and 3 give explicit, non-recursive answers to the *Giambelli problem* for the classical and quantum cohomology of partial flag manifolds.

Suppose that V is a complex vector space of dimension n and choose integers $0 = a_0 < a_1 < \cdots < a_p < a_{p+1} = n$. Let $\mathfrak{X}$ be the partial flag variety which parametrizes quotients

$$V \twoheadrightarrow F_{a_p} \twoheadrightarrow \cdots \twoheadrightarrow F_{a_1} \tag{11}$$

with $\mathrm{rank}(F_{a_k}) = a_k$ for each k. We will also use (11) to denote the tautological sequence of quotient bundles over $\mathfrak{X}$, and define $Q_k = \mathrm{Ker}(F_{a_k} \to F_{a_{k-1}})$, for $1 \leqslant k \leqslant p+1$. According to Borel [Bo], the cohomology ring $H^*(\mathfrak{X}; \mathbb{Z})$ is presented as the polynomial ring in the Chern classes $c_i(Q_k)$ for all i and k, modulo the relation

$$c(Q_1)c(Q_2)\cdots c(Q_{p+1}) = 1\,.$$

Fix a complete flag $G_\bullet$ of subspaces of V, and let $S(a)$ denote the subset of S_n consisting of permutations w compatible with $\{a_k\}$. For each $w \in S(a)$, there is a Schubert variety $\Omega_w \subset \mathfrak{X}$, defined as the locus of $x \in \mathfrak{X}$ such that

$$\mathrm{rank}(G_j(x) \to F_i(x)) \leqslant r_w(i,j), \quad \text{for} \quad i \in \{a_1, \ldots a_p\} \quad \text{and} \quad 1 \leqslant j \leqslant n.$$

The *Schubert classes* $[\Omega_w] \in H^{2\ell(w)}(\mathfrak{X}; \mathbb{Z})$ for $w \in S(a)$ form a natural 'geometric basis' for the cohomology ring of $\mathfrak{X}$. The following Giambelli formula, which is a direct consequence of Corollary 3, writes these classes as polynomials in the 'algebraic generators' for $H^*(\mathfrak{X}; \mathbb{Z})$ given by the $c_i(Q_k)$.

Giambelli I. *We have*

$$[\Omega_w] = \sum_\lambda c_\lambda\, s_{\lambda^1}(Q_1) \cdots s_{\lambda^p}(Q_p), \tag{12}$$

where the sum is over sequences of partitions $\lambda = (\lambda^1, \ldots, \lambda^p)$ and c_λ is the quiver coefficient of Corollary 3.

Alternatively, we can use Corollary 2 to express the Schubert class $[\Omega_w]$ as a polynomial in the special Schubert classes $c_i(F_{a_k})$. Notice that the two approaches are identical for Grassmannians.

Giambelli II. *We have*

$$[\Omega_w] = \sum_{\nu} \widetilde{c}_\nu \, s_{\nu^p}(F_{a_p}) \, s_{\nu^{p-1}}(F_{a_{p-1}} - F_{a_p}) \cdots s_{\nu^1}(F_{a_1} - F_{a_2}) \,, \tag{13}$$

where the sum is over sequences of partitions $\nu = (\nu^1, \ldots, \nu^p)$ *and* $\widetilde{c}_\nu = c_{w,\lambda}^{(n-1)}$, *where* $\lambda = (\lambda^1, \ldots, \lambda^{2n-3})$ *is given by*

$$\lambda^i = \begin{cases} \nu^k & \text{if } i = 2n-1-a_{k+1} \\ \emptyset & \text{otherwise.} \end{cases}$$

When comparing the above two Giambelli formulas, recall that the the quiver coefficients c_λ which appear in (12) correspond to the permutation $w_0 w^{-1} w_0$. The equivalence of (12) and (13) can also be checked directly, by using the Chern class identities $c(Q_k) = c(F_{a_k} - F_{a_{k-1}})$. See [LS1] and [So] for a Pieri formula for flag manifolds which complements Giambelli II.

Following [FGP] and [C-F, §3.2], we recall that equation (13) can be used to obtain a quantum Giambelli formula which holds in the small quantum cohomology ring $QH^*(\mathfrak{X})$. For this, one simply replaces all the special Schubert classes which appear in the Schur determinants $s_{\nu^k}(F_{a_k} - F_{a_{k+1}})$ in (13) with the corresponding quantum classes, as in loc. cit. (compare also with [F3, Prop. 4.3]).

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