RATIONAL CONNECTEDNESS IMPLIES FINITENESS OF QUANTUM K-THEORY

ANDERS S. BUCH, PIERRE–EMMANUEL CHAPUT, LEONARDO C. MIHALCEA, AND NICOLAS PERRIN

ABSTRACT. Let X be any generalized flag variety with Picard group of rank one. Given a degree d, consider the Gromov-Witten variety of rational curves of degree d in X that meet three general points. We prove that, if this Gromov-Witten variety is rationally connected for all large degrees d, then the structure constants of the small quantum K-theory ring of X vanish for large degrees.

1. Introduction

The (small) quantum K-theory ring $\mathrm{QK}(X)$ of a smooth complex projective variety X is a generalization of both the Grothendieck ring $K(X)$ of algebraic vector bundles on X and the small quantum cohomology ring of X. The ring $\mathrm{QK}(X)$ was defined by Givental [9] when X is a rational homogeneous space and by Lee [11] in general. In this paper we study this ring when X is a complex projective rational homogeneous space with $\mathrm{Pic}(X) = \mathbb{Z}$. Equivalently, we have $X = G/P$ where G is a complex semisimple algebraic group and $P \subset G$ is a maximal parabolic subgroup. The product in $\mathrm{QK}(X)$ of two arbitrary classes $\alpha, \beta \in K(X)$ is a power series

$$\alpha \star \beta = \sum_{d \geq 0} (\alpha \star \beta)_d \, q^d \,,$$

where each coefficient $(\alpha \star \beta)_d \in K(X)$ is defined using the K-theory ring of the Kontsevich moduli space $\overline{\mathcal{M}}_{0,3}(X,d)$ of stable maps to X of degree d. For general homogeneous spaces it is an open problem if this power series can have infinitely many non-zero terms. The product $\alpha \star \beta$ is known to be finite if X is a Grassmann variety of type A [4]. More generally, when X is any cominuscule homogeneous space, it was proved by the authors in [3] that all products in $\mathrm{QK}(X)$ are finite. Let $d_X(2)$ denote the smallest possible degree of a rational curve connecting two general points in X. The main theorem of [3] states that $(\alpha \star \beta)_d = 0$ whenever X is cominuscule and $d > d_X(2)$, which is the best possible bound.

Given three general points $x, y, z \in X$, let $M_d(x, y, z) \subset \overline{\mathcal{M}}_{0,3}(X, d)$ denote the *Gromov-Witten variety* of stable maps that send the three marked points to x, y, and z. We will assume that this variety is rationally connected for all sufficiently large degrees d. Let d_{rc} be a positive integer such that $M_d(x, y, z)$ is rationally connected for $d \geq d_{\mathrm{rc}}$. We also let d_{cl} be the smallest length of a chain of lines connecting two general points in X. Our main result is the following theorem.

Date: October 18, 2014.

2000 *Mathematics Subject Classification.* Primary 14N35; Secondary 19E08, 14N15, 14M15, 14M20, 14M22.

The first author was supported in part by NSF grant DMS-1205351.

The third author was supported in part by NSA Young Investigator Award H98230-13-1-0208.

Theorem 1. *We have* $(\alpha \star \beta)_d = 0$ *for all* $d \geq d_{\rm rc} + d_{\rm cl}$.

The Gromov-Witten varieties $M_d(x, y, z)$ of large degrees are known to be rational when X is a cominuscule homogeneous space, an orthogonal Grassmannian $\mathrm{OG}(m, N)$ for $m \neq \frac{N}{2} - 1$, or any adjoint variety of type different from A or G_2. This was proved in [4] for Grassmannians of type A and in [5] in all other cases. Theorem 1 therefore establishes the finiteness of quantum K-theory for many new spaces. The *orthogonal Grassmannian* $\mathrm{OG}(m, N)$ is the variety of isotropic m-dimensional subspaces in the vector space $\mathbb{C}^N$ equipped with a non-degenerate symmetric bilinear form; these varieties account for all spaces G/P where G is a group of type B_n or D_n and P is a maximal parabolic subgroup. The variety $X = G/P$ is called *adjoint* if it is isomorphic to the closed orbit of the adjoint action of G on $\mathbb{P}(\mathrm{Lie}(G))$.

Remark 1.1. We thank Jason Starr for sending us an outline of an argument that uses the results of [6, 7] to prove that the Gromov-Witten varieties $M_d(x, y, z)$ of large degrees are rationally connected when X is any projective rational homogeneous space with $\mathrm{Pic}(X) = \mathbb{Z}$. As a consequence, Theorem 1 can be applied to all such spaces. We also thank Starr for making us aware of [6, Lemma 15.8].

2. Stable maps and Gromov-Witten varieties

We recall here some notation and results from [3]. Let $X = G/P$ be a homogeneous space defined by a semisimple complex linear algebraic group G and a parabolic subgroup $P \subset G$. Let $B \subset P$ be a Borel subgroup. Recall that a *Schubert variety* in X is an orbit closure of a Borel subgroup of G. Equivalently, it is a G-translate of the closure of a B-orbit in X; the latter orbit closure is a *B-stable Schubert variety*. Given an effective degree $d \in H_2(X; \mathbb{Z})$ and an integer $n \geq 0$, the Kontsevich moduli space $\overline{\mathcal{M}}_{0,n}(X, d)$ parametrizes the isomorphism classes of n-pointed stable (genus zero) maps $f : C \to X$ with $f_*[C] = d$, and comes with a total evaluation map $\mathrm{ev} = (\mathrm{ev}_1, \dots, \mathrm{ev}_n) : \overline{\mathcal{M}}_{0,n}(X, d) \to X^n := X \times \cdots \times X$. Here a map is called *stable* if its automorphism group is finite, i.e. each of its contracted components has at least 3 special points. A detailed construction of this space can be found in the survey [8].

Let $\mathbf{d} = (d_0, d_1, \dots, d_r)$ be a sequence of effective classes $d_i \in H_2(X; \mathbb{Z})$, let $\mathbf{e} = (e_0, \dots, e_r) \in \mathbb{N}^{r+1}$, and set $|\mathbf{d}| = \sum d_i$ and $|\mathbf{e}| = \sum e_i$. Let $M_{\mathbf{d},\mathbf{e}} \subset \overline{\mathcal{M}}_{0,|\mathbf{e}|}(X, |\mathbf{d}|)$ be the closure of the locus of stable maps $f : C \to X$ defined on a chain C of $r + 1$ projective lines, such that the i-th projective line contains e_i marked points (numbered from $1 + \sum_{j<i} e_j$ to $\sum_{j \leq i} e_j$) and the restriction of f to this component has degree d_i. To ensure that these maps are indeed stable we assume that $e_i \geq 1 + \delta_{i,0} + \delta_{i,r}$ whenever $d_i = 0$. Moreover, we will assume that $e_0 > 0$ and $e_r > 0$. Set $\mathcal{Z}_{\mathbf{d},\mathbf{e}} = \mathrm{ev}(M_{\mathbf{d},\mathbf{e}}) \subset X^{|\mathbf{e}|}$. Given subvarieties $\Omega_1, \dots, \Omega_m$ of X with $m \leq |\mathbf{e}|$, define a boundary Gromov-Witten variety by $M_{\mathbf{d},\mathbf{e}}(\Omega_1, \dots, \Omega_m) = \bigcap_{i=1}^m \mathrm{ev}_i^{-1}(\Omega_i) \subset M_{\mathbf{d},\mathbf{e}}$. We also write $\Gamma_{\mathbf{d},\mathbf{e}}(\Omega_1, \dots, \Omega_m) = \mathrm{ev}_{|\mathbf{e}|}(M_{\mathbf{d},\mathbf{e}}(\Omega_1, \dots, \Omega_m)) \subset X$. If no sequence $\mathbf{e}$ is specified, we will use $\mathbf{e} = (3)$ when $r = 0$ and $\mathbf{e} = (2, 0, \dots, 0, 1)$ when $r > 0$. This convention will be used only when $d_i \neq 0$ for $i > 0$. For this reason the sequence $\mathbf{d} = (d_0, \dots, d_r)$ will be called a *stable sequence of degrees* if $d_i \neq 0$ for $i > 0$.

An irreducible variety Y has *rational singularities* if there exists a desingularization $\pi : \widetilde{Y} \to Y$ such that $\pi_* \mathcal{O}_{\widetilde{Y}} = \mathcal{O}_Y$ and $R^i \pi_* \mathcal{O}_{\widetilde{Y}} = 0$ for all $i > 0$. An arbitrary variety has rational singularities if its irreducible components have rational singularities, are disjoint, and have the same dimension. We need the following result from [1, Lemma 3].

Lemma 2.1 (Brion)**.** *Let Z and S be varieties and let $\pi : Z \to S$ be a morphism. If Z has rational singularities, then the same holds for the general fibers of π.*

A morphism $f : Y \to Z$ of varieties is a *locally trivial fibration* if each point $z \in Z$ has an open neighborhood $U \subset Z$ such that $f^{-1}(U) \cong U \times f^{-1}(z)$ and f is the projection to the first factor. The following result is obtained by combining Propositions 2.2 and 2.3 in [3].

Proposition 2.2. *Let $B \subset G$ be a Borel subgroup, let Y be a B-variety, let $\Omega \subset X$ be a B-stable Schubert variety, and let $f : Y \to \Omega$ be a dominant B-equivariant map. Then f is a locally trivial fibration over the dense open B-orbit $\Omega^\circ \subset \Omega$.*

It was proved in [3, Prop. 3.7] that $M_{\mathbf{d},\mathbf{e}}$ is unirational and has rational singularities. Lemma 2.1 therefore implies that $M_{\mathbf{d},\mathbf{e}}(x_1, \dots, x_m)$ has rational singularities for all points $(x_1, \dots, x_m)$ in a dense open subset of $(\mathrm{ev}_1 \times \cdots \times \mathrm{ev}_m)(M_{\mathbf{d},\mathbf{e}}) \subset X^m$. Proposition 2.2 applied to the map $\mathrm{ev}_1 : M_{\mathbf{d},\mathbf{e}} \to X$ shows that $M_{\mathbf{d},\mathbf{e}}(x)$ is unirational for all points $x \in X$. Finally, [3, Lemma 3.9(a)] states that the variety $\mathcal{Z}_{d,2} = \mathrm{ev}(M_{d,2}) \subset X^2$ is rational and has rational singularities for any effective degree $d \in H_2(X;\mathbb{Z})$,

Proposition 2.3. *The variety $M_{\mathbf{d},\mathbf{e}}(x, y)$ is unirational for all points (x, y) in a dense open subset of the image $(\mathrm{ev}_1 \times \mathrm{ev}_2)(M_{\mathbf{d},\mathbf{e}}) \subset X^2$.*

Proof. Set $\Omega = \mathrm{ev}_2(M_{\mathbf{d},\mathbf{e}}(1.P)) \subset X$. Since $M_{\mathbf{d},\mathbf{e}}(1.P)$ is irreducible and P-stable, it follows that Ω is a P-stable Schubert variety. Let $U \subset \Omega$ be the dense open P-orbit. It follows from Proposition 2.2 that $\mathrm{ev}_2 : M_{\mathbf{d},\mathbf{e}}(1.P) \to \Omega$ is a locally trivial fibration over U. Since $M_{\mathbf{d},\mathbf{e}}(1.P)$ is unirational, this implies that $M_{\mathbf{d},\mathbf{e}}(1.P, x)$ is unirational for all $x \in U$. Finally notice that $(\mathrm{ev}_1 \times \mathrm{ev}_2)(M_{\mathbf{d},\mathbf{e}}) = G \times^P \Omega = (G \times \Omega)/P$, where P acts by $(g, x).p = (gp, p^{-1}.x)$, and $M_{\mathbf{d}}(x, y)$ is unirational for all points (x, y) in the dense open subset $G \times^P U \subset G \times^P \Omega$. □

Remark 2.4. It is proved in [6, Lemma 15.8] that, if $\mathbf{d} = (1^d) = (1, 1, \dots, 1)$ with d large, $\mathbf{e} = (1, 0^{d-2}, 1)$, and $\mathrm{Pic}(X) = \mathbb{Z}$, then the general fibers of $\mathrm{ev} : M_{\mathbf{d},\mathbf{e}} \to X^2$ are rationally connected. This also follows from Proposition 2.3. A more general statement is proved in [3, Prop. 3.2].

3. Rationally connected Gromov-Witten varieties

An algebraic variety Z is *rationally connected* if two general points $x, y \in Z$ can be joined by a rational curve, i.e. both x and y belong to the image of some morphism $\mathbb{P}^1 \to Z$. We need the following fundamental result from [10].

Theorem 3.1 (Graber, Harris, Starr)**.** *Let $f : Z \to Y$ be any dominant morphism of complete irreducible complex varieties. If Y and the general fibers of f are rationally connected, then Z is rationally connected.*

We assume from now on that $X = G/P$ is defined by a maximal parabolic subgroup $P \subset G$. Then we have $H_2(X;\mathbb{Z}) = \mathbb{Z}$, so the degree of a curve in X can be identified with an integer. We will further assume that the three-point Gromov-Witten varieties of X of sufficiently high degree are rationally connected. More precisely, assume that there exists an integer $d_{\rm rc}$ such that $M_d(x,y,z)$ is rationally connected for all $d \geq d_{\rm rc}$ and all points (x,y,z) in a dense open subset $U_d \subset X^3$.

For $n \geq 2$ we set $d_X(n) = \min\{d \in \mathbb{N} \mid \mathcal{Z}_{d,n} = X^n\}$. This is the smallest integer such that, given n arbitrary points in X, there exists a curve of degree $d_X(n)$ through all n points. Finally we set $d_{\rm cl} = \min\{d \in \mathbb{N} \mid \mathcal{Z}_{(1^d),(1,0^{d-2},1)} = X^2\}$, where $(1^d) = (1,1,\ldots,1)$ denotes a sequence of d ones. This is the smallest length of a chain of lines connecting two general points in X. Notice that $d_X(3) \leq d_{\rm rc}$ and $d_X(2) \leq d_{\rm cl}$.

Theorem 3.2. *Let $\mathbf{d} = (d_0, d_1, \ldots, d_r)$ be a stable sequence of degrees such that $|\mathbf{d}| \geq d_{\rm rc} + d_{\rm cl} - 1$. Then we have $\mathcal{Z}_{\mathbf{d}} = \mathcal{Z}_{d_0,2} \times X$, and $M_{\mathbf{d}}(x,y,z)$ is rationally connected for all points (x,y,z) in a dense open subset of $\mathcal{Z}_{\mathbf{d}}$.*

Proof. Set $\mathbf{d}' = (d_1,\ldots,d_r)$ and $\mathbf{e}' = (1,0,\ldots,0,1) \in \mathbb{N}^r$. It follows from [3, Prop. 3.6] that $M_{\mathbf{d}}$ is the product over X of the maps $\mathrm{ev}_3 : M_{d_0,3} \to X$ and $\mathrm{ev}_1 : M_{\mathbf{d}',\mathbf{e}'} \to X$. The assumption implies that $d_0 \geq d_{\rm rc}$ or $|\mathbf{d}'| \geq d_{\rm cl}$.

Assume first that $|\mathbf{d}'| \geq d_{\rm cl}$. It then follows from the definition of $d_{\rm cl}$ that $\mathcal{Z}_{\mathbf{d}} = \mathcal{Z}_{d_0,2} \times X$. Let $X^\circ = Pw_0.P \subset X$ be the open P-orbit. By Proposition 2.3 and Lemma 2.1 we may choose a dense open subset $U \subset \mathcal{Z}_{d_0,2}$ such that, for all points $(x,y) \in U$ we have that $M_{d_0}(x,y)$ is unirational, $\Gamma_{d_0}(x,y) \cap X^\circ \neq \emptyset$, and $M_{\mathbf{d}}(x,y,1.P)$ has rational singularities. Let $(x,y) \in U$. We will show that $M_{\mathbf{d}}(x,y,1.P)$ is rationally connected. Let $p : M_{\mathbf{d}}(x,y,1.P) \to M_{d_0}(x,y)$ be the projection. Then the fibers of p are given by $p^{-1}(f) = M_{\mathbf{d}',\mathbf{e}'}(\mathrm{ev}_3(f), 1.P)$. Since the morphism $\mathrm{ev}_1 : M_{\mathbf{d}',\mathbf{e}'}(X, 1.P) \to X$ is surjective and P-equivariant, Proposition 2.2 implies that this map is locally trivial over X°. Since $M_{\mathbf{d}',\mathbf{e}'}(X,1.P)$ is unirational, we deduce that $M_{\mathbf{d}',\mathbf{e}'}(z',1.P)$ is unirational for all $z' \in X^\circ$. This implies that $p^{-1}(f)$ is unirational for all $f \in M_{d_0}(x,y,X^\circ)$, which is a dense open subset of $M_{d_0}(x,y)$ by choice of U. Since the general fibers of p are connected, it follows from Stein factorization that all fibers of p are connected. Therefore $M_{\mathbf{d}}(x,y,1.P)$ is connected. Since this variety also has rational singularities, we deduce that $M_{\mathbf{d}}(x,y,1.P)$ is irreducible. Finally, Theorem 3.1 applied to the map $p : M_{\mathbf{d}}(x,y,1.P) \to M_{d_0}(x,y)$ shows that $M_{\mathbf{d}}(x,y,1.P)$ is rationally connected.

Assume now that $d_0 \geq d_{\rm rc}$. In this case we have $\mathcal{Z}_{\mathbf{d}} = X^3$. Let $U \subset X^3$ be a dense open subset such that $M_{\mathbf{d}}(x,y,z)$ has rational singularities and $M_{d_0}(x,y,z)$ is rationally connected and has rational singularities for all $(x,y,z) \in U$. Using similar arguments, one can show that $M_{\mathbf{d}}(x,y,z)$ is rationally connected for all $(x,y,z) \in U$. This follows from Theorem 3.1 again, applied to the map $q : M_{\mathbf{d}}(x,y,z) \to M_{\mathbf{d}',\mathbf{e}'}(X,z)$. Details are left to the reader. $\square$

4. Quantum K-theory

Let $K(X)$ denote the Grothendieck ring of algebraic vector bundles on X. An introduction to this ring can be found in e.g. [2, §3.3]. For each effective degree $d \in H_2(X;\mathbb{Z})$ we define a class $\Phi_d \in K(X^3)$ by

$$\Phi_d = \sum_{\mathbf{d}=(d_0,\ldots,d_r)} (-1)^r \, \mathrm{ev}_*[\mathcal{O}_{M_{\mathbf{d}}}] \,,$$

where the sum is over all stable sequences of degrees $\mathbf{d}$ such that $|\mathbf{d}| = d$, and $\mathrm{ev} : M_{\mathbf{d}} \to X^3$ is the evaluation map. Let $\pi_i : X^3 \to X$ be the projection to the i-th factor. For $\alpha, \beta \in K(X)$ we set $(\alpha \star \beta)_d = \pi_{3*}(\pi_1^*(\alpha) \cdot \pi_2^*(\beta) \cdot \Phi_d) \in K(X)$.

The quantum K-theory ring of X is an algebra over $\mathbb{Z}[\![q]\!]$, which as a $\mathbb{Z}[\![q]\!]$-module is given by $\mathrm{QK}(X) = K(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\![q]\!]$. The multiplicative structure of $\mathrm{QK}(X)$ is defined by

$$\alpha \star \beta = \sum_d (\alpha \star \beta)_d \, q^d$$

for all classes $\alpha, \beta \in K(X)$, where the sum is over all effective degrees d. A theorem of Givental [9] states that $\mathrm{QK}(X)$ is an associative ring. We note that the definition of $\mathrm{QK}(X)$ given here is different from Givental's original construction; the equivalence of the two definitions follows from [3, Lemma 5.1].[1]

We need the following Gysin formula from [4, Thm. 3.1] (see also [3, Prop. 5.2] for the stated version.)

Proposition 4.1. *Let $f : X \to Y$ be a surjective morphism of projective varieties with rational singularities. If the general fibers of f are rationally connected, then $f_*[\mathcal{O}_X] = [\mathcal{O}_Y] \in K(Y)$.*

Corollary 4.2. *Let $\mathbf{d} = (d_0, \dots, d_r)$ be a stable sequence of degrees such that $|\mathbf{d}| \geq d_{\mathrm{rc}} + d_{\mathrm{cl}} - 1$. Then we have $\mathrm{ev}_*[\mathcal{O}_{M_\mathbf{d}}] = [\mathcal{O}_{\mathcal{Z}_\mathbf{d}}] \in K(X^3)$.*

Proof. This holds because $\mathcal{Z}_\mathbf{d} = \mathcal{Z}_{d_0,2} \times X$ has rational singularities [3, Lemma 3.9], the general fibers of the map $\mathrm{ev} : M_\mathbf{d} \to \mathcal{Z}_\mathbf{d}$ are rationally connected by Theorem 3.2, and $M_\mathbf{d}$ has rational singularities by [3, Prop. 3.7]. □

Theorem 1 is equivalent to the following result.

Theorem 4.3. *We have $\Phi_d = 0$ for all $d \geq d_{\mathrm{rc}} + d_{\mathrm{cl}}$.*

Proof. It follows from Corollary 4.2 that, for $d \geq d_{\mathrm{rc}} + d_{\mathrm{cl}}$ we have

$$\Phi_d = \sum_{\mathbf{d}=(d_0,\dots,d_r)} (-1)^r [\mathcal{O}_{\mathcal{Z}_\mathbf{d}}] \ \in K(X^3)\,,$$

where the sum is over all stable sequences of degrees $\mathbf{d}$ with $|\mathbf{d}| = d$. Since $\mathcal{Z}_\mathbf{d} = \mathcal{Z}_{d_0,2} \times X$, the terms of this sum depend only on d_0. Since $d \geq d_{\mathrm{cl}} + d_{\mathrm{rc}} > d_X(2)$, it follows that $\mathcal{Z}_{(d)} = \mathcal{Z}_{(d-1,1)} = X^3$, so the contributions from the sequences $\mathbf{d} = (d)$ and $\mathbf{d} = (d-1, 1)$ cancel each other out. Now let $0 \leq d' \leq d-2$. For each r with $1 \leq r \leq d - d'$, there are exactly $\binom{d-d'-1}{r-1}$ sequences $\mathbf{d}$ in the sum for which $d_0 = d'$ and the length of $\mathbf{d}$ is $r+1$. Since $\sum_{r=1}^{d-d'} (-1)^r \binom{d-d'-1}{r-1} = 0$, it follows that the corresponding terms cancel each other out. It follows that $\Phi_d = 0$, as claimed. □

Remark 4.4. Theorem 4.3 is true also for the equivariant K-theory ring $\mathrm{QK}_T(X)$ with the same proof.

Remark 4.5. If X is not the projective line, then the proof of Theorem 4.3 shows that $\Phi_d = 0$ for all $d \geq d_{\mathrm{rc}} + d_{\mathrm{cl}} - 1$. It would be interesting to determine the maximal value of d for which $\Phi_d \neq 0$. If X is a cominuscule variety, then this number is equal to $d_X(2)$, hence the maximal power of q that appears in products in the quantum K-theory ring of X is equal to the maximal power that appears in the quantum cohomology ring [3].

---

[1] Lemma 5.1 in [3] is stated only for cominuscule varieties, but its proof works verbatim for any projective rational homogeneous space X.

References

[1] M. Brion, *Positivity in the Grothendieck group of complex flag varieties*, J. Algebra **258** (2002), no. 1, 137–159, Special issue in celebration of Claudio Procesi's 60th birthday. MR 1958901 (2003m:14017)

[2] ______, *Lectures on the geometry of flag varieties*, Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser, Basel, 2005, pp. 33–85. MR 2143072 (2006f:14058)

[3] A. S. Buch, P.-E. Chaput, L. Mihalcea, and N. Perrin, *Finiteness of cominuscule quantum K-theory*, Ann. Sci. Ec. Norm. Super. **46**, fascicule 3 (2013), pag. 477– 494.

[4] A. S. Buch and L. C. Mihalcea, *Quantum K-theory of Grassmannians*, Duke Math. J. **156** (2011), no. 3, 501–538. MR 2772069 (2011m:14092)

[5] P.-E. Chaput and N. Perrin, *Rationality of some Gromov-Witten varieties and application to quantum K-theory*, Commun. Contemp. Math. **13** (2011), no. 1, 67–90. MR 2772579 (2012k:14077)

[6] A. J. de Jong, X. He, and J. M. Starr, *Families of rationally simply connected varieties over surfaces and torsors for semisimple groups*, Publ. Math. Inst. Hautes Études Sci. (2011), no. 114, 1–85. MR 2854858

[7] A. J. de Jong and J. Starr, *Low degree complete intersections are rationally simply connected*, preprint http://www.math.sunysb.edu/∼jstarr/papers/nk1006g.pdf.

[8] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 45–96. MR 1492534 (98m:14025)

[9] A. Givental, *On the WDVV equation in quantum K-theory*, Michigan Math. J. **48** (2000), 295–304, Dedicated to William Fulton on the occasion of his 60th birthday. MR 1786492 (2001m:14078)

[10] T. Graber, J. Harris, and J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67 (electronic). MR 1937199 (2003m:14081)

[11] Y.-P. Lee, *Quantum K-theory. I. Foundations*, Duke Math. J. **121** (2004), no. 3, 389–424. MR 2040281 (2005f:14107)

Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA

E-mail address: `asbuch@math.rutgers.edu`

Domaine Scientifique Victor Grignard, 239, Boulevard des Aiguillettes, Université Henri Poincaré Nancy 1, B.P. 70239, F-54506 Vandoeuvre-lès-Nancy Cedex, France

E-mail address: `pierre-emmanuel.chaput@univ-lorraine.fr`

Department of Mathematics, Virginia Tech University, 460 McBryde, Blacksburg VA 24060, USA

E-mail address: `lmihalce@math.vt.edu`

Mathematisches Institut, Heinrich-Heine-Universität, D-40204 Düsseldorf, Germany

E-mail address: `perrin@math.uni-duesseldorf.de`