POSITIVITY OF MINUSCULE QUANTUM K-THEORY

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Abstract. We prove that the Schubert structure constants of the quantum K-theory ring of any minuscule flag variety or quadric hypersurface have signs that alternate with codimension. We also prove that the powers of the deformation parameter q that occur in the product of two Schubert classes in the quantum cohomology or quantum K-theory ring of a cominuscule flag variety form an integer interval. Our proofs are based on several new results, including an explicit description of the most general non-empty intersection of two Schubert varieties in an arbitrary flag manifold, and a computation of the cohomology groups of any negative line bundle restricted to a Richardson variety in a cominuscule flag variety. We also give a type-uniform proof of the quantum-to-classical theorem, which asserts that the (3-point, genus 0) Gromov-Witten invariants of any cominuscule flag variety are classical triple-intersection numbers on an associated flag variety. Finally, we prove several new results about the geometry and combinatorics related to this theorem.

Contents

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1. Introduction

1.1. **Positivity in quantum K-theory.** Let $X = G/P_X$ be a flag variety defined by a semi-simple complex linear algebraic group G and a parabolic subgroup P_X. The (small) quantum K-theory ring $\mathrm{QK}(X)$ of Givental and Lee [Giv00, Lee01] is a deformation of the K-theory ring $K(X)$ of algebraic vector bundles on X, whose structure constants $N_{u,v}^{w,d}$ encode the arithmetic genera of the (3 pointed, genus zero) Gromov-Witten varieties of X. Here $N_{u,v}^{w,d}$ denotes the coefficient of $q^d \mathcal{O}^w$ in the product $\mathcal{O}^u \star \mathcal{O}^v$ in $\mathrm{QK}(X)$, where $\mathcal{O}^w = [\mathcal{O}_{X^w}]$ denotes a K-theoretic Schubert class of X, and q^d is a monomial in the deformation parameters of $\mathrm{QK}(X)$, encoding a degree $d \in H_2(X, \mathbb{Z})$. The Schubert variety X^w corresponds to a Weyl group element w such that $\mathrm{codim}(X^w, X)$ is equal to the length $\ell(w)$.

The constant $N_{u,v}^{w,d}$ is non-zero only if

$$\ell(w) + \int_d c_1(T_X) \geq \ell(u) + \ell(v)\,,$$

and when this inequality is satisfied with equality, $N_{u,v}^{w,d}$ is the (cohomological) Gromov-Witten invariant $\langle [X^u], [X^v], [X_w] \rangle_d$, equal to the number of parameterized curves $f : \mathbb{P}^1 \to X$ of degree d that map the points $0, 1, \infty$ to general translates of the Schubert varieties X^u, X^v, and X_w; here X_w is the Schubert variety of dimension $\ell(w)$ opposite to X^w. In particular, $N_{u,v}^{w,d}$ is non-negative in this case. More generally, it is conjectured [LM06, BM11, BCMP18a] that the constants $N_{u,v}^{w,d}$ have signs that alternate with codimension, in the sense that

$$(-1)^{\ell(uvw) + \int_d c_1(T_X)} N_{u,v}^{w,d} \geq 0\,. \tag{1}$$

This generalizes the fact that the Schubert structure constants of the K-theory ring $K(X)$ have alternating signs [Buc02, Bri02]. Our main result is a proof of the alternating signs conjecture for $\mathrm{QK}(X)$ when X is a minuscule flag variety or a quadric hypersurface.

Theorem 1.1. *Assume that X is a minuscule flag variety or a quadric hypersurface. Then, $(-1)^{\ell(uvw) + \int_d c_1(T_X)} N_{u,v}^{w,d} \geq 0$.*

The family of *minuscule* flag varieties includes Grassmannians of Lie type A, maximal orthogonal Grassmannians, quadric hypersurfaces of even dimension, and two exceptional spaces of type E called the Cayley plane and the Freudenthal variety. The larger family of *cominuscule* flag varieties also includes Lagrangian Grassmannians and odd dimensional quadrics (see Section 4). We will prove more generally that the constant $N_{u,v}^{w,d}$ has the expected sign (1) whenever X is cominuscule and q^d occurs in the product $[X^u] \star [X^v]$ in the quantum cohomology ring $\mathrm{QH}(X)$.

Earlier examples where alternating signs of the structure constants in quantum K-theory have been proved include the Pieri formula for products with special Schubert classes on Grassmannians of type A [BM11], structure constants associated to 'line' degrees corresponding to certain fundamental weights on any G/P [LM14], Chevalley formulas for products with Schubert divisors on some families of flag varieties [BCMP18a, LNS21, KLNS], and all the structure constants of the quantum K-theory of incidence varieties $\mathrm{Fl}(1, n-1; n)$ of type A [Ros20, Xu21].

1.2. **Powers of q in quantum products.** We also address the problem of finding the powers q^d that occur in the quantum product of two Schubert classes, either in the quantum cohomology ring $\mathrm{QH}(X)$ or the quantum K-theory ring $\mathrm{QK}(X)$. When X is a Grassmannian of type A, the smallest power $d_{\min}(u,v)$ of q in the product $[X^u]\star[X^v] \in \mathrm{QH}(X)$ was determined by Fulton and Woodward [FW04], as the number of diagonal units the (dual) Young diagram of X^u must be translated in order to contain the Young diagram of X^v. Postnikov [Pos05] gave a similar rule for the largest degree $d_{\max}(u,v)$, and also proved that the powers of q that occur form an integer interval. This answers the question for the quantum cohomology of Grassmannians of type A.

For an arbitrary flag variety X, it is not clear if a quantum product $[X^u]\star[X^v] \in \mathrm{QH}(X)$ contains a minimal or maximal power of q, since the group $H_2(X,\mathbb{Z})$ is linearly ordered only when it has rank 1. It turns out that $[X^u]\star[X^v]$ always contains a minimal power q^d [Pos05, BCLM20], where $d = d_{\min}(u,v)$ is the (unique) minimal degree of a rational curve connecting two general translates of the Schubert varieties X^u and X^v. The corresponding product $\mathcal{O}^u \star \mathcal{O}^v$ in $\mathrm{QK}(X)$ contains the same minimal power q^d. However, a quantum cohomology product $[X^u]\star[X^v]$ may not contain a unique maximal power of q, and the powers that occur may not form a convex subset in the natural partial order of $H_2(X,\mathbb{Z})$. For example, the q-degrees in the square of $[X^{164532}]$ in $\mathrm{QH}(\mathrm{Fl}(\mathbb{C}^6))$ do not form a convex subset, and no unique maximal degree exists.

Any cominuscule flag variety X has Picard rank one, so all quantum products automatically contain a maximal power of q. We will show that the interval property also holds when X is cominuscule. More precisely, let $\mathcal{B} = \{q^d[X^u]\}$ denote the natural $\mathbb{Z}$-basis of $\mathrm{QH}(X)$. It was proved in [Bel04, CMP09] that any product $[\text{point}]\star[X^v]$ belongs to $\mathcal{B}$ when X is cominuscule. Define a partial order on $\mathcal{B}$ by $q^e[X^v] \leq q^d[X^u]$ if and only if the Schubert varieties X_u and X^v are connected by a rational curve of degree $d-e$.

Theorem 1.2. *Assume that X is cominuscule. Then the powers q^d that occur in $[X^u]\star[X^v] \in \mathrm{QH}(X)$ form an integer interval. More precisely, q^d occurs in $[X^u]\star[X^v]$ if and only if $[X^v] \leq q^d[X_u] \leq [\text{point}]\star[X^v]$.*

Postnikov's description of the extreme powers of q in a quantum product on the Grassmannian $\mathrm{Gr}(m,n)$ involves order ideals in the *cylinder* $\mathbb{Z}^2/(-m,n-m)\mathbb{Z}$ that extends the usual $m\times(n-m)$-rectangle of boxes associated with the Grassmannian. Theorem 1.2 can be interpreted as a type-uniform generalization of this construction. In fact, $\mathcal{B}$ turns out to be a distributive lattice, and the join-irreducible elements in $\mathcal{B}$ can be identified with a set of boxes in the plane that specializes to Postnikov's cylinder in type A. Examples are provided in Section 7.2. Isomorphic partially ordered sets have been constructed in [Hag04, Gre13], where they are used to study to minuscule representations.

In quantum K-theory it is known that the powers of q in any product $\mathcal{O}^u\star\mathcal{O}^v$ are bounded above [BCMP13, BCMP16, Kat, ACT22]; this is not apparent from Givental's definition of the product in $\mathrm{QK}(X)$. Theorem 1.2 has the following generalization.

Theorem 1.3. (a) *Assume that X is minuscule. Then q^d occurs in $\mathcal{O}^u \star \mathcal{O}^v$ if and only if q^d occurs in $[X^u] \star [X^v]$.*

(b) *Assume that X is cominuscule. Then the powers of q that occur in $\mathcal{O}^u \star \mathcal{O}^v$ form an integer interval. The smallest power matches the smallest power in $[X^u] \star [X^v]$, and the largest power is at most one larger than the largest power in $[X^u] \star [X^v]$.*

In Definition 8.2 we give a combinatorial definition of an *exceptional degree* of a product $\mathcal{O}^u \star \mathcal{O}^v$ in the quantum K-theory ring of any cominuscule flag variety X. Exceptional degrees occur only when X is not minuscule, and even in this case, most products have no exceptional degrees (see Table 4). If $\mathcal{O}^u \star \mathcal{O}^v$ has an exceptional degree, then this degree is $d_{\max}(u,v)+1$. We prove that if q^d occurs in $\mathcal{O}^u \star \mathcal{O}^v$, then either $d_{\min}(u,v) \leq d \leq d_{\max}(u,v)$, or $d = d_{\max}(u,v)+1$ is an exceptional degree. In particular, this means that $N^{w,d}_{u,v}$ has the expected sign whenever d is not an exceptional degree. We conjecture that q^d occurs in $\mathcal{O}^u \star \mathcal{O}^v$ whenever d is an exceptional degree.

1.3. **Strategy of proof.** To illustrate the main strategy in our proofs, fix a cominuscule flag variety X and let $M_d = \overline{\mathcal{M}}_{0,3}(X,d)$ denote the Kontsevich moduli space of 3-pointed stable maps to X of degree d and genus zero [FP97]. Let $M_d(X_u, X^v) = \mathrm{ev}_1^{-1}(X_u) \cap \mathrm{ev}_2^{-1}(X^v) \subset M_d$ denote the *Gromov-Witten variety* of stable maps that send the first two marked points to the Schubert varieties X_u and X^v, respectively. The image $\Gamma_d(X_u, X^v) = \mathrm{ev}_3(M_d(X_u,X^v)) \subset X$ is a two-pointed *curve neighborhood*, equal to the closure of the union of all rational curves of degree d in X that meet X_u and X^v. We also let $M_{d-1,1} \subset M_d$ denote the divisor of stable maps $f : C \to X$ for which the domain has (at least) two components, $C = C_1 \cup C_2$, such that C_1 contains the first two marked points, C_2 contains the third marked point, and the restrictions of f to C_1 and C_2 have degrees $d-1$ and 1, respectively. Set $M_{d-1,1}(X_u,X^v) = M_{d-1,1} \cap M_d(X_u,X^v)$ and $\Gamma_{d-1,1}(X_u,X^v) = \mathrm{ev}_3(M_{d-1,1}(X_u,X^v))$. In other words, $\Gamma_{d-1,1}(X_u,X^v)$ is the closure of the set of points in X that are connected by a line to a rational curve of degree $d-1$ from X_u to X^v.

Let $\mathcal{O}_u = [\mathcal{O}_{X_u}]$ and $\mathcal{O}^v = [\mathcal{O}_{X^v}]$ be two opposite Schubert classes. It follows from [BCMP18a, Prop. 3.2] that the product $\mathcal{O}_u \star \mathcal{O}^v \in \mathrm{QK}(X)$ is given by[1]

$$\mathcal{O}_u \star \mathcal{O}^v = \sum_{w,d\geq 0} N^{w,d}_{u^\vee,v}\, q^d\, \mathcal{O}^w = \sum_{d\geq 0} (\mathcal{O}_u \star \mathcal{O}^v)_d\, q^d\,,$$

where the classes $(\mathcal{O}_u \star \mathcal{O}^v)_d \in K(X)$ are determined by

$$(\mathcal{O}_u \star \mathcal{O}^v)_d = (\mathrm{ev}_3)_*[\mathcal{O}_{M_d(X_u,X^v)}] - (\mathrm{ev}_3)_*[\mathcal{O}_{M_{d-1,1}(X_u,X^v)}]\,. \tag{2}$$

It was proved in [BCMP18b, Thm. 4.1] that $\Gamma_d(X_u,X^v)$ is a projected Richardson variety in X, and the restricted map $\mathrm{ev}_3 : M_d(X_u,X^v) \to \Gamma_d(X_u,X^v)$ is cohomologically trivial, that is, $(\mathrm{ev}_3)_*\mathcal{O}_{M_d(X_u,X^v)} = \mathcal{O}_{\Gamma_d(X_u,X^v)}$ and $R^j(\mathrm{ev}_3)_*\mathcal{O}_{M_d(X_u,X^v)} = 0$ for $j>0$. In particular, we have $(\mathrm{ev}_3)_*[\mathcal{O}_{M_d(X_u,X^v)}] = [\mathcal{O}_{\Gamma_d(X_u,X^v)}]$ in $K(X)$. Since projected Richardson varieties have rational singularities [BC12, KLS14], it follows from a theorem of Brion [Bri02] that the expansion of $[\mathcal{O}_{\Gamma_d(X_u,X^v)}]$ in the Schubert basis of $K(X)$ has alternating signs in the sense of Theorem 1.1.

[1]The dual Weyl group element $u^\vee$ satisfies $\mathcal{O}^{u^\vee} = \mathcal{O}_u$.

The further restriction $\mathrm{ev}_3 : M_{d-1,1}(X_u, X^v) \to \Gamma_{d-1,1}(X_u, X^v)$ is not as well understood; for example, we do not know if $\Gamma_{d-1,1}(X_u, X^v)$ has rational singularities. Our strategy is to establish the following two properties.

(i) The general fibers of the map $\mathrm{ev}_3 : M_{d-1,1}(X_u, X^v) \to \Gamma_{d-1,1}(X_u, X^v)$ are cohomologically trivial.

(ii) The variety $\Gamma_{d-1,1}(X_u, X^v)$ is either equal to $\Gamma_d(X_u, X^v)$ or a divisor in $\Gamma_d(X_u, X^v)$.

The first property (i) implies, by using a result of Kollár [Kol86] (see Corollary 8.13), that $(\mathrm{ev}_3)_*[\mathcal{O}_{M_{d-1,1}(X_u,X^v)}]$ is equal to the class of a resolution of singularities of $\Gamma_{d-1,1}(X_u, X^v)$. In particular, if $\Gamma_{d-1,1}(X_u, X^v) = \Gamma_d(X_u, X^v)$, then $(\mathcal{O}_u \star \mathcal{O}^v)_d = 0$. Otherwise, property (ii) predicts that $\Gamma_{d-1,1}(X_u, X^v)$ is a divisor in $\Gamma_d(X_u, X^v)$. In this case Brion's theorem [Bri02] implies that the expansion of $(\mathrm{ev}_3)_*[\mathcal{O}_{M_{d-1,1}(X_u,X^v)}]$ has signs that are opposite to the signs of $[\mathcal{O}_{\Gamma_d(X_u,X^v)}]$, so that all signs are compatible in the difference (2). Theorem 1.1 is therefore a consequence of properties (i) and (ii). In addition, q^d occurs in the product $\mathcal{O}_u \star \mathcal{O}^v$ if and only if $\Gamma_{d-1,1}(X_u, X^v) \subsetneq \Gamma_d(X_u, X^v)$.

We show that property (ii) is always true. More precisely, $\Gamma_{d-1,1}(X_u, X^v)$ is empty for $d \leq d_{\min}(u^\vee, v)$, is a (non-empty) divisor in $\Gamma_d(X_u, X^v)$ for $d_{\min}(u^\vee, v) < d \leq d_{\max}(u^\vee, v)$, and is equal to $\Gamma_d(X_u, X^v)$ for $d > d_{\max}(u^\vee, v)$. We also show that (i) is true, *except* when d is an exceptional degree of $\mathcal{O}_u \star \mathcal{O}^v$. In this case the general fibers of the map $\mathrm{ev}_3 : M_{d-1,1}(X_u, X^v) \to \Gamma_{d-1,1}(X_u, X^v)$ have arithmetic genus one (see Theorem 8.26 and Corollary 10.5), which explains the exceptional behavior of exceptional degrees.

1.4. **The quantum-to-classical construction.** Our proofs rely on the geometric construction underlying the *quantum equals classical* theorem for cominuscule flag varieties [Buc03, BKT03, CMP08, BM11, CP11], which we proceed to discuss when $X = \mathrm{Gr}(m, n)$ is the Grassmannian of m-planes in $\mathbb{C}^n$. Given any stable map $f : C \to X$, let $\mathrm{Ker}(f) \subset \mathbb{C}^n$ be the intersection of the m-planes contained in the image $f(C)$, and let $\mathrm{Span}(f) \subset \mathbb{C}^n$ be the linear span of these m-planes. For f in a dense open subset of $M_d = \overline{\mathcal{M}}_{0,3}(X, d)$, with $d \leq \min(m, n-m)$, we have $\dim \mathrm{Ker}(f) = m - d$ and $\dim \mathrm{Span}(f) = m + d$, that is, $(\mathrm{Ker}(f), \mathrm{Span}(f))$ is a point in the two-step flag variety $Y_d = \mathrm{Fl}(m-d, m+d; n)$. Define the three-step flag variety $Z_d = \mathrm{Fl}(m-d, m, m+d; n)$, and let $p_d : Z_d \to X$ and $q_d : Z_d \to Y_d$ be the projections. The quantum equals classical theorem states that any (3 point, genus zero) Gromov-Witten invariant of X is given by

$$\langle \Omega_1, \Omega_2, \Omega_3 \rangle_d = \int_{Y_d} q_{d*} p_d^*(\Omega_1) \cdot q_{d*} p_d^*(\Omega_2) \cdot q_{d*} p_d^*(\Omega_1) \,.$$

Define a rational map $\varphi : M_d \dashrightarrow Z_d$ by $\varphi(f) = (\mathrm{Ker}(f), \mathrm{ev}_3(f), \mathrm{Span}(f))$, and define subvarieties of Z_d by

$$Z_d(X_u, X^v) = \overline{\varphi(M_d(X_u, X^v))} \ \text{ and } \ Z_{d-1,1}(X_u, X^v) = \overline{\varphi(M_{d-1,1}(X_u, X^v))} \,.$$

We will show that (completions of) the general fibers of the restricted maps $\varphi : M_d(X_u, X^v) \dashrightarrow Z_d(X_u, X^v)$ and $\varphi : M_{d-1,1}(X_u, X^v) \dashrightarrow Z_{d-1,1}(X_u, X^v)$ are cohomologically trivial. As a consequence, we can replace $M_{d-1,1}(X_u, X^v)$ with $Z_{d-1,1}(X_u, X^v)$ in property (i). Furthermore, $Z_d(X_u, X^v)$ is a Richardson variety in Z_d, and $Z_{d-1,1}(X_u, X^v)$ is the inverse image of a projected Richardson variety in Y_d. Geometric results about Schubert varieties can therefore be utilized for studying the fibers of the map $p_d : Z_{d-1,1}(X_u, X^v) \to \Gamma_{d-1,1}(X_u, X^v)$.

The quantum-to-classical construction can be generalized to any cominuscule flag variety X by replacing kernel-span pairs $\omega = (K, S)$ with certain well-behaved subvarieties $\Gamma_\omega \subset X$, which in type A are the sub-Grassmannians $\Gamma_\omega = \mathrm{Gr}(d, S/K) \subset \mathrm{Gr}(m, n)$. These subvarieties correspond to points in a related flag variety $Y_d = G/P_{Y_d}$, which in turn defines the incidence variety $Z_d = G/(P_X \cap P_{Y_d}) = \{(\omega, x) \in Y_d \times X \mid x \in \Gamma_\omega\}$. This provides a type-independent framework for studying properties (i) and (ii).

1.5. **Semi-transversal intersections.** In order to establish the required geometric properties of (especially) fibers of maps related to the quantum-to-classical construction, we prove a number of new results about intersections of Schubert varieties in arbitrary flag varieties that are not in general position. In particular, given two opposite Schubert varieties with empty intersection, we define and study a *semi-transversal intersection* obtained when these varieties are moved towards each other until they just meet, using the group action. In fact, semi-transversal intersections can be defined for subvarieties of any variety with a group action, but in this generality it is not guaranteed that a semi-transversal intersection exists. We show that the semi-transversal intersection of two Schubert varieties always exists, is a Richardson variety, and we give explicit descriptions of the defining Weyl group elements. For example, if $X = G/P_X$ is a cominuscule flag variety, then the semi-transversal intersection of X_u with X^v is the Richardson variety $X_u \cap X^{u \cap v}$, where $u \cap v$ denotes the join operation on the set of minimal length Weyl group elements, corresponding to the intersection of Young diagrams in type A. We also prove the following result about the fibers of a projection of a Schubert or Richardson variety to a smaller flag variety.

Theorem 1.4. *Let $\pi : Z \to X$ be a projection of flag varieties. Each fiber $\pi^{-1}(x) \cong P_X/P_Z$ is again a flag variety.*

(a) *Let $Z_u \subset Z$ be a Schubert variety. Then $Z_u \cap \pi^{-1}(x)$ a (reduced) Schubert variety in $\pi^{-1}(x)$ for all $x \in \pi(X_u)$.*

(b) *Let $Z_u^v = Z_u \cap Z^v \subset Z$ be a Richardson variety. Then $Z_u^v \cap \pi^{-1}(x)$ is a Richardson variety in $\pi^{-1}(x)$ for all x in a dense open subset of $\pi(Z_u^v)$.*

The general fibers are given by explicitly determined Weyl group elements (see Theorem 2.8 and Theorem 2.10).

We apply these results to the projection $p_d : Z_d \to X$ of the quantum-to-classical construction. For each $x \in X$, the fiber $F_d = p_d^{-1}(x) \cong P_X/P_{Z_d}$ is a product of cominuscule flag varieties. If x is a general point of $\Gamma_d(X_u, X^v)$, then $R = F_d \cap Z_d(X_u, X^v)$ is a Richardson variety in F_d given by explicitly determined Weyl group elements. The most interesting case of property (i) happens in the range of degrees $d_{\max}(u^\vee, v) < d \leq \min(d_{\max}(u^\vee), d_{\max}(v))$, where $d_{\max}(v)$ denotes the unique power of q in $[\text{point}] \star [X^v]$. In this case, $\Gamma_{d-1,1}(X_u, X^v) = \Gamma_d(X_u, X^v)$. We show that $D = F_d \cap Z_{d-1,1}(X_u, X^v)$ is a Cartier divisor in R, obtained as the intersection of R with a divisor in F_d. We then prove that D is cohomologically trivial if and only if d is not an exceptional degree. This is done by explicitly computing the cohomology groups of any negative line bundle restricted to a Richardson variety in any cominuscule flag variety (Theorem 4.9). A special case is the following statement.

Theorem 1.5. *Let X be a minuscule flag variety, let $R \subset X$ be a Richardson variety of positive dimension, and let $\mathcal{O}_X(-1) \subset \mathcal{O}_X$ be the ideal sheaf of the Schubert divisor in X. Then $H^i(R, \mathcal{O}_X(-1)) = 0$ for all i.*

Our results can also be used to prove that the Seidel representation of the fundamental group $\pi_1(\mathrm{Aut}(X))$ on the localized quantum cohomology ring $\mathrm{QH}(X)_{q=1}$, as studied in [Bel04, CMP09], has a natural generalization to the quantum K-theory ring in the cominuscule case. We plan to discuss this elsewhere together with applications to Pieri formulas in quantum K-theory.

1.6. **Organization.** This paper is organized as follows. In Section 2 we fix our notation for flag varieties and related combinatorics and prove several results about intersections of Schubert varieties in special position that are valid for arbitrary flag varieties. In particular, we introduce the notion of semi-transversal intersections. In Section 3 we give short proofs of some related results about projections of Richardson varieties, which were first obtained in [KLS14]. In Section 4 we introduce our notation for cominuscule flag varieties. We also compute the (top) cohomology group of any negative line bundle restricted to a Richardson variety in a cominuscule flag variety, as a representation of the maximal torus $T \subset G$. This allows us to determine when the intersection of a Richardson variety with an effective Cartier divisor is cohomologically trivial. Section 5 gives a detailed account of the quantum equals classical theorem, focusing on Gromov-Witten invariants of degrees no larger than the *diameter* of a cominuscule variety X, that is, the smallest degree of a rational curve connecting two general points. We take this opportunity to provide a type-uniform proof of this theorem, something that has so far not been available in the literature. At the same time we further develop the geometry and combinatorics of the quantum-to-classical construction. Section 6 provides explicit descriptions of the general fibers of several maps of varieties related to this construction. Section 7 proves that the q-degrees in the quantum cohomology product of two cominuscule Schubert classes form an integer interval. We also construct our generalization of Postnikov's cylinder, which provides a combinatorial description of the minimal and maximal degrees in a quantum product. Section 8 proves the results about alternating signs and q-degrees in the quantum K-theory ring of a cominuscule flag variety. The proofs of some technical facts are postponed to Section 9. Finally, Section 10 proves that the general fibers of the rational map $M_{d-1,1}(X_u, X^v) \dashrightarrow Z_{d-1,1}(X_u, X^v)$ have cohomologically trivial completions.

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2. Intersections and fibers of Schubert and Richardson varieties

In this section we fix our notation for flag varieties. In addition, we prove some results about intersections of Schubert and Richardson varieties in special position that are valid for arbitrary flag varieties over any algebraically closed field. In this paper, a *variety* is always reduced but not necessarily irreducible. By a *point* we will always mean a closed point. The *fibers* of a morphism are understood to be fibers over closed points, and the *general fibers* mean the fibers over all closed points in a dense open subset of the target.

2.1. **Flag varieties.** Let G be a connected linear algebraic group, and fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset G$. The opposite Borel subgroup $B^- \subset G$ is defined by $B \cap B^- = R_u(G)T$, where $R_u(G)$ is the unipotent radical. Let Φ be the root system of (G,T), with positive roots Φ^+ and simple roots $\Delta \subset \Phi^+$ given by B. This means that Φ is the set of roots of the reductive quotient $G/R_u(G)$, see [Spr98, §7.4.3]. Let $W = N_G(T)/Z_G(T)$ be the Weyl group of G. The reflection along a root $\alpha \in \Phi$ is denoted by s_α.

A complete homogeneous G-variety will be called a *flag variety* of G. Any such flag variety X contains a unique B-invariant point. We denote the parabolic subgroup stabilizing this point by $P_X \subset G$ and the point itself by $1.P_X$. We identify X with the quotient G/P_X. Each element $g \in G$ defines a point $g.P_X = g.(1.P_X)$ in X. Let Φ_X be the root system of (the reductive quotient of) P_X, and set $\Phi_X^+ = \Phi^+ \cap \Phi_X$ and $\Delta_X = \Delta \cap \Phi_X$. Let $W_X = N_{P_X}(T)/Z_{P_X}(T)$ be the Weyl group of P_X, and let $W^X \subset W$ be the subset of minimal representatives of the cosets in W/W_X. Each element $u \in W$ defines a T-fixed point $u.P_X \in X$ and the *Schubert varieties* $X_u = \overline{Bu.P_X}$ and $X^u = \overline{B^- u.P_X}$. We denote the corresponding *Schubert cells* by $\mathring{X}_u = Bu.P_X$ and $\mathring{X}^u = B^- u.P_X$. These Schubert varieties and cells depend only on the coset uW_X in W/W_X, and for $u \in W^X$ we have $\dim(X_u) = \operatorname{codim}(X^u, X) = \ell(u)$.

Let $\leq$ denote the Bruhat order on W. For $u, v \in W^X$ we then have $v \leq u \Leftrightarrow X_v \subset X_u \Leftrightarrow X_u \cap X^v \neq \emptyset$. In this case the intersection $X_u^v = X_u \cap X^v$ is called a *Richardson variety*; this variety is reduced, irreducible, rational, and $\dim(X_u^v) = \ell(u) - \ell(v)$ [Ric92]. The *Richardson cell* $\mathring{X}_u^v = \mathring{X}_u \cap \mathring{X}^v$ is a dense open subset of X_u^v. Any translate of X_u^v will be called a Richardson variety. In other words, a non-empty closed subvariety $\Omega \subset X$ is a Richardson variety if and only if $\Omega = g.X_u^v$ for some $u, v \in W^X$ and $g \in G$. Similarly, arbitrary translates of Schubert varieties will be called Schubert varieties.

Each element $u \in W$ has a unique factorization $u = u^X u_X$ for which $u^X \in W^X$ and $u_X \in W_X$, called the *parabolic factorization* of u with respect to P_X. This factorization is *reduced* in the sense that $\ell(u) = \ell(u^X) + \ell(u_X)$. The parabolic factorization of the longest element $w_0 \in W$ is $w_0 = w_0^X w_{0,X}$, where w_0^X and $w_{0,X}$ are the longest elements in W^X and W_X, respectively. The Poincaré dual element of $u \in W^X$ is $u^\vee = w_0 u w_{0,X} \in W^X$, which satisfies $X^{u^\vee} = w_0.X_u$.

Suppose Y is an additional flag variety of G such that $P_X \subset P_Y$, and let $\pi : X \to Y$ be the projection. We then have $(u_Y)^X = (u^X)_Y$ for any $u \in W$, so this element of $W_Y \cap W^X$ may be written as u_Y^X without ambiguity. The parabolic factorizations of u with respect to both P_Y and P_X can be simultaneously expressed as $u = u^Y u_Y^X u_X$. Notice also that $\pi^{-1}(Y^u) = X^u$ and $\pi^{-1}(Y_u) = X_{u w_{0,Y}^X}$ for any $u \in W^Y$.

We need the *Hecke product* on W, which by definition is the unique associative monoid product satisfying

$$u \cdot s_\beta = \begin{cases} us_\beta & \text{if } u < us_\beta, \\ u & \text{if } u > us_\beta \end{cases}$$

for all $u \in W$ and $\beta \in \Delta$. Equivalently, the Hecke product $u \cdot v$ of $u, v \in W$ is given by $\overline{B(u \cdot v)B} = \overline{BuBvB}$ [Spr98, §8.3]. In particular we have $u.X_v \subset X_{u \cdot v}$. The product of u and v is reduced (i.e. $\ell(uv) = \ell(u) + \ell(v)$) if and only if $uv = u \cdot v$. Several other useful properties of the Hecke product can be found in e.g. [BM15, §3].

The *left weak Bruhat order* on W is defined by $u \leq_L w$ if and only if $\ell(wu^{-1}) = \ell(w) - \ell(u)$. Equivalently, there exists $x \in W$ such that $w = xu$ is a reduced factorization of w. We also need the P_X-*Bruhat order* $\leq_X$ on W, which we define by

$$v \leq_X u \quad \text{ if and only if } \quad v \leq u \;\text{ and }\; u_X \leq_L v_X \,.$$

It follows from Corollary 2.12 below that this definition is equivalent to the definition given in [KLS14, §2] (see also [BS98]).

Lemma 2.1. *For $x, u \in W$ we have $u \leq_L x \cdot u$ and $(x \cdot u)u^{-1} \leq x$.*

Proof. By definition of the Hecke product, there exists $x' \leq x$ such that $x \cdot u = x'u$ and the product $x'u$ is reduced. The lemma follows from this. □

Given a positive root $\alpha \in \Phi^+ \smallsetminus \Phi_X$ there exists a unique T-stable curve $C \subset X$ through $1.P_X$ and $s_\alpha.P_X$. For any simple root $\beta \in \Delta \smallsetminus \Delta_X$ we then have $\int_C [X^{s_\beta}] = (\alpha^\vee, \omega_\beta)$, where $\alpha^\vee$ denotes the coroot of α and ω_β is the fundamental weight corresponding to β (see [FW04, §3]).

An action of an algebraic group H on a variety Y is called *split* if there exists a morphism $s : U \to H$ defined on a dense open subset $U \subset Y$, and a point $y_0 \in U$, such that $s(y).y_0 = y$ for all $y \in U$. Many actions encountered in the study of Schubert varieties are split, including the action of B on a Schubert variety X_u and the action of $B \times B$ on a double Bruhat cell BuB. If $f : Z \to Y$ is any equivariant morphism of H-varieties such that the action of H on Y is split, then f is a locally trivial fibration over the dense open orbit in Y [BCMP13, Prop. 2.3]. In fact, the map $\varphi : U \times f^{-1}(y_0) \to f^{-1}(U)$ defined by $\varphi(y, z) = s(y).z$ is an isomorphism. We will make repeated use of this fact throughout this paper.

2.2. **Semi-transversal intersections.** Let Ω_1 and Ω_2 be closed subsets of a flag variety X of the group G. It is customary to study the intersection of general translates of these varieties, that is, any intersection of the form $\Omega_1 \cap g.\Omega_2$, where g is a general element of G. In particular, the product of (Chow) cohomology classes is given by $[\Omega_1] \cdot [\Omega_2] = [\Omega_1 \cap g.\Omega_2]$. In this section we consider the situation where the intersection of general translates of Ω_1 and Ω_2 is empty. We then seek to understand non-empty intersections of the form $\Omega_1 \cap g.\Omega_2$ that are as general as possible. Such intersections will be called *semi-transversal* intersections of Ω_1 and Ω_2 (when they exist). Intuitively, a semi-transversal intersection is obtained by moving general translates of Ω_1 and Ω_2 towards each other until they just meet. Semi-transversal intersections make sense for actions of arbitrary algebraic groups, so we will formulate our definition in this setting.

Let X be an algebraic variety and let G be an algebraic group acting on X. For any subset $\Omega \subset X$ we let $G_\Omega = \{g \in G \mid g.\Omega = \Omega\}$ denote the stabilizer. Given two closed subsets $\Omega_1, \Omega_2 \subset X$, we define a subset of G by

$$G(\Omega_1, \Omega_2) = \{g \in G \mid \Omega_1 \cap g.\Omega_2 \neq \emptyset\}\,.$$

This set is stable under the action of $G_{\Omega_1} \times G_{\Omega_2}$ defined by $(a_1, a_2).g = a_1 g a_2^{-1}$, so we have $G_{\Omega_1} G_{\Omega_2} \subset G(\Omega_1, \Omega_2)$ if and only if $\Omega_1 \cap \Omega_2 \neq \emptyset$. Notice also that $G(\Omega_2, \Omega_1) = G(\Omega_1, \Omega_2)^{-1}$, and $G(g_1.\Omega_1, g_2.\Omega_2) = g_1 G(\Omega_1, \Omega_2) g_2^{-1}$ for $g_1, g_2 \in G$. If X is a complete variety, then $G(\Omega_1, \Omega_2)$ is closed in G; this follows because $G(\Omega_1, \Omega_2) = p(\mu^{-1}(\Omega_1))$, where $p : G \times \Omega_2 \to G$ is the projection and $\mu : G \times \Omega_2 \to X$ is defined by the action.

Definition 2.2. We will say that Ω_1 and Ω_2 *meet semi-transversally* if $G_{\Omega_1} G_{\Omega_2}$ is a dense subset of $G(\Omega_1, \Omega_2)$. A *semi-transversal intersection* of Ω_1 and Ω_2 is any subscheme of the form $g_1.\Omega_1 \cap g_2.\Omega_2$ for which $g_1.\Omega_1$ and $g_2.\Omega_2$ meet semi-transversally (with $g_1, g_2 \in G$).

If Ω_1 and Ω_2 meet semi-transversally, then for all g in a dense open subset of $G(\Omega_1, \Omega_2)$, the intersection $\Omega_1 \cap g.\Omega_2$ is a translate of $\Omega_1 \cap \Omega_2$, so $\Omega_1 \cap \Omega_2$ is as general as possible among non-empty intersections. A semi-transversal intersection of Ω_1 and Ω_2 exists if and only if $G(\Omega_1, \Omega_2)$ contains a dense orbit for the action of $G_{\Omega_1} \times G_{\Omega_2}$, in which case $\Omega_1 \cap g.\Omega_2$ is a semi-transversal intersection whenever g belongs to this orbit. Any semi-transversal intersection of Ω_1 and Ω_2 is a G-translate of any other. Notice that $\Omega_1 \cap \Omega_2$ may be a semi-transversal intersection even though Ω_1 and Ω_2 fail to meet semi-transversally. The condition that Ω_1 and Ω_2 meet semi-transversally is stronger because it concerns the relative position of the two varieties and not just their intersection. When the group G is not clear from the context, we will write "G-semi-transversal" to clarify the action.

Example 2.3. (a) Let G act trivially on X. Then Ω_1 meets Ω_2 semi-transversally if and only if $\Omega_1 \cap \Omega_2 \neq \emptyset$, in which case $\Omega_1 \cap \Omega_2$ is the only semi-transversal intersection.

(b) Let $\mathrm{GL}(3)$ act on $\mathbb{P}^2$. Then a line and a conic meet semi-transversally if and only if they have two points in common.

(c) Let $\mathrm{GL}(4)$ act on $\mathbb{P}^3$. Now a line and an irreducible curve of degree two meet semi-transversally if and only if their intersection is a single reduced point.

(d) Let $G = \mathrm{PGL}(2) \times \mathrm{PGL}(2) \rtimes S_2$ be the automorphism group of $X = \mathbb{P}^1 \times \mathbb{P}^1$, and set $L = \mathbb{P}^1 \times \{0\}$. Then G_L is connected while $G(L, L)$ is disconnected, so no semi-transversal intersection exists of L with itself. However, if we restrict the action to the identity component $G^0 = \mathrm{PGL}(2) \times \mathrm{PGL}(2)$, then L meets itself semi-transversally.

(e) Let $\mathrm{PGL}(n+1)$ act on $\mathbb{P}^n$, let $H \subset \mathbb{P}^n$ be a hyperplane, and let $S \subset \mathbb{P}^n$ be any hypersurface with finite automorphism group. Then the dimension of $G_H \times G_S$ is smaller than the dimension of $G(H, S) = G$, so no semi-transversal intersection of H and S exists. In particular, transversal intersections may fail to be semi-transversal.

(f) The maximal orthogonal Grassmannian $X = \mathrm{OG}(4, 8)$ is one component of the variety of maximal isotropic subspaces in an orthogonal vector space V of dimension 8. This space X is a flag variety of both $\mathrm{SO}(8)$ and its subgroup $\mathrm{SO}(7)$. A Schubert line in X corresponds to a 2-dimensional isotropic subspace $E \subset V$. Two Schubert lines given by E_1 and E_2 meet SO(8)-semi-transversally if and only if $E_1 + E_2$ is a point of X, whereas they meet SO(7)-semi-transversally only if $E_1 + E_2$ is an isotropic subspace of dimension 3. It is therefore impossible for two Schubert lines to meet semi-transversally for both groups at the same time.

(g) The projective space $\mathbb{P}^{2n-1}$ is a flag variety of both $\mathrm{SL}(2n)$ and its subgroup $\mathrm{Sp}(2n)$ of elements preserving a symplectic form on a vector space V of dimension $2n$. A Schubert line for the action of $\mathrm{Sp}(2n)$ is given by a 2-dimensional isotropic subspace $E \subset V$. Two Schubert lines given by isotropic subspaces E_1 and E_2 meet $\mathrm{SL}(2n)$-semi-transversally if and only if $\dim(E_1 + E_2) = 3$, whereas they meet $\mathrm{Sp}(2n)$-semi-transversally if and only if $\dim(E_1+E_2) = 3$ and E_1+E_2 is not isotropic. Two Schubert lines can therefore meet $\mathrm{SL}(2n)$-semi-transversally without meeting $\mathrm{Sp}(2n)$-semi-transversally.

2.3. **Semi-transversal intersections of Schubert varieties.** We are mostly interested in semi-transversal intersections of Schubert varieties in a flag variety, so assume again that X is a flag variety of a connected linear algebraic group G. Our first result shows that a semi-transversal intersection of two Schubert varieties exists and is a Richardson variety. Notice that $X_u \cap w_0.X_v$ is a semi-transversal intersection of X_u and X_v whenever this intersection is not empty; this happens exactly when $\kappa = w_0$ in the following result.

Theorem 2.4. *Let $u, v \in W^X$, and set $\kappa = u \cdot w_{0,X} \cdot v^{-1}$. Then $G(X_u, X_v) = \overline{B\kappa B}$, $w_0\kappa v \in W^X$, and $X_u \cap \kappa.X_v = X_u \cap w_0.X_{w_0\kappa v}$ as (reduced and irreducible) subschemes of X. In particular, the Richardson variety $X_u \cap w_0.X_{w_0\kappa v}$ is a semi-transversal intersection of X_u and X_v.*

Proof. Let $\mu : G \times X_v \to X$ be the map defined by $\mu(h, x) = h.x$. If we let G act on the left factor of $G \times X_v$, then μ is G-equivariant, so μ is a locally trivial fibration by [BCMP13, Prop. 2.3]. It follows that $\mu^{-1}(X_u) \subset G \times X_v$ is a closed irreducible subvariety. Let $p : G \times X_v \to G$ be the first projection and let $B \times B$ act on $G \times X_v$ by $(b_1, b_2).(h, x) = (b_1 h b_2^{-1}, b_2.x)$. Then p is proper and $B \times B$-equivariant, and since $p^{-1}(g) \cap \mu^{-1}(X_u) = \{g\} \times (g^{-1}.X_u \cap X_v) \cong X_u \cap g.X_v$, it follows that $G(X_u, X_v) = p(\mu^{-1}(X_u)) \subset G$ is closed, irreducible, and $B \times B$-stable. Since $\kappa v \leq u w_{0,X}$ by Lemma 2.1, we obtain $\kappa v.P_X \in X_u \cap \kappa.X_v \neq \emptyset$, so $B\kappa B \subset G(X_u, X_v)$. On the other hand, if $w \in W$ is any element such that $X_u \cap w.X_v \neq \emptyset$, then let $v'.P_X \in w^{-1}.X_u \cap X_v$ be any T-fixed point, with $v' \in W^X$. Then $v' \leq v$ and $wv' \leq u w_{0,X}$, which implies $w = (wv')(v')^{-1} \leq (wv') \cdot (v')^{-1} \leq u w_{0,X} \cdot v^{-1} = \kappa$. This shows that $G(X_u, X_v) = \overline{B\kappa B}$.

Since $p : \mu^{-1}(X_u) \to G(X_u, X_v)$ is $B \times B$-equivariant, this map is locally trivial over the open orbit $B\kappa B$. In particular, the intersection $X_u \cap \kappa.X_v$ is reduced. Since $\ell(\kappa v) = \ell(\kappa) - \ell(v)$ by Lemma 2.1, we have $\ell(w_0\kappa v) = \ell(w_0\kappa) + \ell(v)$. It follows that $w_0\kappa.X_v \subset X_{w_0\kappa v}$, so we have $X_u \cap \kappa.X_v \subset X_u \cap w_0.X_{w_0\kappa v}$. Using that $\mu : G \times X_v \to X$ is locally trivial, we get $\dim(\mu^{-1}(X_u)) = \dim(P_X) + \ell(u) + \ell(v)$, and since $p : \mu^{-1}(X_u) \to G(X_u, X_v)$ is locally trivial over $B\kappa B$, we obtain $\dim(X_u \cap \kappa.X_v) = \ell(u) + \ell(w_{0,X}) + \ell(v) - \ell(\kappa)$. On the other hand, the dimension of the Richardson variety $X_u \cap w_0.X_{w_0\kappa v}$ is bounded by $\dim(X_u \cap w_0.X_{w_0\kappa v}) \leq \ell(u) + \ell(w_0\kappa v) - \dim(X) = \ell(u) + \ell(w_{0,X}) + \ell(v) - \ell(\kappa)$. We deduce that $X_u \cap \kappa.X_v = X_u \cap w_0.X_{w_0\kappa v}$ and that $\dim(X_{w_0\kappa v}) = \ell(w_0\kappa v)$, so that $w_0\kappa v \in W^X$. □

The following corollary gives a geometric characterization of the weak left Bruhat order that we have not seen before.

Corollary 2.5. *Let $u, v \in W^X$. The following are equivalent.* (1) *There exists $g \in G$ such that $X_u \cap g.X_v$ is a single point.* (2) *The product $u \cdot w_{0,X} \cdot v^{-1}$ is reduced.* (3) *We have $u \leq_L v^\vee$.*

Proof. Set $\kappa = u \cdot w_{0,X} \cdot v^{-1}$. It follows from Theorem 2.4 that $X_u \cap \kappa.X_v$ is a point if and only if $w_0 \kappa v = u^\vee = w_0 u w_{0,X}$, or $\kappa = u w_{0,X} v^{-1}$, so (1) and (2) are equivalent. Conditions (2) and (3) are equivalent because we have $\ell(v w_{0,X} u^{-1}) = \ell(v w_{0,X}) + \ell(u)$ if and only if $\ell(v^\vee u^{-1}) = \ell(v^\vee) - \ell(u)$. ☐

Remark 2.6. Using the Stein factorization of the map $p : \mu^{-1}(X_u) \to G(X_u, X_v)$, it follows that $X_u \cap g.X_v$ is connected for all $g \in G(X_u, X_v)$. However, the intersection $X_u \cap g.X_v$ is not always irreducible, and Jesper Thomsen has shown us an example where $X_u \cap g.X_v$ fails to be reduced [Tho]. For example, the intersection of two Schubert divisors in the Grassmannian $\mathrm{Gr}(2,4)$ may be a union of two projective planes, and two Schubert divisors in the quadric hypersurface $Q^3 \subset \mathbb{P}^4$ can meet in a double line.

Remark 2.7. If X is a flag variety of two different groups G and H, and $\Omega_1, \Omega_2 \subset X$ are Schubert varieties with respect to both groups, then one can show that any semi-transversal intersection of Ω_1 and Ω_2 for the action of H is also a semi-transversal intersection for the action of G, up to translation by an automorphism of X. In fact, there are only three families of flag varieties of groups of distinct Lie types, namely odd-dimensional projective spaces $\mathbb{P}^{2n-1} = A_{2n-1}/P_1 = C_n/P_1$, maximal orthogonal Grassmannians $\mathrm{OG}(n, 2n) = D_n/P_n = B_{n-1}/P_{n-1}$, and the 5-dimensional quadric $Q^5 = B_3/P_1 = G_2/P_2$. The first two families consist of (co)minuscule flag varieties, in which case the claim follows from Proposition 4.5. The last case $B_3/P_1 = G_2/P_2$ has been checked from Theorem 2.4 with help from a computer. We will not need this fact in the following.

2.4. **Fibers of Schubert varieties.** Let $Y = G/P_Y$ be an additional flag variety of G such that $P_X \subset P_Y$, and let $\pi : X \to Y$ be the projection. Then $F = \pi^{-1}(1.P_Y) = P_Y/P_X$ is a flag variety of P_Y. The Schubert varieties in F are the B-orbit closures $F_w = X_w$ for $w \in W_Y$, as well as their P_Y-translates. Similarly, any fiber $\pi^{-1}(g.P_Y) = g.F$ for $g \in G$ is a flag variety of gP_Yg^{-1}. Given a subvariety $\Omega \subset X$, the fibers of the restricted map $\pi : \Omega \to \pi(\Omega)$ will be called fibers of Ω. Our next result shows that, if Ω is a Schubert variety in X, then any fiber $\Omega \cap \pi^{-1}(y)$ with $y \in \pi(\Omega)$ is a Schubert variety in $\pi^{-1}(y)$.

Theorem 2.8. *Any intersection $X_u \cap \pi^{-1}(y)$ defined by $u \in W^X$ and $y \in \pi(X_u)$ is a (reduced) Schubert variety in $\pi^{-1}(y)$. Let $u = u^Y u_Y$ be the parabolic factorization with respect to P_Y. Any fiber $X_u \cap \pi^{-1}(y)$ given by $y \in \mathring{Y}_u$ is a translate of $X_u \cap u.F = u^Y.F_{u_Y}$.*

Proof. We may assume that G is reductive. Let $L \subset P_Y$ be the Levi subgroup containing T and set $B_L = B \cap L$. We then have $F_w = \overline{B_L w.P_X}$ for each $w \in W_Y$. Since π is B-equivariant, we may assume that $y = v.P_Y \in \pi(X_u)$ is a T-fixed point, given by an element $v \in W^Y$. Then $v \leq u$. Since $v \in W^Y$ we have $v.\alpha > 0$ for each positive root α of L, and this implies that $vB_Lv^{-1} \subset B$ and therefore $B_L \subset v^{-1}Bv \cap L$. Since $v^{-1}Bv \cap L$ is a Borel subgroup of L, we obtain $B_L = v^{-1}Bv \cap L$. It follows that $v^{-1}.X_u \cap F$ is a closed B_L-stable subset of F, so this set is a union of B_L-stable Schubert varieties in F. The set of T-fixed points in $v^{-1}.X_u \cap F$ is $\{z.P_X \mid z \in W_Y \text{ and } vz \leq uw_{0,X}\}$. It follows from

[KLS14, Prop. 2.1] that the set $\{z \in W_Y \mid vz \leq uw_{0,X}\}$ has a unique maximal element, say $z' \in W_Y$. This implies that $v^{-1}.X_u \cap F = F_{z'}$ (as sets), hence $X_u \cap \pi^{-1}(v.P_Y) = v.F_{z'}$ is a translate of this Schubert variety. Notice also that, if $y \in \mathring{Y}_u$, then $v = u^Y$, $z' = u_Y w_{0,X}$, and $F_{z'} = F_{u_Y}$. To see that $X_u \cap \pi^{-1}(v.P_Y)$ is reduced, set $Z = X_u \cap \pi^{-1}(Y_v)$. Since Z is an intersection of B-stable Schubert varieties, it follows that Z is a reduced union of Schubert varieties in X, see [BK05, §2]. Since the map $\pi : Z \to Y_v$ has irreducible fibers by the above argument, it follows that Y_v is dominated by a unique irreducible component Z' of Z, and $Z' \subset X$ is a B-stable Schubert variety. Since $\pi : Z' \to Y_v$ is B-equivariant, it is locally trivial over the open cell $\mathring{Y}_v \subset Y_v$. We conclude that $X_u \cap \pi^{-1}(v.P_Y) = Z' \cap \pi^{-1}(v.P_Y)$ is reduced. □

Remark 2.9. The parabolic factorization of $u \in W^X$ commutes with dualization in the sense that $w_0 u w_{0,X} = (w_0 u^Y w_{0,Y})(w_{0,Y} u_Y w_{0,X})$. It follows that the general fibers of the projection $\pi : X^u \to \pi(X^u)$ are translates of the Schubert variety F^{u_Y}.

2.5. **Fibers of Richardson varieties.** We finally consider the fibers $R \cap \pi^{-1}(y)$ of a Richardson variety $R \subset X$ under the projection of flag varieties $\pi : X \to Y$. While the fibers of a Richardson variety may fail to be irreducible [BR12, Ex. 3.1], we will show that $R \cap \pi^{-1}(y)$ is a Richardson variety in $\pi^{-1}(y)$ for all points y in a dense open subset of $\pi(R)$. Some very special cases of this were proved in [BR12]. The projection $\pi : R \to \pi(R)$ was also studied in [BC12, KLS14], were it was proved that this map is cohomologically trivial and that the projected Richardson variety $\pi(R)$ is Cohen-Macaulay with rational singularities.

Theorem 2.10. *Let $R = X_u \cap w_0.X_v$ be a Richardson variety in X, with $u, v \in W^X$. Let $u = u^Y u_Y$ and $v = v^Y v_Y$ be the parabolic factorizations with respect to P_Y. Then for all points y in a dense open subset of $\pi(R)$, the fiber $R \cap \pi^{-1}(y)$ is a G-translate of a semi-transversal intersection of F_{u_Y} and F_{v_Y} in $F = \pi^{-1}(1.P_Y)$.*

Proof. Using [Spr98, Lemma 8.3.6] we may choose two morphisms $\phi_1 : \mathring{Y}_u \to B$ and $\phi_2 : \mathring{Y}_v \to B$ such that $\phi_1(y)u.P_Y = y$ for all $y \in \mathring{Y}_u$ and $\phi_2(y)v.P_Y = y$ for all $y \in \mathring{Y}_v$. For each element $w \in W$ we choose a fixed representative $\dot{w} \in N_G(T)$. Set $\pi(R)^0 = \pi(R) \cap \mathring{Y}_u \cap w_0.\mathring{Y}_v$ and define $\psi : \pi(R)^0 \to G$ by

$$\psi(y) = (\dot{u}^Y)^{-1}\, \phi_1(y)^{-1}\, \dot{w}_0\, \phi_2(\dot{w}_0^{-1}.y)\, \dot{v}^Y \,.$$

Since $\dot{w}_0 \phi_2(\dot{w}_0^{-1}.y)\dot{v}^Y.P_Y = y = \phi_1(y)\dot{u}^Y.P_Y$, we have $\psi(y) \in P_Y$ for all $y \in \pi(R)^0$. Theorem 2.8 implies that $X_u \cap \pi^{-1}(y) = \phi_1(y)u^Y.F_{u_Y}$ and $w_0.X_v \cap \pi^{-1}(y) = \dot{w}_0\phi_2(\dot{w}_0^{-1}.y)v^Y.F_{v_Y}$. It follows that translation by $\phi_1(y)\dot{u}^Y$ maps the triple of varieties $(F_{u_Y}, \psi(y).F_{v_Y}, F)$ to $(X_u \cap \pi^{-1}(y), w_0.X_v \cap \pi^{-1}(y), \pi^{-1}(y))$. We deduce that $F_{u_Y} \cap \psi(y).F_{v_Y} \neq \emptyset$ for all $y \in \pi(R)^0$, hence $\psi(y) \in P_Y(F_{u_Y}, F_{v_Y})$. Since Theorem 2.4 shows that $F_{u_Y} \cap g.F_{v_Y}$ is a semi-transversal intersection in F for all elements g in a dense open subset of $P_Y(F_{u_Y}, F_{v_Y})$, it is enough to show that $X_u \cap \pi^{-1}(y)$ meets $w_0.X_v \cap \pi^{-1}(y)$ semi-transversally in $\pi^{-1}(y)$ for at least one point $y \in \pi(R)^0$.

Fix any point $y' = g.P_Y \in \pi(R)^0$. Then $\pi^{-1}(y') = g.F$ is a flag variety of the group gP_Yg^{-1}, and the set $H = (gP_Yg^{-1})(X_u \cap g.F, w_0.X_v \cap g.F)$ of §2.2 is an irreducible closed subvariety of gP_Yg^{-1} by Theorem 2.4. This theorem also implies that $X_u \cap g.F$ meets $hw_0.X_v \cap g.F$ semi-transversally in $g.F$ for all elements h in a dense open subset $H^0 \subset H$. Since BB^- is a dense open subset of G and

$1 \in BB^- \cap H$, it follows that $BB^- \cap H$ is another dense open subset of H. Let $U^- \subset B^-$ be the unipotent radical, let $\rho : BB^- = B \times U^- \to B$ be the first projection, and define $f : BB^- \to Y$ by $f(h) = \rho(h)^{-1}.y'$. For $h \in BB^-$ we have $\rho(h)^{-1}h \in U^-$, which implies that $X_u \cap w_0.X_v \cap \pi^{-1}(f(h)) = \rho(h)^{-1}.(X_u \cap hw_0.X_v \cap \pi^{-1}(y'))$. It follows that f restricts to a morphism $f : BB^- \cap H \to \pi(R)$. Since $f(1) = y' \in \pi(R)^0$, it follows that $f^{-1}(\pi(R)^0)$ is a dense open subset of H. Finally, let $h \in f^{-1}(\pi(R)^0) \cap H^0$ be any element and set $y = f(h)$. Then we have $y \in \pi(R)^0$, and $X_u \cap \pi^{-1}(y)$ meets $w_0.X_v \cap \pi^{-1}(y)$ semi-transversally in $\pi^{-1}(y)$, as required. □

Corollary 2.11. *The general fibers of a projection $\pi : R \to \pi(R)$ from a Richardson variety are Richardson varieties.*

The following result was proved in [KLS14, Cor. 3.4] with a different but equivalent definition of the P_X-Bruhat order $\leq_X$.

Corollary 2.12. *Let $u, v \in W^X$. The projection $\pi : X_u^v \to \pi(X_u^v)$ is a birational map of non-empty varieties if and only if $v \leq_Y u$.*

Proof. This follows from Theorem 2.10, Corollary 2.5, and Remark 2.9. □

For later use we state the following result, which was proved in [BC12, KLS14].

Theorem 2.13. *Let $\pi : X \to Y$ be a projection of flag varieties and let $R \subset X$ be a Richardson variety.*

(a) *The image $\pi(R)$ is Cohen-Macaulay and has rational singularities.*

(b) *The map $\pi : R \to \pi(R)$ is cohomologically trivial, that is, $\pi_*(\mathcal{O}_R) = \mathcal{O}_{\pi(R)}$ and $R^j \pi_* \mathcal{O}_R = 0$ for $j > 0$.*

3. Projected Richardson varieties

We need some additional results about projections of Richardson varieties that were proved in [KLS14]. In this expository section we give short proofs of these results. The statements of some results deviate slightly from the original versions, for example the bounds on u' and v' in Theorem 3.5 are important for our applications but not immediately clear from [KLS14, Prop. 3.3 and Prop. 3.6]. Another difference is our simpler but equivalent definition of the P_X-Bruhat order from Section 2.1: for $u, v \in W$ we have $v \leq_X u$ if and only if $v \leq u$ and $u_X \leq_L v_X$.

We work over any algebraically closed field. Let $E = G/B$ be the variety of complete flags, let $X = G/P_X$ be any flag variety of G, and let $\pi : E \to X$ be the projection. Given $v, u \in W$ with $v \leq u$, the images in X of the corresponding Richardson variety and Richardson cell in E are denoted by $\Pi_u^v(X) = \pi(E_u^v)$ and $\mathring{\Pi}_u^v(X) = \pi(\mathring{E}_u^v)$.

Lemma 3.1. *Let $u, v \in W$ satisfy $v \leq u$, and let $s \in W_X$ be a simple reflection such that $u < us$ and $v < vs$. Then the following hold.*

(a) $\Pi_u^v(X) = \Pi_{us}^{vs}(X) = \Pi_{us}^v(X)$.

(b) $\mathring{\Pi}_u^v(X) = \mathring{\Pi}_{us}^{vs}(X)$.

(c) $\mathring{\Pi}_{us}^v(X) = \mathring{\Pi}_u^v(X) \cup \mathring{\Pi}_u^{vs}(X)$.

(d) $\pi : \mathring{E}_u^v \to \mathring{\Pi}_u^v(X)$ *is an isomorphism of varieties if and only if* $\pi : \mathring{E}_{us}^{vs} \to \mathring{\Pi}_{us}^{vs}(X)$ *is an isomorphism of varieties.*

Proof. We may assume that P_X is the minimal parabolic subgroup defined by $W_X = \{1, s\}$. We then have $u, v \in W^X$. Part (a) follows from $\pi(E_u^v) = \pi(E_u \cap \pi^{-1}(X^v)) = \pi(E_u) \cap X^v = X_u^v$ and symmetric identities. We also have $\pi(\mathring{E}_{us}^v) \subset \mathring{X}_u^v = \pi(\mathring{E}_u) \cap \mathring{X}^v = \pi(\mathring{E}_u \cap \pi^{-1}(\mathring{X}^v)) = \pi(\mathring{E}_u \cap (\mathring{E}^v \cup \mathring{E}^{vs})) = \pi(\mathring{E}_u^v) \cup \pi(\mathring{E}_u^{vs})$. Given $x \in \mathring{X}_u^v$, the fiber $\pi^{-1}(x) \cong \mathbb{P}^1$ is contained in $\pi^{-1}(\mathring{X}_u^v) = \mathring{E}_{us}^v \cup \mathring{E}_u^v \cup \mathring{E}_{us}^{vs} \cup \mathring{E}_u^{vs}$. Since π is injective on $\mathring{E}_u$ and on $\mathring{E}^{vs}$, $\pi^{-1}(x) \cap (\mathring{E}_u^v \cup \mathring{E}_{us}^{vs} \cup \mathring{E}_u^{vs})$ is finite, so $\pi^{-1}(x) \cap \mathring{E}_{us}^v \neq \emptyset$. This proves (c). A symmetric argument gives $\mathring{X}_u^v = \pi(\mathring{E}_{us}^{vs}) \cup \pi(\mathring{E}_u^{vs})$. Again using that π is injective on $\mathring{E}_u$ and on $\mathring{E}^{vs}$, part (b) follows because $\pi(\mathring{E}_u^v) = \mathring{X}_u^v - \pi(\mathring{E}_u^{vs}) = \pi(\mathring{E}_{us}^{vs})$. Finally, both maps in part (d) are isomorphisms because $\pi : \mathring{E}_u \to \mathring{X}_u$ and $\pi : \mathring{E}^{vs} \to \mathring{X}^{vs}$ are isomorphisms by [Spr98, Lemma 8.3.6]. □

Proposition 3.2. *Let $u, v \in W$ satisfy $v \leq u$. The following are equivalent.*

(a) *We have $v \leq_X u$.*

(b) *The dimension of $\Pi_u^v(X)$ is $\ell(u) - \ell(v)$.*

(c) *The map $\pi : \mathring{E}_u^v \to \mathring{\Pi}_u^v(X)$ is an isomorphism of varieties.*

Proof. If $u \in W^X$, then (a) holds by definition of $\leq_X$, and (b) and (c) hold because $\pi : \mathring{E}_u \to \mathring{X}_u$ is an isomorphism. Otherwise let $s \in W_X$ be a simple reflection such that $us < u$. If $v < vs$, then $v \not\leq_X u$, and Lemma 3.1(a) implies that $\dim \Pi_u^v(X) < \dim E_u^v$. On the other hand, if $vs < v$, then $vs \leq_X us$ holds if and only if $v \leq_X u$, so the result follows from Lemma 3.1(b,d) by induction on $\ell(u_X)$. □

If part (c) of Proposition 3.2 is replaced with the requirement that $\pi : E_u^v \to \pi(E_u^v)$ is birational, then this theorem holds when E is an arbitrary flag variety and $u, v \in W^E$. However, the following example shows that $\pi : \mathring{E}_u^v \to \pi(\mathring{E}_u^v)$ may fail to be injective when $E \neq G/B$ and $v \leq_X u$ in W^E.

Example 3.3. Let $G = \mathrm{GL}(5)$, let $B \subset G$ be the subgroup of upper triangular matrices, and let $B \subset P_X \subset P_Y \subset G$ be the parabolic subgroups such that W_X is generated by s_4 and W_Y is generated by s_1, s_3, s_4. Then $X = G/P_X = \mathrm{Fl}(1, 2, 3; 5)$ and $Y = G/P_Y = \mathrm{Gr}(2, 5)$. Let $\pi : X \to Y$ be the projection and set $v = s_3$ and $u = s_3 s_2 s_1 s_4 s_3 s_2 s_3 = 45213$. Since $u_Y = v_Y = s_3$, it follows from Proposition 3.2 or Corollary 2.12 that $\pi : X_u^v \to \pi(X_u^v)$ is birational. Let $E = G/B = \mathrm{Fl}(5)$ and $v' = s_3 s_4$, and consider the composition of projections

$$\mathring{E}_u^{v'} \longrightarrow \mathring{X}_u^v \xrightarrow{\pi} \pi(\mathring{X}_u^v)\,.$$

Since $u \in W^X$, the map $\mathring{E}_u \to \mathring{X}_u$ is an isomorphism of affine spaces, so the first projection is injective. However, since $u_Y \not\leq_L v'_Y$, it follows from Proposition 3.2 that the composed projection is not injective. We conclude that $\pi : \mathring{X}_u^v \to \pi(\mathring{X}_u^v)$ is not injective.

Let $\sim_X$ denote the equivalence relation on the set $\{(v, u) \in W \times W \mid v \leq u\}$ generated by $(v, u) \sim_X (v, us) \sim_X (vs, us)$ whenever $s \in W_X$ is a simple reflection such that $u < us$ and $v < vs$.

Theorem 3.4. *Let $u, v, u', v' \in W$ satisfy $v \leq u$ and $v' \leq u'$.*

(a) *If $\Pi_u^v(X) = \Pi_{u'}^{v'}(X)$ and $u, u' \in W^X$, then $u = u'$ and $v = v'$.*

(b) *We have* $\Pi_u^v(X) = \Pi_{u'}^{v'}(X)$ *if and only if* $(v,u) \sim_X (v',u')$.

(c) *If* $v \leq_X u$ *and* $v' \leq_X u'$, *then either* $\mathring{\Pi}_u^v(X) = \mathring{\Pi}_{u'}^{v'}(X)$ *or* $\mathring{\Pi}_u^v(X) \cap \mathring{\Pi}_{u'}^{v'}(X) = \emptyset$.

Proof. It follows from Lemma 3.1(a) that $(v,u) \sim_X (v',u')$ implies $\Pi_u^v(X) = \Pi_{u'}^{v'}(X)$. Assume that $u, u' \in W^X$ and $\mathring{\Pi}_u^v(X) \cap \mathring{\Pi}_{u'}^{v'}(X) \neq \emptyset$. Then $\mathring{X}_u \cap \mathring{X}_{u'} \neq \emptyset$, which implies $u = u'$. Since π is injective on $\mathring{E}_u$, we deduce that $v = v'$ as well. This proves (a), and also establishes (b) and (c) when $u, u' \in W^X$. Assume next that $u \notin W^X$. Choose a simple reflection $s \in W_X$ such that $\widetilde{u} = us < u$, and set $\widetilde{v} = (v \cdot s)s$. Then we have $(v,u) \sim_X (\widetilde{v}, \widetilde{u})$, $\Pi_u^v(X) = \Pi_{\widetilde{u}}^{\widetilde{v}}(X)$, and $\widetilde{u}_X < u_X$, so part (b) follows by induction on $\ell(u_X) + \ell(u'_X)$. If $v \leq_X u$, then we must have $\widetilde{v} = vs < v$ and $\widetilde{v} \leq_X \widetilde{u}$. Since Lemma 3.1(b) shows that $\mathring{\Pi}_u^v(X) = \mathring{\Pi}_{\widetilde{u}}^{\widetilde{v}}(X)$, part (c) also follows by induction on $\ell(u_X) + \ell(u'_X)$. □

Theorem 3.5. *Let* $u, v \in W$ *satisfy* $v \leq u$.

(a) *The set* $\mathring{\Pi}_u^v(X)$ *is the disjoint union of some of the sets* $\mathring{\Pi}_{u'}^{v'}(X)$ *for which* $v \leq v' \leq_X u' \leq u$.

(b) *We have* $\Pi_u^v(X) = \bigcup_{v \leq v' \leq_X u' \leq u} \mathring{\Pi}_{u'}^{v'}(X)$.

(c) *There exists* $u', v' \in W$ *such that* $v \leq v' \leq_X u' \leq u$ *and* $\Pi_u^v(X) = \Pi_{u'}^{v'}(X)$.

Proof. We first prove part (a). If $u \in W^X$, then $v \leq_X u$ and the result is clear. Otherwise let $s \in W_X$ be a simple reflection such that $us < u$. If $v < vs$, then Lemma 3.1(c) shows that $\mathring{\Pi}_u^v(X) = \mathring{\Pi}_{us}^v(X) \cup \mathring{\Pi}_{us}^{vs}(X)$, so the result follows by induction on $\ell(u_X)$. The obtained union is automatically disjoint by Theorem 3.4(c). Assume that $vs < v$. Then Lemma 3.1(b) gives $\mathring{\Pi}_u^v(X) = \mathring{\Pi}_{us}^{vs}(X)$, and by induction on $\ell(u_X)$ we can express $\mathring{\Pi}_u^v(X)$ as a union of sets $\mathring{\Pi}_{u'}^{v'}(X)$ for which $vs \leq v' \leq_X u' \leq us$. Any such set $\mathring{\Pi}_{u'}^{v'}(X)$ for which $v \leq v'$ satisfies the requirements of part (a), so assume that $vs \leq v' \leq_X u' \leq us$ and $v \not\leq v'$. Then we must have $v' < v's$. Since $v' \leq_X u'$, we also obtain $u' < u's$, so Lemma 3.1(b) shows that $\mathring{\Pi}_{u'}^{v'}(X) = \mathring{\Pi}_{u's}^{v's}(X)$. Since $v \leq v's \leq_X u's \leq u$, this set has the required form. This completes the proof of (a). Part (b) follows from (a) because E_u^v is the union of all sets $\mathring{E}_{u'}^{v'}$ for which $v \leq v' \leq u' \leq u$, and (c) follows because $\mathring{\Pi}_{u'}^{v'}(X)$ must be dense in $\mathring{\Pi}_u^v(X)$ for some $u', v' \in W$ with $v \leq v' \leq_X u' \leq u$. □

Corollary 3.6. *If* $v \leq u$ *in* W^X, *then* $X_u^v = \Pi_u^v(X)$ *and* $\mathring{X}_u^v = \mathring{\Pi}_{uw_{0,X}}^v(X)$.

Proof. The first equality is true because $X_u^v = \pi(E_u) \cap X^v = \pi(E_u \cap \pi^{-1}(X^v)) = \Pi_u^v(X)$. For the second, notice first that $\mathring{\Pi}_{uw_{0,X}}^v(X) \subset \pi(\mathring{E}_{uw_{0,X}}) \cap \pi(\mathring{E}^v) = \mathring{X}_u^v$. We also have $\mathring{X}_u^v = \mathring{X}_u \cap \pi(\mathring{E}^v) = \pi(\pi^{-1}(\mathring{X}_u) \cap \mathring{E}^v) = \bigcup_{x \in W_X} \mathring{\Pi}_{ux}^v(X)$. Since Lemma 3.1(c) implies that $\mathring{\Pi}_{ux}^v(X) \subset \mathring{\Pi}_{uw_{0,X}}^v(X)$, we deduce that $\mathring{X}_u^v \subset \mathring{\Pi}_{uw_{0,X}}^v(X)$. □

4. Cominuscule flag varieties

4.1. **Schubert varieties.** In the remainder of this paper we let $X = G/P_X$ be a *cominuscule flag variety* defined over $\mathbb{C}$. This means that P_X is a maximal parabolic subgroup of G, and the unique simple root γ in $\Delta \smallsetminus \Delta_X$ is cominuscule, that is, when the highest root of Φ is expressed as a linear combination of simple roots, the coefficient of γ is one. If in addition the root system Φ is simply laced, then X is also called *minuscule*. It was proved by Proctor that the Bruhat order on W^X is a distributive lattice that agrees with the left weak Bruhat order [Pro84]. Stembridge has proved that all elements of W^X are *fully commutative*, which means that any reduced expression of an element of W^X can be obtained from any other by interchanging commuting simple reflections [Ste96]. We proceed to summarize the facts we need in more detail, following the notation from [BCMP18a]. Proofs of our claims can be found in [Pro84, Ste96, Per07, BS16].

The root lattice $\mathrm{Span}_{\mathbb{Z}}(\Delta)$ has a partial order defined by $\alpha' \leq \alpha$ if and only if $\alpha - \alpha'$ is a sum of positive roots. Set $\mathcal{P}_X = \{\alpha \in \Phi \mid \alpha \geq \gamma\}$, with the induced partial order. For any element $u \in W$ we let $I(u) = \{\alpha \in \Phi^+ \mid u.\alpha < 0\}$ denote the inversion set. We then have $\ell(u) = |I(u)|$, and $u \in W^X$ if and only if $I(u) \subset \mathcal{P}_X$. Moreover, the assignment $u \mapsto I(u)$ restricts to a bijection between the elements of W^X and the (lower) order ideals of $\mathcal{P}_X$. This bijection is an order isomorphism in the sense that $v \leq u$ if and only if $I(v) \subset I(u)$. The order ideals in $\mathcal{P}_X$ generalize the Young diagrams known from the Schubert calculus of classical Grassmannians. For this reason the roots in $\mathcal{P}_X$ will sometimes be called *boxes*. An order ideal in $\mathcal{P}_X$ will be called a *straight shape*, and a difference between order ideals is called a *skew shape*.

Given a straight shape $\lambda \subset \mathcal{P}_X$, let $\lambda = \{\alpha_1, \alpha_2, \dots, \alpha_{|\lambda|}\}$ be any *increasing ordering* of its elements, i.e. $\alpha_i < \alpha_j$ implies $i < j$. Then the element of W^X corresponding to λ is the product of reflections $w_\lambda = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_{|\lambda|}}$ (see e.g. [BS16, Thm. 2.4]). Given $v, u \in W^X$ with $v \leq u$, we will use the notation $u/v = uv^{-1} \in W$. Since the Bruhat order on W^X agrees with the left weak Bruhat order [Pro84] (see also [Ste96, Thm. 7.1]), we have $\ell(u/v) = \ell(u) - \ell(v)$.

For any root $\alpha \in \mathcal{P}_X$, consider the shape $\lambda(\alpha) = \{\alpha' \in \mathcal{P}_X \mid \alpha' < \alpha\}$, and set $\delta(\alpha) = w_{\lambda(\alpha)}.\alpha$. Then $s_{\delta(\alpha)} = w_{\lambda(\alpha)} s_\alpha w_{\lambda(\alpha)}^{-1} = w_{\lambda(\alpha) \cup \{\alpha\}} / w_{\lambda(\alpha)}$ has length one. It follows that $\delta : \mathcal{P}_X \to \Delta$ is a labeling of the boxes in $\mathcal{P}_X$ by simple roots. Examples of this labeling are provided in Table 1.

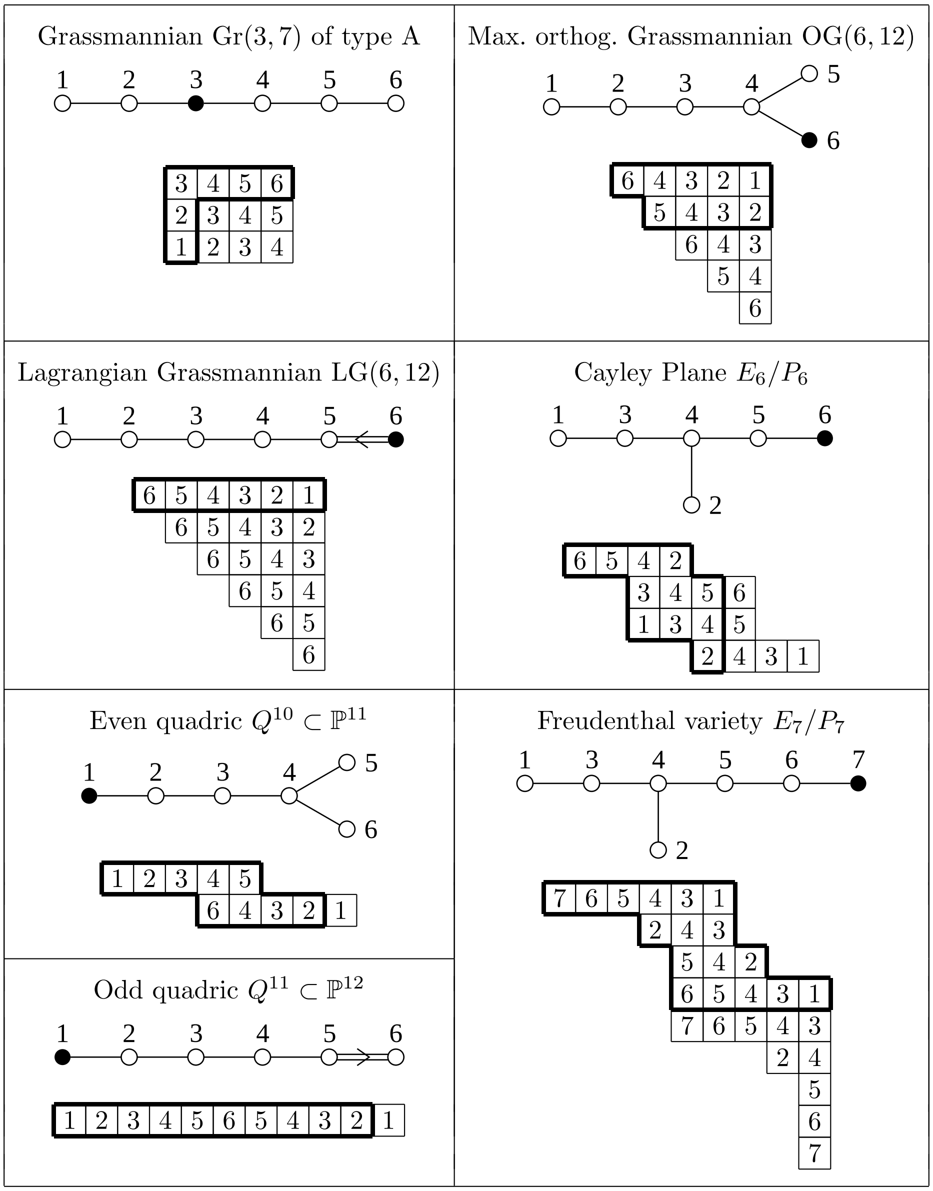
The element u/v depends only on the skew shape $I(u) \smallsetminus I(v)$. If $I(u) \smallsetminus I(v) = \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$ is any increasing ordering, then $u/v = s_{\delta(\alpha_\ell)} \cdots s_{\delta(\alpha_2)} s_{\delta(\alpha_1)}$ is a reduced expression for u/v. In the special case $v = 1$, every reduced expression for u can be obtained in this way. We will say that u/v is a *rook strip* if this element of W is a product of commuting simple reflections. Equivalently, no pair of roots in $I(u) \smallsetminus I(v)$ are comparable by the partial order $\leq$ on $\mathcal{P}_X$. We call u/v a *short rook strip* if it is a product of commuting reflections defined by short simple roots. Notice that if the root system Φ is simply laced, then all roots are long by convention, so u/v is a short rook strip if and only if $u = v$.

The following result holds for any (reduced and crystallographic) root system Φ.

Lemma 4.1. *Let Φ be a root system with associated Weyl group W, let $w \in W$, and let $\alpha' \lessdot \alpha$ be a covering relation in $I(w)$. Then $\alpha - \alpha' \in \Phi^+$.*

Proof. We can write $\alpha = \alpha' + \beta_1 + \cdots + \beta_k$, with $\beta_i \in \Phi^+$ and $\beta_i + \beta_j \notin \Phi$ for all i and j. By [GP20, Thm. 2.4] we may assume $\alpha' + \beta_1 + \cdots + \beta_{k-1} \in \Phi^+$, and [GP20, Thm. 2.5] implies that $\alpha' + \beta_k \in \Phi^+$. Using that $w.\alpha'$ and $w.\alpha$ are negative roots, it follows that $\alpha' + \beta_1 + \cdots + \beta_{k-1}$ or $\alpha' + \beta_k$ belongs to $I(w)$. In both cases, since $\alpha' \lessdot \alpha$ is a covering relation, we deduce that $k = 1$. $\square$

Table 1. Partially ordered sets $\mathcal{P}_X$ of cominuscule varieties with $I(z_1)$ highlighted (see Definition 5.2). In each case the partial order is given by $\alpha' \leq \alpha$ if and only if α' is weakly north-west of α.



Lemma 4.2. *Let $u, v \in W^X$ satisfy $v \leq u$.*

(a) *The action of v defines an order isomorphism $v : I(u) \smallsetminus I(v) \to I(u/v)$.*

(b) *Let $\alpha \in \Phi$. Then α is a minimal box of $\mathcal{P}_X \smallsetminus I(v)$ if and only if $\alpha \geq \gamma$ and $v.\alpha \in \Delta$. In this case we have $v.\alpha = \delta(\alpha)$.*

Proof. We have $v.(I(u) \smallsetminus I(v)) \subset I(u/v)$ by definition of the inversion sets, and both sets have the same cardinality. The map $\alpha \mapsto v.\alpha$ is order-preserving on all of $\mathcal{P}_X$ since $v.\beta > 0$ for all $\beta \in \Delta \smallsetminus \{\gamma\}$. To prove that $v^{-1} : I(u/v) \to I(u) \smallsetminus I(v)$ also preserves the order, let $\alpha' \lessdot \alpha$ be a covering relation in $I(u/v)$. Then $v^{-1}.\alpha - v^{-1}.\alpha' = v^{-1}(\alpha - \alpha') \in \Phi$ is a root by Lemma 4.1, so $v^{-1}.\alpha'$ and $v^{-1}.\alpha$ are comparable elements in $\mathcal{P}_X$. Since v is order-preserving and $\alpha' < \alpha$, we deduce that $v^{-1}.\alpha' < v^{-1}.\alpha$. This proves (a). If α is a minimal box of $\mathcal{P}_X \smallsetminus I(v)$, then $\lambda(\alpha) \subset I(v)$, and any box $\alpha' \in I(v) \smallsetminus \lambda(\alpha)$ is incomparable to α, hence $s_{\alpha'}.\alpha = \alpha$ by [BS16, Lemma 2.2]. This implies that $v.\alpha = w_{\lambda(\alpha)}.\alpha = \delta(\alpha) \in \Delta$. On the other hand, the conditions $\alpha \geq \gamma$ and $v.\alpha \in \Delta$ imply that $\alpha \in \mathcal{P}_X \smallsetminus I(v)$. Let $\alpha' \in \mathcal{P}_X \smallsetminus I(v)$ be any minimal box such that $\alpha' \leq \alpha$. Since $0 < v.\alpha' \leq v.\alpha \in \Delta$, we must have $\alpha' = \alpha$, which proves (b). □

Remark 4.3. (a) If $\alpha_1 \neq \alpha_2 \in \mathcal{P}_X$ are incomparable boxes, then Lemma 4.2(b) implies that $\delta(\alpha_1) = w_\lambda.\alpha_1 \neq w_\lambda.\alpha_2 = \delta(\alpha_2)$ where $\lambda = \lambda(\alpha_1) \cup \lambda(\alpha_2)$. In addition, we have $(\delta(\alpha_1), \delta(\alpha_2)) = (\alpha_1, \alpha_2) = 0$ by [BS16, Lemma 2.2].

(b) If $\alpha_1 \lessdot \alpha_2$ is a covering relation in $\mathcal{P}_X$, then $(\alpha_1, \alpha_2) > 0$. In fact, since α_1 is a maximal box of $\lambda(\alpha_2)$, we obtain $(\alpha_1, \alpha_2) = (w_{\lambda(\alpha_2)}.\alpha_1, w_{\lambda(\alpha_2)}.\alpha_2) = (-\delta(\alpha_1), \delta(\alpha_2)) \geq 0$. If $(\alpha_1, \alpha_2) = 0$, then $s_{\delta(\alpha_1)}$ and $s_{\delta(\alpha_2)}$ are commuting simple reflections. Set $\lambda = \lambda(\alpha_2) \smallsetminus \{\alpha_1\}$. Since $s_{\delta(\alpha_1)} s_{\delta(\alpha_2)} w_\lambda \in W^X$ is a reduced product, it follows that $s_{\delta(\alpha_2)} w_\lambda \in W^X$ and $\lambda \subsetneq I(s_{\delta(\alpha_2)} w_\lambda) \subsetneq \lambda \cup \{\alpha_1, \alpha_2\}$. But then $I(s_{\delta(\alpha_2)} w_\lambda) = \lambda \cup \{\alpha_1\}$, a contradiction.

Lemma 4.4. (a) *The action $w_{0,X} : \mathcal{P}_X \to \mathcal{P}_X$ is an order-reversing involution, and $\delta(w_{0,X}.\alpha) = -w_0.\delta(\alpha)$ is the Cartan involution of $\delta(\alpha)$ for each $\alpha \in \mathcal{P}_X$.*
(b) *The Poincaré dual element of $u \in W^X$ is determined by $I(u^\vee) = I(w_0 u w_{0,X}) = \mathcal{P}_X \smallsetminus w_{0,X}.I(u)$.*

Proof. The action of $w_{0,X}$ is an order-reversing involution on $\mathcal{P}_X$ since it does not change the coefficient of γ, and $w_{0,X}.\beta < 0$ for each $\beta \in \Delta \smallsetminus \{\gamma\}$. For $u \in W^X$ and $\alpha \in \Phi^+$ we deduce that $w_0 u w_{0,X}.\alpha < 0$ holds if and only if $w_{0,X}.\alpha \in \mathcal{P}_X \smallsetminus I(u)$, which proves (b). Since $w_{0,X}.\alpha$ is a maximal box of $\mathcal{P}_X \smallsetminus w_{0,X}.\lambda(\alpha)$, it follows from Lemma 4.2(b) that $\delta(w_{0,X}.\alpha) = -w^\vee_{\lambda(\alpha)}.(w_{0,X}.\alpha) = -w_0 w_{\lambda(\alpha)}.\alpha = -w_0.\delta(\alpha)$, which completes the proof of (a). □

The Bruhat order on W^X is a distributive lattice, with join and meet operations defined by $I(u \cup v) = I(u) \cup I(v)$ and $I(u \cap v) = I(u) \cap I(v)$. Notice that $(u \cup v)/v = u/(u \cap v)$. It follows that $u/(u \cap v)$ and $v/(u \cap v)$ are commuting elements of W, as their product in either order is $(u \cup v)/(u \cap v)$. We also have $(u \cap v)^\vee = u^\vee \cup v^\vee$ and $(u \cup v)^\vee = u^\vee \cap v^\vee$.

Proposition 4.5. *Let $u, v \in W^X$. The Richardson variety $X_u^{u \cap v}$ is a semi-transversal intersection of X_u and X^v in X.*

Proof. Set $z = v^\vee = w_0 v w_{0,X}$ and $\kappa = u \cdot w_{0,X} \cdot z^{-1}$. It follows from Theorem 2.4 that $X_u \cap w_0.X_{w_0 \kappa z} = X_u^{\kappa z w_{0,X}}$ is a semi-transversal intersection of X_u and X^v. We must therefore show that, for all $u, z \in W^X$ we have

$$(u \cdot w_{0,X} \cdot z^{-1}) z w_{0,X} = u \cap z^\vee \,.$$

Set $u' = u \cap z^\vee$. Since $z^\vee w_{0,X} z^{-1} = w_0$ is a reduced product and $u' \leq_L z^\vee$, it follows that $u' \cdot w_{0,X} \cdot z^{-1} = u' w_{0,X} z^{-1}$ is also a reduced product, so we obtain $(u' \cdot w_{0,X} \cdot z^{-1}) z w_{0,X} = u' = u \cap z^\vee$. Let $\alpha \in I(u) \smallsetminus I(z^\vee)$ and set $\beta = \delta(\alpha)$. It suffices to show that $s_\beta \cdot (u' w_{0,X} z^{-1}) = u' w_{0,X} z^{-1}$. We may assume that $s_\beta u' w_{0,X} >$

$u'w_{0,X}$, in which case $s_\beta u' \in W^X$. We then have $I(s_\beta u') = I(u') \cup \{\alpha'\}$ for some root α' with $\delta(\alpha') = u'.\alpha' = \beta$. Since $s_\beta u' \not\leq z^\vee$ we have $\alpha' \notin I(z^\vee)$. It follows that $(u'w_{0,X}z^{-1})^{-1}.\beta = zw_{0,X}.\alpha' = w_0 z^\vee.\alpha' < 0$, and hence $s_\beta \cdot (u'w_{0,X}z^{-1}) = u'w_{0,X}z^{-1}$, as required. □

4.2. **Cohomology of negative line bundles on Richardson varieties.** Let $K_T(X)$ denote the T-equivariant K-theory ring of X, see e.g. [BCMP18a, §2.1] and the references therein. Pullback along the structure morphism $X \to \{\text{point}\}$ makes $K_T(X)$ an algebra over the ring $K_T(\text{point})$ of virtual representations of T. Let $\chi_X : K_T(X) \to K_T(\text{point})$ be the pushforward along the structure morphism. The Schubert classes in $K_T(X)$ are denoted by $\mathcal{O}^v = [\mathcal{O}_{X^v}]$ and $\mathcal{O}_u = [\mathcal{O}_{X_u}]$. Let $J \subset \mathcal{O}_X$ be the ideal sheaf of the Schubert divisor X^{s_γ}. Then J^{-1} is the ample generator of $\text{Pic}(X)$. In addition, J inherits a structure of T-equivariant line bundle from $\mathcal{O}_X$. An equivalent definition is $J = (G \times^P \mathbb{C}_{\omega_\gamma}) \otimes \mathbb{C}_{-\omega_\gamma}$, see [BCMP18a, §4.1]. Let J_v denote the restriction of J to the T-fixed point $v.P_X$.

Given any integer $p \in \mathbb{Z}$, we set $p' = p - \frac{1}{2}$. The half-integers $\frac{1}{2}\mathbb{Z}$ is then the set of primed and unprimed integers.

Definition 4.6. Let $\mathcal{S} \subset \mathcal{P}_X$ be a skew shape. A *decreasing primed tableau* of shape $\mathcal{S}$ is a labeling $\mathcal{T} : \mathcal{S} \to \frac{1}{2}\mathbb{Z}$ such that (i) $\alpha' < \alpha$ in $\mathcal{S}$ implies $\mathcal{T}(\alpha') > \mathcal{T}(\alpha)$, and (ii) $\mathcal{T}(\alpha) \in \mathbb{Z}$ for all long boxes $\alpha \in \mathcal{S}$.

Given any labeling $\widehat{\mathcal{T}} : \mathcal{P}_X \to \frac{1}{2}\mathbb{Z}$ of $\mathcal{P}_X$, such that $\widehat{\mathcal{T}}(\alpha) \in \mathbb{Z}$ for each long box $\alpha \in \mathcal{P}_X$, define the weight

$$\lambda(\widehat{\mathcal{T}}) = \sum_{\alpha \in \mathcal{P}_X} \mathcal{T}(\alpha)\,(\omega_\gamma, \alpha^\vee)\,\delta(\alpha)\,.$$

Here ω_γ denotes the fundamental weight corresponding to the cominuscule simple root γ. Notice that, if $\widehat{\mathcal{T}}(\alpha)$ is not an integer, then α is a short root, hence $(\omega_\gamma, \alpha^\vee) = 2$.

Let $u, v \in W^X$ satisfy $v \leq u$, let $m \in \mathbb{Z}$, and let $a \in \frac{1}{2}\mathbb{Z}$. Given a decreasing primed tableau $\mathcal{T}$ of shape $I(u) \smallsetminus I(v)$, let $\mathcal{T}[m] : \mathcal{P}_X \to \frac{1}{2}\mathbb{Z}$ denote the extension of $\mathcal{T}$ that maps all boxes of $I(v)$ to m and maps all boxes of $\mathcal{P}_X \smallsetminus I(u)$ to 0, see Example 4.11. Using this notation, we define a representation of T by

$$C^u_{v,[a,m)} = \bigoplus_{\mathcal{T}} \mathbb{C}_{-\lambda(\mathcal{T}[m])}\,,$$

where the sum is over all decreasing primed tableaux $\mathcal{T}$ of shape $I(u) \smallsetminus I(v)$ with labels in $[a, m)$, i.e. $a \leq \mathcal{T}(\alpha) < m$ for all $\alpha \in I(u) \smallsetminus I(v)$.

Lemma 4.7. *Let $u, v \in W^X$, $m, p \in \mathbb{Z}$ and $a \in \frac{1}{2}\mathbb{Z}$, and assume that $v \leq u$, $a \leq m$, and $p \geq 0$. Then*

$$C^u_{v,[a,m+p)} \cong \bigoplus_{w \in W^X :\, v \leq w \leq u} C^w_{v,[0,p)} \otimes_{\mathbb{C}} C^u_{w,[a,m)}\,.$$

Proof. Given a decreasing primed tableau $\mathcal{T}$ of shape $I(u) \smallsetminus I(v)$ with labels in $[a, m+p)$, let $\mathcal{T}''$ be the tableau consisting of the boxes with labels smaller than m, and let $\mathcal{T}'$ be the tableau obtained by subtracting m from all boxes with labels greater than or equal to m. Then $\mathcal{T}'$ has shape $I(w) \smallsetminus I(v)$ and $\mathcal{T}''$ has shape $I(u) \smallsetminus I(w)$ for a unique element $w \in W^X$ with $v \leq w \leq u$, $\mathcal{T}'$ has labels in $[0, p)$, $\mathcal{T}''$ has labels in $[a, m)$, and we have $\mathcal{T}[m+p] = \mathcal{T}'[p] + \mathcal{T}''[m]$ with pointwise addition. The assumption $a \leq m$ ensures that the assignment $\mathcal{T} \mapsto (\mathcal{T}', \mathcal{T}'')$ has a well defined inverse map. The lemma follows from this. $\square$

The following identities generalize Theorems 3.6 and 3.7 from [BCMP18a]. A more general Chevalley formula that holds in the K-theory of arbitrary flag varieties was proved in [LP07, Thm. 13.1].

Proposition 4.8. (a) *For $v \in W^X$ and $m \geq 0$ we have in $K_T(X)$ that*

$$[J]^m \cdot \mathcal{O}^v \;=\; \sum_{u \in W^X :\, v \leq u} (-1)^{\ell(u/v)}\, [C^u_{v,[0,m)}]\, \mathcal{O}^u \,.$$

(b) *For $v \leq u$ in W^X and $m \geq 1$ we have in $K_T(\text{point})$ that*

$$\chi(X^v_u, J^m) \;=\; (-1)^{\ell(u/v)}\, [C^u_{v,[\frac{1}{2},m)}]\,.$$

Proof. Part (a) is clear for $m = 0$ and is equivalent to [BCMP18a, Thm. 3.6] for $m = 1$. For $m \geq 2$ it follows by induction using Lemma 4.7. Let $\mathcal{I}^w \in K_T(X)$ be dual to $\mathcal{O}_w$, i.e. $\chi_X(\mathcal{O}_u \cdot \mathcal{I}^w) = \delta_{u,w}$ for $u \in W^X$. By [BCMP18a, Lemma 3.5] we have $\mathcal{I}^w = \sum_u (-1)^{\ell(u/w)} \mathcal{O}^u$, the sum over all $u \geq w$ for which u/w is a rook strip. Using that $C^u_{v,[0,m)} = \bigoplus_w C^w_{v,[\frac{1}{2},m)}$, with the sum over all $w \in W^X$ for which $v \leq w \leq u$ and u/w is a rook strip, it follows that part (a) for $m \geq 1$ is equivalent to the identity

$$[J]^m \cdot \mathcal{O}^v \;=\; \sum_{w \in W^X : v \leq w} (-1)^{\ell(w/v)}\, [C^w_{v,[\frac{1}{2},m)}]\, \mathcal{I}^w \,.$$

Part (b) follows by multiplying both sides by $\mathcal{O}_u$ and applying χ_X. $\square$

Theorem 4.9. *Let $u, v \in W^X$ satisfy $v \leq u$ and let $m \geq 1$. Then $H^i(X^v_u, J^m) = 0$ for all $0 \leq i < \dim(X^v_u) = \ell(u/v)$. Moreover, we have $H^{\ell(u/v)}(X^v_u, J^m) \cong C^u_{v,[\frac{1}{2},m)}$ as representations of T.*

Proof. To prove the vanishing of cohomology groups, we may assume that $X = G/P_X$ is defined over an algebraically closed field of positive characteristic [BK05, §1.6]. Then [BK05, Thm. 2.3.1] together with [BK05, Lemma 1.1.8] applied to the projection $G/B \to X$ shows that X^v_u is Frobenius split. Since J^{-1} is ample and X^v_u is Cohen-Macaulay and irreducible, the Kodaira vanishing theorem for split varieties [BK05, Thm. 1.2.9] implies that $H^i(X^v_u, J^m) = 0$ for $i < \dim(X^v_u)$. We therefore have $\chi(X^v_u, J^m) = (-1)^{\ell(u/v)} [H^{\ell(u/v)}(X^v_u, J^m)]$, so the result follows from Proposition 4.8. $\square$

Remark 4.10. Using standard monomial theory, it is possible to compute the cohomology groups of the restriction of any ample line bundle on G/P to a Richardson variety; see [BL03, Thm. 3] or [LL03, Thm. 20]. However, we have not seen the computation of the (top) cohomology of negative line bundles in the literature, and this cannot be deduced using Serre duality since the canonical sheaf of a Richardson variety is not a line bundle in general.

Example 4.11. Let $X = \mathrm{LG}(3,6) = C_3/P_3$ be the Lagrangian Grassmannian of maximal isotropic subspaces in a complex symplectic vector space of dimension 6. Let $\Delta = \{\beta_1, \beta_2, \beta_3\}$ be the set of simple roots, where $\gamma = \beta_3$ is the long root. The labeling $\delta : \mathcal{P}_X \to \Delta$ is given by the following diagram, where the upper-left box represents γ and the bottom-right box represents the highest root of Φ.

β_3	β_2	β_1
	β_3	β_2
		β_3

Set $v = s_2 s_3$ and $u = s_2 s_3 s_1 s_2 s_3$. Then we have

$$H^3(X_u^v, J^{\otimes 2}) \;\cong\; C^u_{v,[\frac{1}{2},2)} = \mathbb{C}_{-2\beta_1-5\beta_2-3\beta_3} \oplus \mathbb{C}_{-3\beta_1-5\beta_2-3\beta_3}\,.$$

The extensions $\mathcal{T}[2]$ of the decreasing primed tableaux $\mathcal{T}$ corresponding to the summands are displayed below. The coefficient of the simple root β_i in the weight obtained from of each tableau is the (negative) sum of the half-integers in the i-th diagonal, multiplied by 2 if β_i is short.

2	2	1
	1	1′
		0

2	2	2′
	1	1′
		0

An algebraic variety D is called *cohomologically trivial* if $H^0(D, \mathcal{O}_D) = \mathbb{C}$ and $H^i(D, \mathcal{O}_D) = 0$ for $i > 0$.

Corollary 4.12. *Let $D \subset X_u^v$ be an effective Cartier divisor of class $[D] = m[X^{s_\gamma}]|_{X_u^v}$. Then D is cohomologically trivial if and only if there are no decreasing primed tableaux of shape $I(u) \smallsetminus I(v)$ with labels in $[\frac{1}{2}, m) \cap \frac{1}{2}\mathbb{Z}$.*

Proof. The Richardson variety X_u^v is cohomologically trivial, as it is rational with rational singularities. The long exact sequence of cohomology groups derived from $0 \to J^m|_{X_u^v} \to \mathcal{O}_{X_u^v} \to \mathcal{O}_D \to 0$ then shows that D is cohomologically trivial if and only if $H^i(X_u^v, J^m) = 0$ for all i. The result therefore follows from Theorem 4.9. □

Example 4.13. Let X_u^v be a Richardson variety of positive dimension and let $D \subset X_u^v$ be a Cartier divisor. If $[D] = [X^{s_\gamma}]|_{X_v^u}$, then D is cohomologically trivial if and only if u/v is not a short rook strip. In particular, D is cohomologically trivial if X is minuscule. If $[D] = 2[X^{s_\gamma}]|_{X_u^v}$ and X is minuscule, then D is cohomologically trivial if and only if u/v is not a rook strip.

If $X = X_1 \times \cdots \times X_k$ is a product of cominuscule flag varieties, then the Schubert varieties in X are given by sequences $(\lambda_1, \dots, \lambda_k)$ of order ideals $\lambda_i \subset \mathcal{P}_{X_i}$. Such a sequence can be identified with an order ideal in the disjoint union $\mathcal{P}_X = \mathcal{P}_{X_1} \coprod \cdots \coprod \mathcal{P}_{X_k}$. We will consider a point as a product of (zero) cominuscule varieties with associated set $\mathcal{P}_{\{\text{point}\}} = \emptyset$. The results in this section hold for products of cominuscule varieties with minor modifications. In Lemma 4.2(b), the condition $\alpha \geq \gamma$ can be replaced with $\alpha \in \mathcal{P}_X$. In the results of Section 4.2, J should be the ideal sheaf of the union of the Schubert divisors $X^{s_{\gamma_i}}$ for $1 \leq i \leq k$.

5. The quantum to classical principle

5.1. **Introduction.** The quantum to classical principle allows Gromov-Witten invariants of certain flag varieties to be computed as classical intersection numbers on related flag varieties. The goal in this section is to derive the quantum-to-classical theorem with as little type-by-type checking as possible. In addition we will develop the associated combinatorics and geometry in order to support the main results of this paper. We will restrict our discussion to Gromov-Witten invariants of cominuscule flag varieties of small degrees. Here a degree d is considered *small* if q^d occurs in a product of two Schubert classes in the small quantum cohomology ring $\mathrm{QH}(X)$. Equivalently, d is less than or equal to the *diameter* $d_X(2)$ defined in Section 5.2. Our main references include [Buc03, BKT03, CMP08, BKT09, BM11, CP11].

The first version of the quantum-to-classical theorem applied to the enumerative (cohomological) Gromov-Witten invariants of classical Grassmannians. Let $X = \mathrm{Gr}(m, n)$ be the Grassmannians of m-dimensional vector subspaces of $\mathbb{C}^n$. A rational curve $C \subset X$ has a *kernel* and a *span* defined by [Buc03]

$$\mathrm{Ker}(C) = \bigcap_{V \in C} V \qquad ; \qquad \mathrm{Span}(C) = \sum_{V \in C} V\,.$$

If C is a general curve of small degree d, then one can show that $\dim \mathrm{Ker}(C) = m-d$ and $\dim \mathrm{Span}(C) = m + d$, which means that $(\mathrm{Ker}(C), \mathrm{Span}(C))$ is a point in the two-step flag variety $Y_d = \mathrm{Fl}(m - d, m + d; n)$. Given three Schubert varieties $\Omega_1, \Omega_2, \Omega_3 \subset X$ in general position, the Gromov-Witten invariant $\langle[\Omega_1], [\Omega_2], [\Omega_3]\rangle_d$ is the number of rational curves of degree d meeting these Schubert varieties (assuming that this number is finite). The quantum-to-classical theorem states that the map $C \mapsto (\mathrm{Ker}(C), \mathrm{Span}(C))$ gives a bijection between the counted curves and the intersection of three Schubert varieties in Y_d. As a consequence, the Gromov-Witten invariant $\langle[\Omega_1], [\Omega_2], [\Omega_3]\rangle_d$ is equal to a classical Schubert structure constant of $H^*(Y_d)$ [BKT03].

Subsequent work [CMP08] demonstrated that the quantum-to-classical theorem can be understood in a type-independent way if the points (K, S) of Y_d are replaced with the corresponding subvarieties of X defined by $\mathrm{Gr}(d, S/K) = \{V \in X \mid K \subset V \subset S\}$. Indeed, these subvarieties of X are non-singular Schubert varieties, and also cominuscule flag varieties themselves. They can also be described as unions of rational curves of degree d that pass through two given points in X. Such Schubert varieties will be called *primitive* cominuscule varieties in this paper, see Section 5.4.

The quantum-to-classical theorem was extended to non-enumerative (equivariant and K-theoretic) Gromov-Witten invariants in [BM11] by showing that the moduli space $\overline{\mathcal{M}}_{0,3}(X, d)$ of stable maps to X is birational to the space $\{(K, S, V_1, V_2, V_3)\}$ of kernel-span pairs $(K, S) \in Y_d$ together with 3 additional points $V_i \in \mathrm{Gr}(d, S/K)$. Indeed, given a general 5-tuple of this type, there exists a unique rational curve $C \subset \mathrm{Gr}(d, S/K) \subset X$ of degree d which contains the three points V_1, V_2, V_3.

While we only discuss Gromov-Witten invariants of small degrees $d \leq d_X(2)$, the definition of the quantum K-theory ring $\mathrm{QK}(X)$ also depends on Gromov-Witten invariants of higher degrees. Such Gromov-Witten invariants can be computed with similar methods, granted that certain Gromov-Witten varieties of large degrees are rational; this was proved in [BM11] for Grassmannians of type A and in [CP11] for cominuscule varieties of other Lie types (see also [BCMP13, Remark 3.4]).

5.2. **Curve neighborhoods.** Let $X = G/P_X$ be a cominuscule flag variety defined over $\mathbb{C}$. For any non-negative integer $d \geq 0$ we let $M_d = \overline{\mathcal{M}}_{0,3}(X, d)$ denote the Kontsevich moduli space of 3-pointed stable maps to X of degree d and genus zero [FP97]. The evaluation map $\mathrm{ev}_i : M_d \to X$, defined for $1 \leq i \leq 3$, sends a stable map to the image of the i-th marked point in its domain. Given classes $\Omega_1, \Omega_2, \Omega_3 \in H^*(X; \mathbb{Z})$, the corresponding cohomological Gromov-Witten invariant of X of degree d is defined by

$$\langle \Omega_1, \Omega_2, \Omega_3 \rangle_d = \int_{M_d} \mathrm{ev}_1^*(\Omega_1) \cdot \mathrm{ev}_2^*(\Omega_2) \cdot \mathrm{ev}_3^*(\Omega_3)\,.$$

More generally, three K-theory classes $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in K_T(X)$ define the K-theoretic Gromov-Witten invariant

$$I_d(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = \chi_{M_d}(\mathrm{ev}_1^*(\mathcal{F}_1) \cdot \mathrm{ev}_2^*(\mathcal{F}_2) \cdot \mathrm{ev}_3^*(\mathcal{F}_3))\,.$$

Given subvarieties $\Omega_1, \Omega_2 \subset X$, let $M_d(\Omega_1, \Omega_2) = \mathrm{ev}_1^{-1}(\Omega_1) \cap \mathrm{ev}_2^{-1}(\Omega_2)$ denote the *Gromov-Witten variety* of stable maps that send the first two marked points to the given subvarieties. Let $\Gamma_d(\Omega_1, \Omega_2) = \mathrm{ev}_3(M_d(\Omega_1, \Omega_2))$ be the union of all stable curves of degree d in X that connect Ω_1 and Ω_2. We also consider the special cases $M_d(\Omega_1) = \mathrm{ev}_1^{-1}(\Omega_1)$ and $\Gamma_d(\Omega_1) = \mathrm{ev}_3(M_d(\Omega_1))$.

Define the *degree distance* $\mathrm{dist}(x, y)$ between two points $x, y \in X$ to be the smallest degree of a rational curve $C \subset X$ with $x, y \in C$. This is the minimal degree d for which $\Gamma_d(x, y) \neq \emptyset$. We need the following key result from [CMP08]. A type-independent proof based on properties satisfied by all flag manifolds was given in [BM15, §5.4].

Theorem 5.1. *Let $u \in W^X$. Then $\mathrm{dist}(1.P_X, u.P_X)$ is the number of occurrences of s_γ in any reduced expression for u.*

Equivalently, $\mathrm{dist}(1.P_X, u.P_X)$ is the number of boxes $\alpha \in I(u)$ with label $\delta(\alpha) = \gamma$. Define the *diameter* of X to be the integer $d_X(2) = \mathrm{dist}(1.P_X, w_0.P_X)$. The subset of boxes in $\mathcal{P}_X$ labeled by γ is totally ordered by Remark 4.3(a). We denote these boxes by

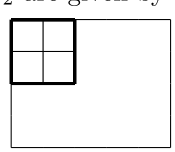
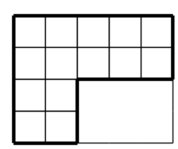
$$\delta^{-1}(\gamma) = \{\widetilde{\alpha}_1 < \widetilde{\alpha}_2 < \cdots < \widetilde{\alpha}_{d_X(2)}\}\,.$$

Definition 5.2. For $0 \leq d \leq d_X(2)$ we define elements κ_d and z_d in W^X by

$$I(\kappa_d) = \{\alpha \in \mathcal{P}_X \mid \alpha \leq \widetilde{\alpha}_d\} \quad \text{and} \quad I(z_d) = \{\alpha \in \mathcal{P}_X \mid \alpha \not\geq \widetilde{\alpha}_{d+1}\}\,.$$

We set $\kappa_0 = z_0 = 1$ and $z_{d_X(2)} = w_0^X$.

Example 5.3. Let $X = \mathrm{Gr}(4, 9)$ be the Grassmannian of 4-planes in $\mathbb{C}^9$. Then the elements κ_2 and z_2 are given by the following shapes:

$I(\kappa_2) =$  and $I(z_2) =$ .

The shapes $I(z_1)$ for a representative selection of cominuscule flag varieties are displayed in Table 1.

Lemma 5.4. *We have $\kappa_d^{-1} = \kappa_d$, $(z_d w_{0,X})^{-1} = z_d w_{0,X}$, and $z_d^\vee z_d = w_0^X$. In addition, $\Gamma_d(1.P_X) = X_{z_d}$.*

Proof. It follows from Theorem 5.1 that κ_d and $z_d w_{0,X}$ are the unique minimal and maximal elements of the set $\{w \in W \mid \mathrm{dist}(1.P_X, w.P_X) = d\}$. Since $\mathrm{dist}(1.P_X, w.P_X) = \mathrm{dist}(1.P_X, w^{-1}.P_X)$ for any element $w \in W$, we deduce that κ_d and $z_d w_{0,X}$ are self-inverse. We then obtain

$$z_d^\vee z_d = w_0 z_d w_{0,X} z_d = w_0 (z_d w_{0,X})^{-1} z_d = w_0 w_{0,X} = w_0^X\,.$$

The identity $\Gamma_d(1.P_X) = X_{z_d}$ follows from Theorem 5.1 since $\Gamma_d(1.P_X)$ is a Schubert variety by [BCMP13, Cor. 3.3(a)]. $\square$

Remark 5.5. We have $W_X \kappa_d W_X = \{u \in W \mid \operatorname{dist}(1.P_X, u.P_X) = d\}$.

Lemma 5.6. *The orbits of the diagonal action of G on $X \times X$ are given by $\mathring{\mathcal{Z}}_{d,2} = \{(x_1, x_2) \in X \times X \mid \operatorname{dist}(x_1, x_2) = d\}$, for $0 \leq d \leq d_X(2)$.*

Proof. Each set $\mathring{\mathcal{Z}}_{d,2}$ is stable under the action of G. Given $(x_1, x_2) \in \mathring{\mathcal{Z}}_{d,2}$, we can choose $g \in G$ such that $g.x_1 = 1.P_X$, and then choose $b \in B$ such that $bg.x_2 = u.P_X$ is a T-fixed point, with $u \in W^X$. Since $\operatorname{dist}(1.P_X, u.P_X) = d$, Theorem 5.1 implies that $u/\kappa_d \in W_X$. The lemma follows from this because $(u/\kappa_d)^{-1} bg.(x_1, x_2) = (1.P_X, \kappa_d.P_X)$. □

Lemma 5.7. *We have $(\alpha, \gamma^\vee) = 1$ for $\alpha \in I(z_1) \smallsetminus \{\gamma\}$, and $(\alpha, \gamma^\vee) = 0$ for $\alpha \in \mathcal{P}_X \smallsetminus I(z_1)$.*

Proof. Notice that $(\alpha, \gamma^\vee) \geq 0$ for every $\alpha \in \mathcal{P}_X$, since otherwise the coefficient of γ in the root $s_\gamma.\alpha = \alpha - (\alpha, \gamma^\vee)\gamma$ is at least 2. In addition, if $\alpha', \alpha \in \mathcal{P}_X$ satisfy $\alpha' \leq \alpha$, then $(\alpha', \gamma^\vee) \geq (\alpha, \gamma^\vee)$, as $\alpha - \alpha'$ is a non-negative linear combination of $\Delta \smallsetminus \{\gamma\}$. Finally, since γ is a long root, we have $(\alpha, \gamma^\vee) \leq 1$ for any root $\alpha \neq \gamma$. It is therefore enough to show that $(\alpha, \gamma^\vee) \neq 0$ for $\alpha \in I(z_1)$ and that $(\widetilde{\alpha}_2, \gamma^\vee) = 0$. If $\alpha \in I(z_1) \cup \{\widetilde{\alpha}_2\}$ is any root with $\alpha \neq \gamma$, then we have $\delta(\alpha) = ys_\gamma.\alpha = y.(\alpha - (\alpha, \gamma^\vee)\gamma) \in \Delta$ for some $y \in W_X$. Since the action of y does not change the coefficient of γ, we deduce that $(\alpha, \gamma^\vee) = 0$ if and only if $\delta(\alpha) = \gamma$, as required. □

Corollary 5.8. *We have $\int_{X_{s_\gamma}} c_1(T_X) = \ell(z_1) + 1$.*

Proof. By [FW04, Lemma 3.5] we have $\int_{X_{s_\gamma}} c_1(T_X) = \sum_{\alpha \in \mathcal{P}_X} (\alpha, \gamma^\vee)$, so the corollary follows from Lemma 5.7. □

Proposition 5.9. (a) *The map $z_d : \mathcal{P}_X \smallsetminus I(z_d) \to I(z_d^\vee)$ defined by $\alpha \mapsto z_d.\alpha$ is an order isomorphism, and $\delta(z_d.\alpha) = \delta(\alpha)$ for each $\alpha \in \mathcal{P}_X \smallsetminus I(z_d)$.*

(b) *The map $-\kappa_d : I(\kappa_d) \to I(\kappa_d)$ defined by $\alpha \mapsto -\kappa_d.\alpha$ is an order-reversing involution, and $\delta(-\kappa_d.\alpha) = \delta(\alpha)$ for each $\alpha \in I(\kappa_d)$.*

Proof. Since $z_d^\vee = w_0^X / z_d$ by Lemma 5.4, it follows from Lemma 4.2(a) that $z_d : \mathcal{P}_X \smallsetminus I(z_d) \to I(z_d^\vee)$ is an order-preserving bijection. Since $z_d^{-1} = w_{0,X} z_d w_{0,X}$, the inverse bijection is also order-preserving. Let $\alpha \in \mathcal{P}_X \smallsetminus I(z_d)$ and set $\lambda = \lambda(\alpha) \cup I(z_d)$. Since α is a minimal box of $\mathcal{P}_X \smallsetminus \lambda$, we have $\delta(\alpha) = w_\lambda.\alpha$. Write $\lambda = I(z_d) \coprod \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$, where the roots are listed in increasing order. Then $w_\lambda = z_d s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_\ell}$. Using that z_d is an order isomorphism, we obtain $\lambda(z_d.\alpha) = \{z_d.\alpha_1, z_d.\alpha_2, \ldots, z_d.\alpha_\ell\}$, with the roots listed in increasing order, hence

$$w_{\lambda(z_d.\alpha)} z_d = s_{z_d.\alpha_1} s_{z_d.\alpha_2} \cdots s_{z_d.\alpha_\ell} z_d = z_d s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_\ell} z_d^{-1} z_d = w_\lambda \,,$$

and $\delta(z_d.\alpha) = w_{\lambda(z_d.\alpha)}.(z_d.\alpha) = w_\lambda.\alpha = \delta(\alpha)$. This proves part (a).

For $\alpha \in I(\kappa_d)$ we have $-\kappa_d.\alpha \in \Phi^+$ and $\kappa_d.(-\kappa_d.\alpha) = -\alpha < 0$, hence $-\kappa_d.\alpha \in I(\kappa_d)$. The map $-\kappa_d : I(\kappa_d) \to I(\kappa_d)$ is order-reversing because $\kappa_d.\beta > 0$ for $\beta \in \Delta \smallsetminus \{\gamma\}$. Given any element $u \leq \kappa_d$ we have $I(u\kappa_d) = -\kappa_d.(I(\kappa_d) \smallsetminus I(u))$; in fact, the containment $\supseteq$ follows from the definition of inversion sets, and both sides have the same cardinality because $(u\kappa_d)^{-1} = \kappa_d/u$. Now choose $u \leq \kappa_d$ such that α is a minimal box of $I(\kappa_d) \smallsetminus I(u)$. Then $-\kappa_d.\alpha$ is a maximal box of $I(u\kappa_d)$, so it follows from Lemma 4.2(b) that $\delta(-\kappa_d.\alpha) = -u\kappa_d.(-\kappa_d.\alpha) = u.\alpha = \delta(\alpha)$. This proves part (b). □

Proposition 5.10. *The element $z_1 s_\gamma = z_1/\kappa_1$ permutes $\mathcal{P}_X$ and satisfies $w_{0,X}(z_1 s_\gamma) w_{0,X} = (z_1 s_\gamma)^{-1}$. The action of $z_1 s_\gamma$ on $\mathcal{P}_X$ has the following properties.*
(a) *$z_1 s_\gamma : \mathcal{P}_X \smallsetminus I(z_1) \to I(z_1^\vee)$ is an order isomorphism, and $\delta(z_1 s_\gamma.\alpha) = \delta(\alpha)$ for all $\alpha \in \mathcal{P}_X \smallsetminus I(z_1)$.*
(b) *$z_1 s_\gamma : I(z_1) \smallsetminus \{\gamma\} \to w_{0,X}.(I(z_1) \smallsetminus \{\gamma\})$ is an order isomorphism.*
(c) *We have $z_1 s_\gamma.\widetilde{\alpha}_d = \widetilde{\alpha}_{d-1}$ for $2 \leq d \leq d_X(2)$, and $z_1 s_\gamma.\gamma = w_{0,X}.\gamma$ is the highest root of Φ^+.*

Proof. Since $z_1 s_\gamma \in W_X$ we have $z_1 s_\gamma.\mathcal{P}_X = \mathcal{P}_X$, and Lemma 5.7 implies

$$z_1 s_\gamma.\alpha = \begin{cases} z_1.\alpha & \text{if } \alpha \in \mathcal{P}_X \smallsetminus I(z_1); \\ z_1.\alpha - z_1.\gamma & \text{if } \alpha \in I(z_1) \smallsetminus \{\gamma\}; \\ -z_1.\gamma & \text{if } \alpha = \gamma \end{cases}$$

for any $\alpha \in \mathcal{P}_X$. Part (a) therefore follows from Proposition 5.9(a). In particular, we have $z_1 s_\gamma.\widetilde{\alpha}_d = \widetilde{\alpha}_{d-1}$ for $d \geq 2$. Since $z_1 s_\gamma.\alpha < z_1 s_\gamma.\gamma$ for every $\alpha \in I(z_1) \smallsetminus \{\gamma\}$, we deduce that $z_1 s_\gamma.\gamma = w_{0,X}.\gamma$ is the maximal box of $\mathcal{P}_X$, which proves part (c). Using Lemma 5.4 we also obtain $z_1 s_\gamma w_{0,X} = z_1 w_{0,X} s_{w_{0,X}.\gamma} = w_{0,X} z_1^{-1} s_{z_1.\gamma} = w_{0,X}(z_1 s_\gamma)^{-1}$. Finally, $z_1 s_\gamma$ is order preserving on $I(z_1) \smallsetminus \{\gamma\}$ because $z_1.\beta > 0$ for each $\beta \in \Delta \smallsetminus \{\gamma\}$, and the identity $(z_1 s_\gamma)^{-1} = w_{0,X}(z_1 s_\gamma) w_{0,X}$ shows that the inverse map is also order preserving. This proves part (b). □

Corollary 5.11. *We have $(z_1 s_\gamma)^d.\alpha = z_d.\alpha$ for each $\alpha \in \mathcal{P}_X \smallsetminus I(z_d)$.*

Proof. Noting that $\mathcal{P}_X \smallsetminus I(z_d) = \{\alpha \in \mathcal{P}_X \mid \widetilde{\alpha}_{d+1} \leq \alpha \leq w_{0,X}.\widetilde{\alpha}_1\}$ and $I(z_d^\vee) = \{\alpha \in \mathcal{P}_X \mid \widetilde{\alpha}_1 \leq \alpha \leq w_{0,X}.\widetilde{\alpha}_{d+1}\}$, it follows from Proposition 5.10 that $(z_1 s_\gamma)^d : \mathcal{P}_X \smallsetminus I(z_d) \to I(z_d^\vee)$ is an order isomorphism that preserves the labeling δ. The result therefore follows from Proposition 5.9(a) and Remark 4.3(a). □

Remark 5.12. We will show in Corollary 7.19 that $z_d/\kappa_d = (z_1/\kappa_1)^d = (z_1 s_\gamma)^d$. Together with Corollary 5.11, this implies that $\kappa_d.\alpha = \alpha$ for all $\alpha \in \mathcal{P}_X \smallsetminus I(z_d)$. Proposition 6.2 implies that $z_d/\kappa_d : I(z_d) \smallsetminus I(\kappa_d) \to I(\kappa_d^\vee) \smallsetminus I(z_d^\vee)$ is an order isomorphism, which generalizes Proposition 5.10(b). Using Proposition 5.9(b), it follows that $z_d/\kappa_d : I(\kappa_d) \to w_{0,X}.I(\kappa_d)$ is an order isomorphism. These remarks will not be used in the following.

For $1 \leq d \leq d_X(2)$ we define $\mathcal{S}_d = (I(z_d) \smallsetminus I(z_{d-1})) \cup (I(\kappa_d) \smallsetminus I(\kappa_{d-1}))$.

Proposition 5.13. *We have $\mathcal{S}_d = \{\alpha \in \mathcal{P}_X \mid (\alpha, \widetilde{\alpha}_d) > 0\}$ for $1 \leq d \leq d_X(2)$, and $z_1 s_\gamma.\mathcal{S}_d = \mathcal{S}_{d-1}$ for $2 \leq d \leq d_X(2)$.*

Proof. The second identity follows from the first identity together with Proposition 5.10(c). We must therefore show that $(\alpha, \widetilde{\alpha}_d) > 0$ if and only if $\alpha \in \mathcal{S}_d$, for any $\alpha \in \mathcal{P}_X$. If α and $\widetilde{\alpha}_d$ are not comparable in the partial order on $\mathcal{P}_X$, then $(\alpha, \widetilde{\alpha}_d) = 0$ and $\alpha \notin \mathcal{S}_d$. If $\alpha \geq \widetilde{\alpha}_d$, then Proposition 5.9(a) shows that $z_{d-1}.\widetilde{\alpha}_d = \gamma$, and also that $\alpha \in \mathcal{S}_d$ if and only if $z_{d-1}.\alpha \in I(z_1)$, so the claim follows from Lemma 5.7, noting that $(\alpha, \widetilde{\alpha}_d) = (z_{d-1}.\alpha, \gamma)$. Finally, if $\alpha \leq \widetilde{\alpha}_d$, then Proposition 5.9(b) shows that $-\kappa_d.\widetilde{\alpha}_d = \gamma$, and also that $\alpha \in \mathcal{S}_d$ if and only if $-\kappa_d.\alpha \in I(z_1)$, so the claim again follows from Lemma 5.7, this time noting that $(\alpha, \widetilde{\alpha}_d) = (-\kappa_d.\alpha, \gamma)$. □

Corollary 5.14. *Let $0 \leq d \leq d_X(2)$.*
(a) *We have $\dim M_d = \dim(X) + \ell(z_d) + \ell(\kappa_d)$.*
(b) *The variety $M_d(1.P_X, \kappa_d.P_X)$ is irreducible of dimension $\ell(\kappa_d)$.*

Proof. Since $\ell(z_d)+\ell(\kappa_d)-\ell(z_{d-1})-\ell(\kappa_{d-1}) = \#\mathcal{S}_d+1$ for $d\geq 1$, it follows from Proposition 5.13 and Corollary 5.8 that $\ell(z_d)+\ell(\kappa_d) = d\int_{X_\gamma} c_1(T_X)$. This proves part (a). Since $\mathrm{ev}_1 : M_d \to X$ is a locally trivial fibration, it follows that $M_d(1.P_X)$ is irreducible of dimension $\ell(z_d)+\ell(\kappa_d)$. Part (b) follows from this, using that $\mathrm{ev}_2 : M_d(1.P_X) \to X_{z_d}$ is a locally trivial fibration over a dense open subset of $X_{z_d} = \overline{P_X\kappa_d.P_X}$ that contains $\kappa_d.P_X$. □

5.3. **Incidence varieties.** Given any flag variety $Y = G/P_Y$, the *incidence variety* of X and Y is the flag variety $Z = G/P_Z$ defined by $P_Z = P_X \cap P_Y$. Let $p : Z \to X$ and $q : Z \to Y$ be the projections and set $F = p^{-1}(1.P_X) = P_X/P_Z$ and $\Gamma = q^{-1}(1.P_Y) = P_Y/P_Z$. For example, if $X = \mathrm{Gr}(m,n)$ and $Y = \mathrm{Fl}(m-d,m+d;n)$, then $Z = \mathrm{Fl}(m-d,m,m+d;n)$, $\Gamma = \mathrm{Gr}(d,2d)$, and $F = \mathrm{Gr}(m-d,m)\times \mathrm{Gr}(d,n-m)$. For $\omega \in Y$ we write $\Gamma_\omega = p(q^{-1}(\omega)) \subset X$. We identify Z with the subvariety $\{(\omega,x) \in Y \times X \mid x \in \Gamma_\omega\}$ of $Y \times X$. The restricted maps $p : \Gamma \to p(\Gamma)$ and $q : F \to q(F)$ are isomorphisms, hence $p(\Gamma) \subset X$ and $q(F) \subset Y$ are non-singular Schubert varieties. More precisely we have $F = Z_{w^Z_{0,X}}$ and $q(F) = Y_{w^Z_{0,X}}$, and also $\Gamma = Z_\kappa$ and $p(\Gamma) = X_\kappa$, where $\kappa = w^Z_{0,Y}$. In our applications of this construction we have $\kappa = \kappa_d$ for some degree d, see Corollary 5.20 (but κ is not related to Theorem 2.4).

If $\gamma \notin \Delta_Y$, then $P_Y \subset P_X$ and Γ is a point. Assume that $\gamma \in \Delta_Y$. Since $\Delta_Y \smallsetminus \Delta_Z = \{\gamma\}$ consists of a cominuscule simple root, it follows that Γ is a cominuscule flag variety. The corresponding partially ordered set is given by $\mathcal{P}_\Gamma = I(\kappa) = \Phi^+_Y \smallsetminus \Phi_Z = \mathcal{P}_X \cap \Phi_Y$. The labeling $\mathcal{P}_\Gamma \to \Delta_Y$ is the restriction of the labeling $\delta : \mathcal{P}_X \to \Delta$, and a curve $C \subset \Gamma$ has the same degree in Γ as in X. The variety $\Gamma = P_Y/(P_Y \cap P_X)$ depends only on the connected component of (the Dynkin diagram of) Δ_Y that contains γ.

If $S \subset \Delta$ is any subset that is connected in the Dynkin diagram of Φ, then the sum of all simple roots in S is a root in Φ. Given $\beta \in \Delta$, let $[\gamma,\beta]$ denote the smallest connected subset of Δ that contains γ and β. A simple root $\beta \in \Delta \smallsetminus \Delta_Y$ is an *essential excluded root* of Y if $[\gamma,\beta] \subset \Delta_Y \cup \{\beta\}$, i.e. β is connected to the component of γ in Δ_Y. The group P_Y is contained in the stabilizer of X_κ in G, and is equal to this stabilizer if and only if all roots in $\Delta \smallsetminus \Delta_Y$ are essential excluded roots of Y.

Remark 5.15. Assume that P_Y is the stabilizer of X_κ in G. Then $\Delta_Y = \{\beta \in \Delta \mid (\kappa.\omega_\gamma,\beta^\vee) \leq 0\}$. In fact, if $\beta \in \Delta$ and $(\kappa.\omega_\gamma,\beta^\vee) > 0$, then $\alpha = \kappa^{-1}.\beta$ must be a minimal box of $\mathcal{P}_X \smallsetminus I(\kappa)$ by Lemma 4.2(b), which implies that $\beta \in \Delta \smallsetminus \Delta_Y$. On the other hand, if $\beta \in \Delta \smallsetminus \Delta_Y$, then let α be the sum of the simple roots in the interval $[\gamma,\beta]$. Then α is a minimal box of $\mathcal{P}_X \smallsetminus I(\kappa)$, $(\omega_\gamma,\alpha^\vee) > 0$, and Lemma 4.2(b) implies that $\beta = \kappa.\alpha$.

5.4. **Primitive cominuscule varieties.** The cominuscule flag variety X will be called *primitive* if the excluded cominuscule simple root γ is invariant under the Cartan involution, that is, $\gamma = -w_0.\gamma$. The list of all primitive cominuscule varieties is contained in Table 2.

Proposition 5.16. *Let X be a cominuscule flag variety of diameter $d = d_X(2)$, and let $\rho \in \Phi^+$ be the highest root. The following conditions are equivalent and hold if and only if X is primitive.* (1) $\gamma = -w_0.\gamma$. (2) $\delta(\rho) = \gamma$. (3) $\kappa_d = z_d$. (4) $(w_0^X)^{-1} = w_0^X$. (5) $\dim(M_d) = 3\dim(X)$.

Table 2. Primitive cominuscule varieties.

X	$d_X(2)$
$\mathrm{Gr}(d,2d)$	d
$\mathrm{LG}(d,2d)$	d
$\mathrm{OG}(2d,4d)$	d
Q^N	2
E_7/P_7	3

Proof. Condition (1) is our definition of primitive. Lemma 4.4 shows that $\delta(\rho) = \delta(w_{0,X}.\gamma) = -w_0.\delta(\gamma) = -w_0.\gamma$, so (1) is equivalent to (2). The implication (2) $\Rightarrow$ (3) is clear from the definitions, (3) $\Rightarrow$ (4) holds because $\kappa_d^{-1} = \kappa_d$ and $z_d = w_0^X$, and (4) $\Rightarrow$ (2) holds because (4) implies that $(w_0^X)^{-1} \in W^X$. Finally, (5) is equivalent to (3) by Corollary 5.14(a). □

The following identity is a consequence of the structure theorems for quantum cohomology proved in [Ber97, BKT03, KT03, KT04, CMP08]. It can also be checked by constructing the unique rational curve through three general points in each case. We sketch how this is done in the proof.

Theorem 5.17. *Let X be a primitive cominuscule variety of diameter $d = d_X(2)$. Then $\langle \mathrm{point}, \mathrm{point}, \mathrm{point}\rangle_d = 1$.*

Proof. Assume first that $V, V', V'' \subset \mathbb{C}^{2d}$ are three general points in the primitive Grassmannian $\mathrm{Gr}(d,2d)$ of type A. Choose any basis $\{v_1,\dots,v_d\}$ of V, and write $v_i = v_i' + v_i''$ for each i, with $v_i' \in V'$ and $v_i'' \in V''$. Then the only rational curve of degree d through V, V', V'' is $C = \{\langle sv_1' + tv_1'', \dots, sv_d' + tv_d''\rangle \mid (s:t) \in \mathbb{P}^1\}$, see [BKT03, Prop. 1] for details. The same construction works for the Lagrangian Grassmannian $\mathrm{LG}(d,2d)$ and the maximal orthogonal Grassmannian $\mathrm{OG}(2d,4d)$, see [BKT03, Prop. 2 and Prop. 4]. If $V, V', V'' \subset \mathbb{C}^{N+2}$ are general points in the quadric $Q^N = \mathrm{OG}(1,N+2)$, then $E = V \oplus V' \oplus V''$ is an orthogonal vector space of dimension 3, and the unique curve of degree 2 through V, V', V'' is $C = \mathbb{P}(E) \cap Q^N$. A similar construction of the unique cubic curve through three general points of the Freudenthal variety E_7/P_7 can be found in [CMP08, Lemma 5]. □

Lemma 5.18. *Let X be a primitive cominuscule variety. For $0 \le d \le d_X(2)$ we have $\kappa_d^\vee = z_{d_X(2)-d}$.*

Proof. This follows from Lemma 4.4, noting that $w_{0,X}.\widetilde{\alpha}_d = \widetilde{\alpha}_{d_X(2)-d+1}$. □

In the following we will consider a single point to be a primitive cominuscule flag variety of diameter zero.

Proposition 5.19. *Let X be any cominuscule variety and $0 \le d \le d_X(2)$. There exists a unique largest parabolic subgroup $P_{Y_d} \subset G$ containing B such that $\Gamma_d = P_{Y_d}/(P_X \cap P_{Y_d})$ is a primitive cominuscule variety of diameter d. In addition, $F_d = P_X/(P_X \cap P_{Y_d})$ is a product of cominuscule varieties.*

Proof. This must be checked from the classification of cominuscule flag varieties, but only the associated Dynkin diagrams need to be considered. For $d = 0$ we have $Y_0 = X$ and $\Gamma_0 = F_0 = \{\mathrm{point}\}$. If X is a primitive cominuscule variety and $d = d_X(2)$, then $Y_d = F_d = \{\mathrm{point}\}$ and $\Gamma_d = X$. The choice of P_{Y_d} in all other cases is indicated in Table 3, where the roots of $\Delta \smallsetminus \Delta_{Y_d}$ are colored gray and γ is colored black. $\square$

Let X be a cominuscule variety. For $0 \leq d \leq d_X(2)$ we define $Y_d = G/P_{Y_d}$ by the parabolic subgroup of Proposition 5.19, and we let $Z_d = G/P_{Z_d}$ be the incidence variety defined by $P_{Z_d} = P_{Y_d} \cap P_X$. Let $p_d : Z_d \to X$ and $q_d : Z_d \to Y_d$ denote the projections, with fibers $F_d = P_X/P_{Z_d}$ and $\Gamma_d = P_{Y_d}/P_{Z_d}$. For any point $\omega \in Y_d$ we will use the notation $\Gamma_\omega = p_d q_d^{-1}(\omega) \subset X$. We identify Z_d with its image by the map $q_d \times p_d$, that is $Z_d = \{(\omega, x) \in Y_d \times X \mid x \in \Gamma_\omega\}$. We will also frequently identify Γ_d with $p_d(\Gamma_d) = \Gamma_{1.P_Y}$. Since P_{Y_d} is the stabilizer of $\Gamma_{1.Y_d}$ by the maximality condition of Proposition 5.19, the following result shows that the assignment $\omega \mapsto \Gamma_\omega$ is a bijection from Y_d to the set of all translates of X_{κ_d} in X. Recall that $\Gamma_d(x, y)$ is the union of all stable curves of degree d through x and y, and $\mathring{\mathcal{Z}}_{d,2} = \{(x, y) \in X \times X \mid \operatorname{dist}(x, y) = d\}$.

Corollary 5.20. (a) *We have* $X_{\kappa_d} = \Gamma_d = \Gamma_d(1.P_X, \kappa_d.P_X)$.

(b) *Let* $x, y \in X$. *We have* $\operatorname{dist}(x, y) \leq d$ *if and only if* $x, y \in \Gamma_\omega$ *for some* $\omega \in Y_d$. *The element* ω *is unique if* $\operatorname{dist}(x, y) = d$.

(c) *The function* $\varphi : \mathring{\mathcal{Z}}_{d,2} \to Y_d$ *defined by* $\Gamma_{\varphi(x,y)} = \Gamma_d(x, y)$ *is a morphism of varieties.*

Proof. Since Γ_d is a primitive cominuscule variety of diameter d, it follows that $\widetilde{\alpha}_d$ is the largest root in $\mathcal{P}_{\Gamma_d}$, hence $\mathcal{P}_{\Gamma_d} = I(\kappa_d)$ and $\Gamma_d = X_{\kappa_d}$. Corollary 5.14(b) implies that $\Gamma_d(1.P_X, \kappa_d.P_X) = \operatorname{ev}_3(M_d(1.P_X, \kappa_d.P_X))$ is an irreducible subvariety of X of dimension at most $\ell(\kappa_d)$, and Theorem 5.17 shows that $X_{\kappa_d} \subset \Gamma_d(1.P_X, \kappa_d.P_X)$. This proves part (a). For part (b) we may assume that $x = 1.P_X$ and $y = \kappa_{d'}.P_X$ by Lemma 5.6, where $d' = \operatorname{dist}(x, y)$. If $\operatorname{dist}(x, y) \leq d$, then $x, y \in \Gamma_{1.P_{Y_d}}$ by part (a). On the other hand, since the diameter of Γ_ω is d for each $\omega \in Y_d$, we have $\operatorname{dist}(x, y) \leq d$ whenever $x, y \in \Gamma_\omega$. Finally, if $x, y \in \Gamma_\omega$ and $\operatorname{dist}(x, y) = d$, then Theorem 5.17 applied to Γ_ω shows that $\Gamma_\omega \subset \Gamma_d(x, y) = \Gamma_{1.P_{Y_d}}$, hence $\omega = 1.P_{Y_d}$. Part (b) follows from this. Choose splittings $s_1 : \mathring{X}^1 \to B^-$ and $s_2 : \mathring{X}_{z_d} \to B$ so that $x = s_1(x).P_X$ for all $x \in \mathring{X}^1$, and $y = s_2(y) z_d.P_X$ for all $y \in \mathring{X}_{z_d}$. For all points (x, y) in a dense open subset of $\mathring{\mathcal{Z}}_{d,2}$ we have $(x, y) = s_1(x) s_2(s_1(x)^{-1}.y) z_d \kappa_d.(1.P_X, \kappa_d.P_X)$, so φ is defined by $\varphi(x, y) = s_1(x) s_2(s_1(x)^{-1}.y) z_d \kappa_d.P_{Y_d}$ on this subset. This proves part (c). $\square$

5.5. **A blow-up of the Kontsevich moduli space.** Let $0 \leq d \leq d_X(2)$. The map $q_d : Z_d \to Y_d$ is a locally trivial fibration with fibers $\Gamma_\omega \cong X_{\kappa_d}$, $\omega \in Y_d$. Define a new family $\mathrm{B}\ell_d \to Y_d$ by replacing each fiber Γ_ω with the moduli space $\overline{\mathcal{M}}_{0,3}(\Gamma_\omega, d)$. Since Γ_ω is a subvariety of X, we have $\overline{\mathcal{M}}_{0,3}(\Gamma_\omega, d) \subset M_d$, and these inclusions define a morphism $\pi : \mathrm{B}\ell_d \to M_d$. Equivalently, we have $\mathrm{B}\ell_d = G \times^{P_{Y_d}} \overline{\mathcal{M}}_{0,3}(\Gamma_d, d)$. We will identify $\mathrm{B}\ell_d$ with its image in $Y_d \times M_d$, that is

$$\mathrm{B}\ell_d = \{(\omega, f) \in Y_d \times M_d \mid \operatorname{Image}(f) \subset \Gamma_\omega\}.$$

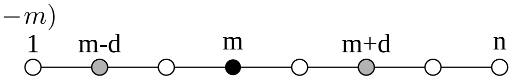
We also define the space

$$Z_d^{(3)} \;=\; Z_d \times_{Y_d} Z_d \times_{Y_d} Z_d \;=\; \{(\omega, x_1, x_2, x_3) \in Y_d \times X^3 \mid x_1, x_2, x_3 \in \Gamma_\omega\}.$$

Define a morphism $\phi : \mathrm{B}\ell_d \to Z_d^{(3)}$ by $\phi(\omega, f) = (\omega, \operatorname{ev}_1(f), \operatorname{ev}_2(f), \operatorname{ev}_3(f))$, and let $e_i : Z_d^{(3)} \to Z_d$ denote the i-th projection. We obtain the following commutative

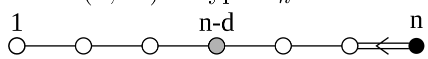
TABLE 3. The quantum-to-classical construction.

Grassmannian $X = \mathrm{Gr}(m, n+1)$ of type A_n
$d_X(2) = \min(m, n+1-m)$



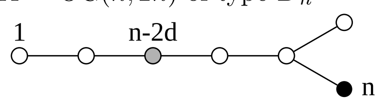
$Y_d = \mathrm{Fl}(m-d, m+d; n+1)$; $\Gamma_d = \mathrm{Gr}(d, 2d)$;
$F_d = \mathrm{Gr}(m-d, m) \times \mathrm{Gr}(d, n+1-m)$

Lagrangian Grassmannian $X = \mathrm{LG}(n, 2n)$ of type C_n
$d_X(2) = n$



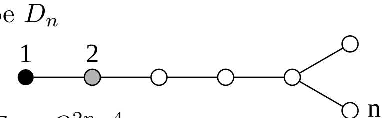
$Y_d = \mathrm{SG}(n-d, 2n)$; $\Gamma_d = \mathrm{LG}(d, 2d)$; $F_d = \mathrm{Gr}(n-d, n)$

Max. orthogonal Grassmannian $X = \mathrm{OG}(n, 2n)$ of type D_n
$d_X(2) = \lfloor n/2 \rfloor$



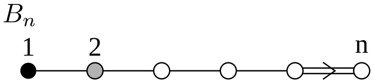
$Y_d = \mathrm{OG}(n-2d, 2n)$; $\Gamma_d = \mathrm{OG}(2d, 4d)$; $F_d = \mathrm{Gr}(n-2d, n)$

Even quadric $X = Q^{2n-2}$ of type D_n
$d_X(2) = 2$



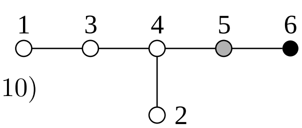
$Y_1 = \mathrm{OG}(2, 2n)$; $\Gamma_1 = \mathbb{P}^1$; $F_1 = Q^{2n-4}$

Odd quadric $X = Q^{2n-1}$ of type B_n
$d_X(2) = 2$

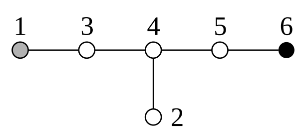


$Y_1 = \mathrm{OG}(2, 2n+1)$; $\Gamma_1 = \mathbb{P}^1$; $F_1 = Q^{2n-3}$

Cayley plane $X = E_6/P_6$
$d_X(2) = 2$

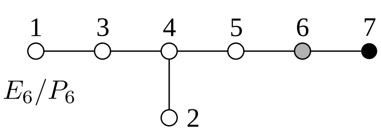


$Y_1 = E_6/P_5$; $\Gamma_1 = \mathbb{P}^1$; $F_1 = \mathrm{OG}(5, 10)$

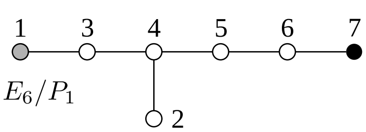


$Y_2 = E_6/P_1$; $\Gamma_2 = Q^8$; $F_2 = Q^8$

Freudenthal variety $X = E_7/P_7$
$d_X(2) = 3$



$Y_1 = E_7/P_6$; $\Gamma_1 = \mathbb{P}^1$; $F_1 = E_6/P_6$



$Y_2 = E_7/P_1$; $\Gamma_2 = Q^{10}$; $F_2 = E_6/P_1$

diagram from [BM11]:

(3)
$$\begin{array}{ccccc}
\mathrm{B}\ell_d & & \xrightarrow{\quad\pi\quad} & & M_d \\
\downarrow{\scriptstyle\phi} & & & & \downarrow{\scriptstyle \mathrm{ev}_i} \\
Z_d^{(3)} & \xrightarrow{e_i} & Z_d & \xrightarrow{p_d} & X \\
 & & \downarrow{\scriptstyle q_d} & & \\
 & & Y_d & &
\end{array}$$

Proposition 5.21. *The maps $\pi : \mathrm{B}\ell_d \to M_d$ and $\phi : \mathrm{B}\ell_d \to Z_d^{(3)}$ are birational.*

Proof. It follows from Theorem 5.17 that ϕ is birational. Since the image of the map $\mathrm{ev}_1 \times \mathrm{ev}_2 : M_d \to X \times X$ is contained in $\mathcal{Z}_{d,2} = \{(x_1, x_2) \mid \mathrm{dist}(x_1, x_2) \leq d\}$, it follows that $U = (\mathrm{ev}_1 \times \mathrm{ev}_2)^{-1}(\mathring{\mathcal{Z}}_{d,2})$ is a dense open subset of M_d. Given any stable map $f \in U$, we have $\mathrm{dist}(\mathrm{ev}_1(f), \mathrm{ev}_2(f)) = d$, so Corollary 5.20 implies that the image of f is contained in Γ_ω for a unique point $\omega \in Y_d$. It follows that $\pi^{-1}(f) = (\omega, f)$, so π is birational. □

The (three point, genus zero) Gromov-Witten invariants of small degrees of a cominuscule flag variety are given by the following result. Generalizations to larger degrees can be found in [BM11, CP11, BCMP18b].

Corollary 5.22. *Let X be cominuscule and $0 \leq d \leq d_X(2)$.*
(a) *For $\Omega_1, \Omega_2, \Omega_3 \in H_T^*(X; \mathbb{Z})$ we have*
$$\langle \Omega_1, \Omega_2, \Omega_3 \rangle_d = \int_{Y_d} q_{d*} p_d^*(\Omega_1) \cdot q_{d*} p_d^*(\Omega_2) \cdot q_{d*} p_d^*(\Omega_3)\,.$$
(b) *For $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in K_T(X)$ we have*
$$I_d(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = \chi_{Y_d}\left(q_{d*} p_d^*(\mathcal{F}_1) \cdot q_{d*} p_d^*(\mathcal{F}_2) \cdot q_{d*} p_d^*(\mathcal{F}_3)\right).$$

Proof. Since all varieties in the diagram (3) have rational singularities, it follows from Proposition 5.21 that $\pi_*[\mathcal{O}_{\mathrm{B}\ell_d}] = [\mathcal{O}_{M_d}]$ and $\phi_*[\mathcal{O}_{\mathrm{B}\ell_d}] = [\mathcal{O}_{Z_d^{(3)}}]$. We obtain
$$\begin{aligned}
\chi_{M_d}(\mathrm{ev}_1^*(\mathcal{F}_1) \cdot \mathrm{ev}_2^*(\mathcal{F}_2) \cdot \mathrm{ev}_3^*(\mathcal{F}_3)) &= \chi_{Z_d^{(3)}}(e_1^* p_d^*(\mathcal{F}_1) \cdot e_2^* p_d^*(\mathcal{F}_2) \cdot e_3^* p_d^*(\mathcal{F}_3)) \\
&= \chi_{Y_d}(q_{d*} p_d^*(\mathcal{F}_1) \cdot q_{d*} p_d^*(\mathcal{F}_2) \cdot q_{d*} p_d^*(\mathcal{F}_3))\,,
\end{aligned}$$
where the first identity follows from the projection formula (twice) together with commutativity of the diagram (3), and the second follows from [BM11, Lemma 3.5]. This proves part (b). Part (a) is proved by repeating the same argument with cohomology classes, or by extracting the initial terms of both sides in part (b), see [BM11, §4.1]. □

For any subvarieties $\Omega_1, \Omega_2 \subset X$ and $0 \leq d \leq d_X(2)$ we define
$$\begin{aligned}
Y_d(\Omega_1, \Omega_2) &= q_d(p_d^{-1}(\Omega_1)) \cap q_d(p_d^{-1}(\Omega_2)) \\
&= \{\omega \in Y_d \mid \Gamma_\omega \cap \Omega_1 \neq \emptyset \text{ and } \Gamma_\omega \cap \Omega_2 \neq \emptyset\}\,, \\
Z_d(\Omega_1, \Omega_2) &= q_d^{-1}(Y_d(\Omega_1, \Omega_2))\,.
\end{aligned}$$
We also define the special cases $Y_d(\Omega_1) = q_d p_d^{-1}(\Omega_1)$ and $Z_d(\Omega_1) = q_d^{-1}(Y_d(\Omega_1))$. Notice that for $\omega \in Y_d$ we have $\Gamma_\omega \cap \Omega_1 \neq \emptyset$ if and only if $\omega \in Y_d(\Omega_1)$.

Corollary 5.23. *We have* $\Gamma_d(\Omega_1,\Omega_2) = p_d(Z_d(\Omega_1,\Omega_2))$. *As a special case we obtain* $\Gamma_d(\Omega_1) = p_d(Z_d(\Omega_1)) = p_d q_d^{-1} q_d p_d^{-1}(\Omega_1)$.

Proof. Since the diagram (3) is commutative and the maps π and ϕ are surjective, we obtain

$$\begin{aligned}\Gamma_d(\Omega_1,\Omega_2) &= \operatorname{ev}_3 \pi\left((\operatorname{ev}_1 \pi)^{-1}(\Omega_1) \cap (\operatorname{ev}_2 \pi)^{-1}(\Omega_2)\right)\\ &= p_d e_3\left((p_d e_1)^{-1}(\Omega_1) \cap (p_d e_2)^{-1}(\Omega_2)\right)\\ &= p_d\left(\{(\omega, x_3) \in Z_d \mid \Gamma_\omega \cap \Omega_1 \neq \emptyset \text{ and } \Gamma_\omega \cap \Omega_2 \neq \emptyset\}\right)\end{aligned}$$

as required. □

6. Fibers of the quantum-to-classical construction

In this section we obtain explicit descriptions of the general fibers of several maps related to the quantum-to-classical construction. These results are required for determining the powers of q that occur in quantum products, as well as for our proof that the structure constants of quantum K-theory have alternating signs.

6.1. Bijections between order ideals.

Lemma 6.1. *For* $0 \leq d \leq d_X(2)$ *we have the identities* $\kappa_d = w_{0,Y_d}^{Z_d}$, $z_d = w_{0,X} w_{0,Y_d}$, $z_d/\kappa_d = w_{0,X}^{Z_d}$, *and* $w_{0,Y_d} = \kappa_d w_{0,Z_d} = w_{0,Z_d}\kappa_d$.

Proof. Since the projection $p_d : q_d^{-1}(1.P_{Y_d}) \to X_{\kappa_d}$ is an isomorphism, it follows that $q_d^{-1}(1.P_{Y_d}) = (Z_d)_{\kappa_d}$ and $w_{0,Y_d}^{Z_d} = \kappa_d$. We also obtain $p_d^{-1}(1.P_X) = (Z_d)_{w_{0,X}^{Z_d}}$ and $Y_d(1.P_X) = (Y_d)_{w_{0,X}^{Z_d}}$, and therefore $Z_d(1.P_X) = (Z_d)_{w_{0,X}^{Z_d} w_{0,Y_d}^{Z_d}}$. Since we have $\operatorname{dist}(z, 1.P_X) = d$ for all points z in a dense open subset of $\Gamma_d(1.P_X)$, it follows from Corollary 5.20 and Corollary 5.23 that $p_d : Z_d(1.P_X) \to \Gamma_d(1.P_X) = X_{z_d}$ is birational. We deduce that $w_{0,X}^{Z_d} w_{0,Y_d}^{Z_d} = z_d$, hence $w_{0,X}^{Z_d} = z_d/\kappa_d$. Finally, we have $\kappa_d w_{0,Z_d} = w_{0,Y_d} = w_{0,Y_d}^{-1} = w_{0,Z_d}\kappa_d$, and $z_d = w_{0,X}^{Z_d}\kappa_d = w_{0,X} w_{0,Z_d}\kappa_d = w_{0,X} w_{0,Y_d}$, which completes the proof. □

The variety $F_d = p_d^{-1}(1.P_X) = P_X/P_{Z_d}$ is a product of cominuscule varieties by Proposition 5.19, and the Schubert varieties in this space (which are products of Schubert varieties in the factors of F_d) are indexed by elements of the set $W^{F_d} \subset W_X$ of minimal representatives of the cosets in W_X/W_{Z_d}. The maximal element in W^{F_d} is $w_{0,X}^{Z_d} = z_d/\kappa_d$, so the elements of W^{F_d} correspond to order ideals in $\mathcal{P}_{F_d} = I(z_d/\kappa_d)$. This subset of Φ^+ is always disjoint from $\mathcal{P}_X$ (see Example 6.3). Notice that if F_d has more than one cominuscule factor, then $\mathcal{P}_{F_d}$ is a disjoint union of the corresponding partially ordered sets. For $\eta \in \mathcal{P}_{F_d}$ we set $\lambda'(\eta) = \{\eta' \in \mathcal{P}_{F_d} \mid \eta' < \eta\}$. Then the labeling $\delta' : \mathcal{P}_{F_d} \to \Delta_X$ is given by $\delta'(\eta) = w'_{\lambda'(\eta)}.\eta$, where $w'_{\lambda'(\eta)}$ is the product of the reflections $s_{\eta'}$ for $\eta' \in \lambda'(\eta)$, in increasing order.

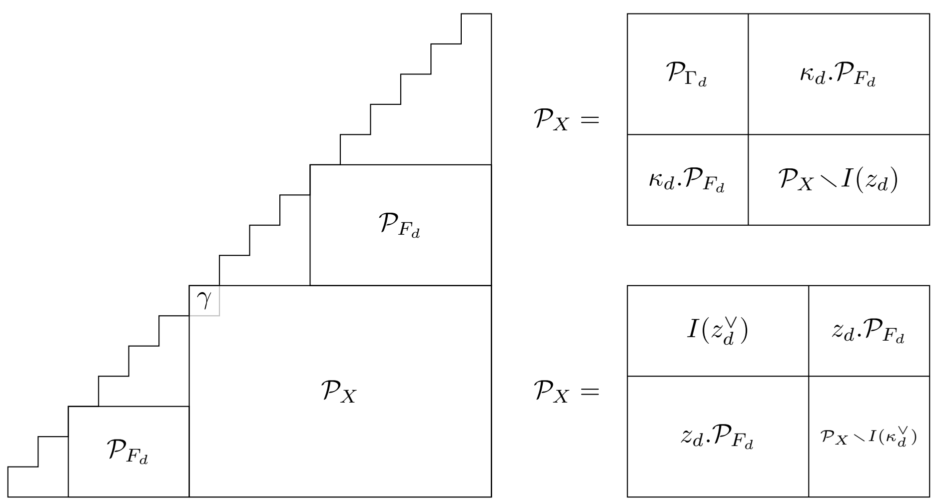
Proposition 6.2. *The following order isomorphisms are obtained by restricting the actions of Weyl group elements.*

(a) $\kappa_d : \mathcal{P}_{F_d} \to I(z_d) \smallsetminus I(\kappa_d)$ *is an order isomorphism, and* $\delta(\kappa_d.\eta) = \delta'(\eta)$ *for each* $\eta \in \mathcal{P}_{F_d}$.

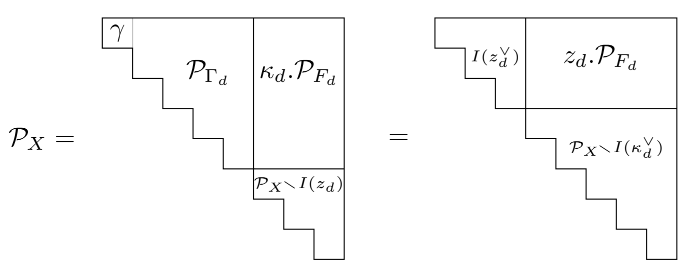
(b) $z_d : \mathcal{P}_{F_d} \to I(\kappa_d^\vee) \smallsetminus I(z_d^\vee)$ *is an order isomorphism, and* $\delta(z_d.\eta) = w_0^X.\delta'(\eta)$ *for each* $\eta \in \mathcal{P}_{F_d}$.

Proof. It follows from Lemma 4.2(a) that $\kappa_d : I(z_d) \smallsetminus I(\kappa_d) \to \mathcal{P}_{F_d}$ is an order-preserving bijection, and since $\kappa_d.\beta > 0$ for each $\beta \in \Delta \smallsetminus \{\gamma\}$, the inverse bijection $\kappa_d^{-1} = \kappa_d$ is also order-preserving. Given $\eta \in \mathcal{P}_{F_d}$, the set $\mu = I(\kappa_d) \cup \kappa_d.\lambda'(\eta)$ is a straight shape in $\mathcal{P}_X$ such that $\kappa_d.\eta$ is a minimal box of $\mathcal{P}_X \smallsetminus \mu$, hence $\delta(\kappa_d.\eta) = w_\mu.(\kappa_d.\eta)$. Using that $w_\mu = w'_{\lambda'(\eta)}\kappa_d$, we obtain $\delta(\kappa_d.\eta) = w'_{\lambda'(\eta)}.\eta = \delta'(\eta)$. This proves part (a). Lemma 4.4(a) applied to F_d shows that $w_{0,Z_d} : \mathcal{P}_{F_d} \to \mathcal{P}_{F_d}$ is an order-reversing involution such that $\delta'(w_{0,Z_d}.\eta) = -w_{0,X}.\delta'(\eta)$. Part (b) therefore follows from Lemma 4.4(a), part (a), and the identities $z_d = w_{0,X}\kappa_d w_{0,Z_d}$ and $w_0 w_{0,X} = w_0^X$. $\square$

Example 6.3. Consider $X = \mathrm{Gr}(7, 17)$ and $d = 4$, so that $\Gamma_d = \mathrm{Gr}(4, 8)$ and $F_d = \mathrm{Gr}(3, 7) \times \mathrm{Gr}(4, 10)$. Let $\Phi^+ = \{e_i - e_j \mid 1 \le i < j \le 17\}$ be the set of positive roots of type A_{16}. We identify each root $e_i - e_j$ with the box in row $17 - i$ and column $j-1$ of a triangular diagram of boxes. Proposition 6.2 shows that $\mathcal{P}_{F_d}$ can be identified with $\kappa_d.\mathcal{P}_{F_d} = I(z_d) \smallsetminus I(\kappa_d)$, and also with $z_d.\mathcal{P}_{F_d} = I(\kappa_d^\vee) \smallsetminus I(z_d^\vee)$. This gives two dissections of $\mathcal{P}_X$. Notice that $\mathcal{P}_{\Gamma_d} = I(\kappa_d)$, and $\mathcal{P}_{F_d}$ can be identified with the disjoint union of $\mathcal{P}_{\mathrm{Gr}(3,7)}$ and $\mathcal{P}_{\mathrm{Gr}(4,10)}$.



Example 6.4. Let $X = \mathrm{LG}(8, 16)$ and $d = 5$. Then Proposition 6.2 provides the following dissections of $\mathcal{P}_X$.



6.2. **Fibers of the quantum-classical diagram.**

Definition 6.5. Given $u, v \in W^X$ and $0 \leq d \leq d_X(2)$, define the Weyl group elements

$$u(d) = (u \cap z_d^\vee) z_d\,, \qquad \widehat{u}_d = (u \cap \kappa_d^\vee)/(u \cap z_d^\vee)\,, \qquad u_d = (w_0^X)^{-1}\, \widehat{u}_d\, w_0^X\,,$$
$$v(-d) = (v \cup z_d)/z_d\,, \qquad \text{and} \qquad v^d = (v \cap z_d)/(v \cap \kappa_d)\,.$$

It was proved in [BCMP18a] that $X_{u(d)} = \Gamma_d(X_u)$ and $X^{v(-d)} = \Gamma_d(X^v)$. We will reprove these statements in Corollary 6.9 below, together with similar geometric interpretations of u_d and v^d. In particular, $u(d)$ and $v(-d)$ belong to W^X. The shape $I(v(-d))$ is obtained from $I(v)$ by removing the boxes in $I(v \cap z_d)$ and moving the remaining boxes north-west until they fit in the upper-left corner of $\mathcal{P}_X$ (see [BCMP18a, §3.2]). Similarly, $I(u(d))$ is obtained by attaching the shape $I(u)$ to the south-east border of $I(z_d)$ and discarding any boxes that do not fit within $\mathcal{P}_X$. More precisely, the following identities follow from Proposition 5.9(a) and Corollary 5.11.

Lemma 6.6. *We have* $I(u(-d)) = (z_1 s_\gamma)^d.(I(u) \smallsetminus I(z_d))$ *and* $I(u(d)) = I(z_d) \cup (z_1 s_\gamma)^{-d}.(I(u) \cap I(z_d^\vee))$.

Our next result shows that u_d and v^d belong to $W^{F_d} = W_X \cap W^{Z_d}$, and the shapes of these elements in $\mathcal{P}_{F_d}$ are determined by $z_d.I(u_d) = I(u) \cap z_d.\mathcal{P}_{F_d}$ and $\kappa_d.I(v^d) = I(v) \cap \kappa_d.\mathcal{P}_{F_d}$ (see Example 6.3, Example 6.4, and Example 6.8). Recall from Section 2.1 that the parabolic factorization of $v \in W$ with respect to P_{Y_d} is denoted by $v = v^{Y_d} v_{Y_d}$.

Proposition 6.7. *Let* $u, v \in W^X$ *and* $0 \leq d \leq d_X(2)$*.*

(a) *We have* $v_{Y_d} = v \cap \kappa_d$ *and* $v^{Y_d} = (v \cup \kappa_d)/\kappa_d$*, and the parabolic factorization of* v^{Y_d} *with respect to* P_X *is* $v^{Y_d} = v(-d) v^d$*.*

(b) *We have* $u_d, v^d \in W^{F_d}$*, with shapes given by* $I(u_d) = z_d^{-1}.I(u) \cap \mathcal{P}_{F_d}$ *and* $I(v^d) = \kappa_d.I(v) \cap \mathcal{P}_{F_d}$*.*

(c) *We have that* $u^\vee(-d) = u(d)^\vee = w_0 u(d) w_{0,X}$ *is dual to* $u(d)$ *in* W^X*, and* $(u^\vee)^d = w_{0,X} u_d w_{0,Z_d}$ *is dual to* u_d *in* W^{F_d}*.*

Proof. The element $\widehat{u}_d$ is by definition the product of the simple reflections $s_{\delta(\alpha)}$ for all boxes α in $I(u) \cap (I(\kappa_d^\vee) \smallsetminus I(z_d^\vee))$, in decreasing order. Proposition 6.2(b) therefore shows that

$$u_d = (w_0^X)^{-1} \widehat{u}_d w_0^X = \prod_{\eta \in z_d^{-1}.I(u) \cap \mathcal{P}_{F_d}} s_{\delta'(\eta)}\,,$$

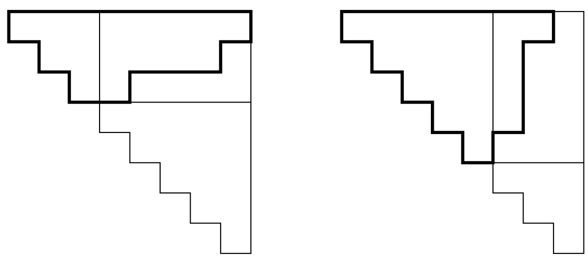
the product in decreasing order. This shows that $u_d \in W^{F_d}$ and $I(u_d) = z_d^{-1}.I(u) \cap \mathcal{P}_{F_d}$. Proposition 6.2(a) similarly shows that $v^d \in W^{F_d}$ and $I(v^d) = \kappa_d.I(v) \cap \mathcal{P}_{F_d}$. This proves part (b).

Since $v \cup \kappa_d \in W^X \subset W^{Z_d}$, the product of $(v \cup \kappa_d)/\kappa_d$ with $w_{0,Y_d} = \kappa_d w_{0,Z_d}$ is reduced, hence $(v \cup \kappa_d)/\kappa_d \in W^{Y_d}$. Since $v \cap \kappa_d \in W_{Y_d}$, we deduce that the parabolic factorization $v = v^{Y_d} v_{Y_d}$ is given by $v^{Y_d} = (v \cup \kappa_d)/\kappa_d$ and $v_{Y_d} = v \cap \kappa_d$. Since $u(d) \leq_L z_d^\vee z_d = w_0^X$, we have $u(d) \in W^X$. The dual element is $w_0 u(d) w_{0,X} = w_0 (u \cap z_d^\vee) w_{0,X} z_d^{-1} = (u^\vee \cup z_d)/z_d = u^\vee(-d)$. This implies that $v(-d) \in W^X$, and since $v^d \in W_X$, it follows that $v^{Y_d} = v(-d) v^d$ is the parabolic factorization of v^{Y_d} with respect to P_X. This proves part (a).

The element $\tau = z_d^\vee/(u\cap z_d^\vee)$ commutes with $\widehat{u}_d$ and satisfies $\tau u(d) = z_d^\vee z_d = w_0^X$. This implies $u_d = (w_0^X)^{-1}\widehat{u}_d w_0^X = u(d)^{-1}\widehat{u}_d u(d)$, hence $w_0 u^\vee(-d)(u^\vee)^d w_{0,Z_d} = w_0(u^\vee \cup \kappa_d)\kappa_d w_{0,Z_d} = (u\cap\kappa_d^\vee)z_d = \widehat{u}_d u(d) = u(d)u_d$. This shows that $u^\vee(-d)(u^\vee)^d$ is dual to $u(d)u_d$ in W^{Z_d}. Since $u^\vee(-d)$ is dual to $u(d)$ in W^X, it follows that $(u^\vee)^d$ is dual to u_d in W^{F_d}, see Remark 2.9. This completes the proof of part (c). □

In examples we denote an element $u \in W^X$ by the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ for which λ_j is the number of boxes in the j-th row of $I(u)$.

Example 6.8. Let $X = \mathrm{LG}(8, 16)$, $u = (8, 6, 2) \in W^X$, $v = u^\vee = (7, 5, 4, 3, 1)$, and set $d = 5$. The shape of $u_d = (5, 4, 1)$ is obtained by intersecting the shape of u with $z_d.\mathcal{P}_{F_d}$, and the shape of $v^d = (2, 1, 1, 1)$ is obtained by intersecting the shape of v with $\kappa_d.\mathcal{P}_{F_d}$, see Example 6.4.



The elements u_d and v^d are dual to each other in W^{F_d}, but the images $z_d.I(u_d)$ and $\kappa_d.I(v^d)$ of their shapes are represented in two different rectangles in $\mathcal{P}_X$. The composed bijection $z_d/\kappa_d : \kappa_d.\mathcal{P}_{F_d} \cong \mathcal{P}_{F_d} \cong z_d.\mathcal{P}_{F_d}$ is given by a transposition when X is a Lagrangian Grassmannian. An expression of an element of W^{F_d} as a partition therefore depends on how the rectangle $\mathcal{P}_{F_d}$ is oriented. Opposite conventions are used in the expressions $u_d = (5, 4, 1)$ and $v^d = (2, 1, 1, 1)$ given above. We also have $u(d) = w_0^X$ and $v(-d) = 1$. Other shifts of u include $u(-2) = (2)$, $u(-1) = (6, 2)$, $u(1) = (8, 7, 6, 2)$, and $u(2) = (8, 7, 6, 5, 2)$.

Corollary 6.9. *Let $u, v \in W^X$ and $0 \le d \le d_X(2)$.*

(a) *The general fibers of the map $q_d : p_d^{-1}(X^v) \to Y_d(X^v)$ are translates of $(\Gamma_d)^{v\cap\kappa_d}$.*

(b) *We have $Y_d(X^v) = (Y_d)^{v(-d)v^d}$, $\Gamma_d(X^v) = X^{v(-d)}$, and the general fibers of the map $p_d : Z_d(X^v) \to \Gamma_d(X^v)$ are translates of $(F_d)^{v^d}$.*

(c) *We have $Z_d(X_u) = (Z_d)_{u(d)u_d}$, $\Gamma_d(X_u) = X_{u(d)}$, and the general fibers of the map $p_d : Z_d(X_u) \to \Gamma_d(X_u)$ are translates of $(F_d)_{u_d}$.*

(d) *The general fibers of the map $p_d : Z_d(X_u, X^v) \to \Gamma_d(X_u, X^v)$ are translates of $(F_d)_{u_d}^{u_d\cap v^d}$.*

Proof. Parts (a) and (b) follow from Theorem 2.8, Corollary 5.23, and Proposition 6.7(a), and Proposition 6.7(c) implies that part (c) is equivalent to part (b), see Remark 2.9. Finally, part (d) follows from Theorem 2.10 and Proposition 4.5 together with parts (b) and (c). □

7. The q-degrees in quantum cohomology products

7.1. **Quantum cohomology.** Let $X = G/P_X$ be a cominuscule flag variety, and let $\mathrm{QH}(X)$ be the (small) quantum cohomology ring of X. As an additive group, this ring is defined as $\mathrm{QH}(X) = H^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$. Multiplication is defined by

$$[X_u] \star [X^v] = \sum_{w,d\geq 0} \langle [X_u],[X^v],[X_w]\rangle_d \, q^d \, [X^w]$$

for $u, v \in W^X$. Let

$$([X_u] \star [X^v])_d = \sum_{w} \langle [X_u],[X^v],[X_w]\rangle_d \, [X^w]$$

denote the coefficient of q^d in this product. The goal of this section is to identify the degrees d for which $([X_u] \star [X^v])_d \neq 0$. In particular, we will show that these degrees form an integer interval.

It follows from [FW04, Thm. 9.1] that the smallest degree d for which $([X_u] \star [X^v])_d \neq 0$ is equal to the *degree distance* between X_u and X^v, defined as the minimal d for which $\Gamma_d(X_u, X^v) \neq \emptyset$. In particular, the quantum product of two Schubert classes is never zero. Let $d_{\min}(u^\vee, v)$ and $d_{\max}(u^\vee, v)$ denote the minimal and maximal degrees for which $([X_u] \star [X^v])_d \neq 0$. We let $d_{\max}(v)$ denote the (unique) number of occurrences of s_γ in a reduced expression for v. Notice that $d = d_{\max}(v)$ is also determined by $\kappa_d \leq v \leq z_d$. The following result implies that $d_{\max}(v)$ is the only power of q that occurs in the product $[\text{point}] \star [X^v]$, that is $d_{\max}(v) = d_{\min}(w_0^X, v) = d_{\max}(w_0^X, v)$. More generally, it was proved in [Bel04, CMP09] that $[\text{point}] \star [X^v] = q^{d_{\max}(v)} \, [X^{w_0^X v}]$ holds in $\mathrm{QH}(X)$.

Proposition 7.1. *We have $([X_u] \star [X^v])_d \neq 0$ if and only if $d_{\min}(u^\vee, v) \leq d \leq \min(d_{\max}(u^\vee), d_{\max}(v))$ and $u_d \leq v^d$. In this case we have $([X_u] \star [X^v])_d = [\Gamma_d(X_u, X^v)]$.*

Proof. Using Corollary 5.22 and the projection formula we obtain

$$\langle [X_u],[X^v],[X_w]\rangle_d = \int_X p_{d*} q_d^* \big(q_{d*} p_d^* [X_u] \cdot q_{d*} p_d^* [X^v] \big) \cdot [X_w] \,,$$

which implies that

$$([X_u] \star [X^v])_d = p_{d*} q_d^* \big(q_{d*} p_d^* [X_u] \cdot q_{d*} p_d^* [X^v] \big) \,.$$

Corollary 6.9(a) shows that $q_{d*} p_d^* [X^v]$ is equal to $[q_d(p_d^{-1}(X^v))]$ for $d \leq d_{\max}(v)$ and is zero otherwise. It follows that $([X_u] \star [X^v])_d$ is non-zero only if $d \leq \min(d_{\max}(u^\vee), d_{\max}(v))$, in which case

$$([X_u] \star [X^v])_d = p_* [Z_d(X_u, X^v)] \,.$$

The proposition therefore follows from Corollary 6.9(d). □

The main technical result of this section is the following lemma, which we will prove after discussing its consequences. Notice that Proposition 6.2(a) shows that $w\kappa_d \in W^X$ and $I(w\kappa_d) = I(\kappa_d) \cup \kappa_d . I(w)$ for each $w \in W^{F_d}$.

Lemma 7.2. *Let $u, v \in W^X$.*

(a) *For $0 < d \leq d_{\max}(u^\vee)$ we have $u_{d-1}\kappa_{d-1} = (u_d \kappa_d)(-1)$.*

(b) *For $0 < d \leq d_{\max}(v)$ we have $v^{d-1}\kappa_{d-1} = (v^d \kappa_d) \cap z_{d-1}$.*

Part (b) of the following lemma will be relevant for our study of quantum K-theory in Section 8.

Lemma 7.3. *Let $u, v \in W^X$ and $0 < d \leq \min(d_{\max}(u^\vee), d_{\max}(v))$.*

(a) *Assume that* $u_d \leq v^d$. *Then* $u_{d-1} \leq v^{d-1}$.

(b) *If* $I(u_d) \smallsetminus I(v^d)$ *is a non-empty rook strip in* $\mathcal{P}_{F_d}$, *then* $d = d_{\max}(u^\vee, v) + 1$.

Proof. The inequality $u_d \leq v^d$ holds if and only if $u_d\kappa_d \leq v^d\kappa_d$. Part (a) therefore follows from Lemma 7.2, using that $(u_d\kappa_d)(-1) \leq u_d\kappa_d$ and $(u_d\kappa_d)(-1) \leq z_d(-1) \leq z_{d-1}$. If $u_d \not\leq v^d$, then it follows from Proposition 7.1 and part (a) that $d > d_{\max}(u^\vee, v)$. If $I(u_d) \smallsetminus I(v^d)$ is a rook strip, then $(u_d\kappa_d)(-1) \leq v^d\kappa_d$, which implies that $u_{d-1} \leq v^{d-1}$ and $d-1 \leq d_{\max}(u^\vee, v)$. This proves part (b). □

The following result was proved in [Pos05] for Grassmannians of type A.

Corollary 7.4. *The q-degrees appearing in a quantum product $[X_u] \star [X^v]$ form an interval, that is* $([X_u] \star [X^v])_d \neq 0$ *if and only if* $d_{\min}(u^\vee, v) \leq d \leq d_{\max}(u^\vee, v)$.

Proof. This follows from Proposition 7.1 and Lemma 7.3. □

The equivariant (small) quantum cohomology ring $\mathrm{QH}_T(X) = H_T^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ is defined like $\mathrm{QH}(X)$, except that equivariant Gromov-Witten invariants are used to define the quantum product $[X_u]_T \star [X^v]_T$, see [Kim95]. Proposition 7.1 is true also for the equivariant quantum product, with the same proof.

Corollary 7.5. *The equivariant quantum product* $[X_u]_T \star [X^v]_T$ *in* $\mathrm{QH}_T(X)$ *contains the same powers of q as the non-equivariant product* $[X_u] \star [X^v]$.

Remark 7.6. It is natural to ask whether the powers of q appearing in an equivariant quantum product of Schubert classes defined by the *same* Borel subgroup form an interval. Based on substantial computer evidence we conjecture that q^d occurs in the product $[X^u]_T \star [X^v]_T$ in $\mathrm{QH}_T(X)$ if and only if $0 \leq d \leq d_{\max}(u, v)$.

7.2. **The minimal and maximal degrees.** We next give a type-uniform description of the minimal and maximal powers of q in the quantum product $[X_u]\star[X^v]$ that generalizes the description for Grassmannians of type A proved in [FW04, Pos05]. The minimal degree $d_{\min}(u^\vee, v)$ is the smallest integer d for which $\Gamma_d(X_u, X^v) \neq \emptyset$. Since $\Gamma_d(X_u, X^v)$ is non-empty if and only if $v \leq u(d)$, the degree $d_{\min}(u^\vee, v)$ can be interpreted as the number of steps the shape $I(u)$ must be shifted in order to contain $I(v)$. This recovers Fulton and Woodward's description in type A [FW04]. Postnikov gave a similar description [Pos05] of the maximal degree $d_{\max}(u^\vee, v)$ in type A, as the number of steps $I(u)$ can be shifted before it no longer fits inside a shape. These descriptions of the powers of q in a quantum product are generalized in Theorem 7.8 and Theorem 7.13. The maximal degree for arbitrary cominuscule varieties is given by the following formula from [CMP07, Thm. 1.2].

Theorem 7.7. *We have* $d_{\max}(u^\vee, v) = d_{\max}(v) - d_{\min}((w_0^X v)^\vee, u)$.

Let $\mathcal{B} = \{q^d[X^u] : u \in W^X, d \in \mathbb{Z}\}$ denote the natural $\mathbb{Z}$-basis of the localized quantum cohomology ring $\mathrm{QH}(X)_q = \mathrm{QH}(X) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q, q^{-1}]$. We define a partial order on $\mathcal{B}$ by

$$q^e[X^v] \leq q^d[X^u] \iff \Gamma_{d-e}(X_u, X^v) \neq \emptyset \iff X_v \subset \Gamma_{d-e}(X_u)\,.$$

Here the sets $\Gamma_{d-e}(X_u, X^v)$ and $\Gamma_{d-e}(X_u)$ can be non-empty only if $d \geq e$. Notice that if $q^f[X^w] \leq q^e[X^v] \leq q^d[X^u]$, then $X_w \subset \Gamma_{e-f}(X_v) \subset \Gamma_{e-f}(\Gamma_{d-e}(X_u)) \subset \Gamma_{d-f}(X_u)$ shows that $q^f[X^w] \leq q^d[X^u]$. The order on $\mathcal{B}$ extends the Bruhat order on W^X, and [FW04, Thm. 9.1] implies that $q^e[X^v] \leq q^d[X^u]$ holds if and only if $q^d[X^u]$ occurs with non-zero coefficient in the expansion of $q^e[X^v] \star q^{d'}[X^w]$ in

$\mathrm{QH}(X)_q$, for some $w \in W^X$ and $d' \geq 0$ [2]. Notice that $[\text{point}] \star [X^v] = q^{d_{\max}(v)}[X^{w_0^X v}]$ is an element of $\mathcal{B}$, as proved in [Bel04, CMP09].

Theorem 7.8. *Let $u, v \in W^X$ and $d \in \mathbb{Z}$. The power q^d occurs in $[X_u] \star [X^v]$ if and only $[X^v] \leq q^d[X^u] \leq [\text{point}] \star [X^v]$.*

Proof. The smallest degree d for which $[X^v] \leq q^d[X^u]$ is $d_{\min}(u^\vee, v)$ by definition of the partial order on $\mathcal{B}$, and the largest degree d for which $q^d[X^u] \leq q^{d_{\max}(v)}[X^{w_0^X v}]$ is $d_{\max}(u^\vee, v)$ by Theorem 7.7. $\square$

Our next goal is to show that $\mathcal{B}$ is a distributive lattice. In the remainder of this section we extend earlier definitions by setting $u(d) = z_d = w_0^X$ and $u(-d) = 1$ for $d \geq d_X(2)$ and $u \in W^X$.

Lemma 7.9. *Let $u, v \in W^X$, $d \in \mathbb{Z}$, and $e \in \mathbb{N}$. The following identities hold.*

$$(u \cup v)(d) = u(d) \cup v(d) \qquad (u \cap v)(d) = u(d) \cap v(d)$$
$$(u \cup v(d))(e) = u(e) \cup v(d+e) \qquad (u(d) \cap v)(-e) = u(d-e) \cap v(-e)$$

Proof. The identities $(u \cup v)(d) = u(d) \cup v(d)$ and $(u \cap v)(d) = u(d) \cap v(d)$ follow from Lemma 6.6, which also implies that $u(-e)(e) = u \cup z_e$. We claim that $u(d)(e) = u(d+e) \cup z_e$. For $d \geq 0$ it follows from Theorem 5.1 that $\Gamma_e(\Gamma_d(X_u)) = \Gamma_{d+e}(X_u)$, which implies $u(d)(e) = u(d+e) = u(d+e) \cup z_e$ by Corollary 6.9(c). A dual argument shows that $u(-d)(-e) = u(-d-e)$ for $d \geq 0$. For $e' \geq 0$ we obtain $u(-e-e')(e) = u(-e')(-e)(e) = u(-e') \cup z_e$ and $u(-e)(e+e') = u(-e)(e)(e') = (u \cup z_e)(e') = u(e') \cup z_{e+e'}$. This proves all cases of $u(d)(e) = u(d+e) \cup z_e$. Using that $z_e \leq u(e)$, we obtain $u(e) \cup v(d+e) = u(e) \cup z_e \cup v(d+e) = u(e) \cup v(d)(e) = (u \cup v(d))(e)$. The last identity of the lemma follows from this, using that $u(d)^\vee = u^\vee(-d)$ by Proposition 6.7(c). $\square$

Proposition 7.10. *The partially ordered set $\mathcal{B} = \{q^d[X^u] : u \in W^X, d \in \mathbb{Z}\}$ is a distributive lattice with meet and join operations given by*

$$q^d[X^u] \cap q^e[X^v] = q^e[X^{u(d-e) \cap v}] \quad \text{and} \quad q^d[X^u] \cup q^e[X^v] = q^d[X^{v(e-d) \cup u}]$$

for $u, v \in W^X$ and $e \leq d$.

Proof. In this proof we denote $q^d[X^u]$ by $[d, u]$ for brevity. Let $u, v, w \in W^X$ and $d, e, f \in \mathbb{Z}$. The partial order on $\mathcal{B} = \mathbb{Z} \times W^X$ is defined by

$$[e, v] \leq [d, u] \;\Leftrightarrow\; \big(e \leq d \text{ and } v \leq u(d-e)\big) \;\Leftrightarrow\; \big(e \leq d \text{ and } v(e-d) \leq u\big)\,.$$

We first show that $[e, u(d-e) \cap v]$ is the greatest lower bound of $[d, u]$ and $[e, v]$ when $e \leq d$. The relations $[e, u(d-e) \cap v] \leq [d, u]$ and $[e, u(d-e) \cap v] \leq [e, v]$ follow from the definition. If $[f, w] \leq [d, u]$ and $[f, w] \leq [e, v]$, then

$$w \leq u(d-f) \cap v(e-f) = (u(d-e) \cap v)(e-f)\,,$$

hence $[f, w] \leq [e, u(d-e) \cap v]$. This proves $[d, u] \cap [e, v] = [e, u(d-e) \cap v]$. Noting that the map $[d, u] \mapsto [-d, u^\vee]$ is an order-reversing involution of $\mathcal{B}$, the expression $[d, u] \cup [e, v] = [d, u \cup v(e-d)]$ for the least upper bound is equivalent to the expression for the greatest lower bound. This shows that $\mathcal{B}$ is a lattice. To prove distributivity, assume again that $e \leq d$, and set $n = \max(e, f)$ and $m = \max(d, f)$.

[2] Our construction defines a partial order on $\mathcal{B} = \{q^d[M^u] : u \in W^M, d \in H_2(M, \mathbb{Z})\}$ for any flag variety M, with the same interpretation in terms of quantum multiplication.

Notice that $u(d-m)(m-n) = u(d-n)$, as we have either $m = d$ or $m = f = n$. Using Lemma 7.9 we obtain

$$
\begin{aligned}
&([d,u] \cup [f,w]) \cap ([e,v] \cup [f,w]) \\
&= [m, u(d-m) \cup w(f-m)] \cap [n, v(e-n) \cup w(f-n)] \\
&= [n, (u(d-m) \cup w(f-m))(m-n) \cap (v(e-n) \cup w(f-n))] \\
&= [n, (u(d-m)(m-n) \cup w(f-n)) \cap (v(e-n) \cup w(f-n))] \\
&= [n, (u(d-n) \cup w(f-n)) \cap (v(e-n) \cup w(f-n))] \\
&= [n, (u(d-n) \cap v(e-n)) \cup w(f-n)] \\
&= [n, (u(d-e) \cap v)(e-n) \cup w(f-n)] \\
&= [e, u(d-e) \cap v] \cup [f,w] \\
&= ([d,u] \cap [e,v]) \cup [f,w] \,.
\end{aligned}
$$

Since this identity is formally equivalent to

$$([d,u] \cup [e,v]) \cap [f,w] = ([d,u] \cap [f,w]) \cup ([e,v] \cap [f,w]) \,,$$

this completes the proof. □

Definition 7.11. An element $\widehat{\alpha} \in \mathcal{B}$ is called *join-irreducible* if $\widehat{\alpha} = \widehat{\alpha}_1 \cup \widehat{\alpha}_2$ implies $\widehat{\alpha} = \widehat{\alpha}_1$ or $\widehat{\alpha} = \widehat{\alpha}_2$, for $\widehat{\alpha}_1, \widehat{\alpha}_2 \in \mathcal{B}$. Let $\widehat{\mathcal{P}}_X \subset \mathcal{B}$ denote the subset of join-irreducible elements. Given $q^d[X^u] \in \mathcal{B}$, set

$$I(q^d[X^u]) = \{\widehat{\alpha} \in \widehat{\mathcal{P}}_X \mid \widehat{\alpha} \leq q^d[X^u]\} \,.$$

For $\alpha \in \mathcal{P}_X$, define $\partial(\alpha) \in \mathbb{N}$, $\xi(\alpha) \in W^X$, and $\tau(\alpha) \in \mathcal{B}$ by

$$
\begin{aligned}
\partial(\alpha) &= \min\{d \geq 0 \mid (z_1 s_\gamma)^{-d}.\alpha \in \mathcal{P}_X \smallsetminus I(z_1^\vee)\} \,, \\
I(\xi(\alpha)) &= \{\alpha' \in \mathcal{P}_X \mid \alpha' \leq \alpha\} \,, \text{ and} \\
\tau(\alpha) &= q^{-\partial(\alpha)}[X^{\xi(\beta)}] \,, \quad \text{where} \quad \beta = (z_1 s_\gamma)^{-\partial(\alpha)}.\alpha \,.
\end{aligned}
$$

The integer $\partial(\alpha)$ exists by Proposition 5.10(a). For example, we have

$$\tau(\gamma) = q^{1-d_X(2)}[X^{\kappa_{d_X(2)}}] \quad \text{and} \quad \tau(\rho) = [\text{point}] \,, \tag{4}$$

where $\rho \in \Phi^+$ denotes the highest root. For any box $\alpha \in \mathcal{P}_X$, we will show in Theorem 7.13 that $[X^{\xi(\alpha)}] = \tau(\alpha) \cup 1$ holds in $\mathcal{B}$, and $[X^{\xi(\alpha)}]$ is join-irreducible if and only if $\alpha \in \mathcal{P}_X \smallsetminus I(z_1^\vee)$. This motivates the definition of $\tau(\alpha)$.

Lemma 7.12. *Let $\alpha \in \mathcal{P}_X$ and $q^d[X^u] \in \mathcal{B}$. We have $\tau(\alpha) \leq q^d[X^u]$ in $\mathcal{B}$ if and only if $\alpha \in I(u(d))$.*

Proof. Set $e = \partial(\alpha)$ and $\beta = (z_1 s_\gamma)^{-e}.\alpha$, so that $\tau(\alpha) = q^{-e}[X^{\xi(\beta)}]$. We then have $\tau(\alpha) \leq q^d[X^u]$ if and only if $-e \leq d$ and $\xi(\beta) \leq u(d+e)$, or equivalently, $d+e \geq 0$ and $\beta \in I(u(d+e)))$. Since $\beta \in \mathcal{P}_X \smallsetminus I(z_1^\vee)$, the condition $d+e \geq 0$ follows from $\beta \in I(u(d+e))$. Using that

$$I(u(d+e)) \cup I(z_e) \;=\; I(u(d)(e)) \;=\; (z_1 s_\gamma)^{-e}\big(I(u(d)) \cap I(z_e^\vee)\big) \cup I(z_e)$$

by Lemma 7.9 and Lemma 6.6, and that $\beta \notin I(z_e)$ and $\alpha \in I(z_e^\vee)$, we deduce that $\beta \in I(u(d+e))$ is equivalent to $\alpha \in I(u(d))$, as required. □

Part (d) of the following result is essentially a consequence of Birkhoff's representation theorem [Bir37] together with Proposition 7.10; we supply a proof since $\widehat{\mathcal{P}}_X$ is an infinite set.

Theorem 7.13. (a) *We have* $[X^{\xi(\alpha)}] = \tau(\alpha) \cup 1$ *in* $\mathcal{B}$ *for each* $\alpha \in \mathcal{P}_X$.

(b) *We have* $\widehat{\mathcal{P}}_X = \{q^d[X^{\xi(\alpha)}] : \alpha \in \mathcal{P}_X \smallsetminus I(z_1^\vee),\, d \in \mathbb{Z}\} \cup \{q^d : d \in \mathbb{Z}\}$.

(c) *The map* $\tau : \mathcal{P}_X \to \tau(\mathcal{P}_X)$ *is an order isomorphism onto an interval in* $\widehat{\mathcal{P}}_X$.

(d) *The map* $q^d[X^u] \mapsto I(q^d[X^u])$ *is an order isomorphism of* $\mathcal{B}$ *with the set of non-empty, proper, lower order ideals in* $\widehat{\mathcal{P}}_X$, *ordered by inclusion.*

Proof. If $q^d = q^e[X^v] \cup q^d[X^u]$, then $e \leq d$ and $v(e-d) \cup u = 1$, hence $q^d[X^u] = q^d$. This shows that q^d is join-irreducible. Let $q^d[X^u] \in \mathcal{B}$ satisfy $u \neq 1$. If $q^d[X^u]$ is join-irreducible, then u is join-irreducible in W^X, so $u = \xi(\alpha)$ for some $\alpha \in \mathcal{P}_X$. If $\alpha \in I(z_1^\vee)$, then $\beta = (z_1 s_\gamma)^{-1}.\alpha \in \mathcal{P}_X$ by Proposition 5.10(a), and it follows from Lemma 6.6 that $u = \xi(\beta)(-1)$. But then $q^d[X^u] = q^d \cup q^{d-1}[X^{\xi(\beta)}]$ is not join-irreducible, a contradiction. On the other hand, assume that $u = \xi(\alpha)$ where $\alpha \in \mathcal{P}_X \smallsetminus I(z_1^\vee)$. If $q^d[X^u] = q^e[X^v] \cup q^d[X^{u'}]$ with $e < d$, then $u = v(e-d) \cup u'$. Since $\alpha \notin I(v(e-d))$, we have $\alpha \in I(u')$, so $u' = u$. This proves part (b).

Let $\alpha', \alpha \in \mathcal{P}_X$, and set $u = \xi((z_1 s_\gamma)^{-\partial(\alpha)}.\alpha)$. Using that $u(-\partial(\alpha)) = \xi(\alpha)$ by Lemma 6.6, it follows from Lemma 7.12 that $\tau(\alpha') \leq \tau(\alpha)$ holds if and only if $\alpha' \in I(\xi(\alpha))$, which is equivalent to $\alpha' \leq \alpha$. This shows that $\tau : \mathcal{P}_X \to \tau(\mathcal{P}_X)$ is an order isomorphism. To see that $\tau(\mathcal{P}_X)$ is an interval in $\widehat{\mathcal{P}}_X$, assume that $\tau(\gamma) \leq q^d[X^{\xi(\alpha)}] \leq \tau(\rho)$, where $\alpha \in \mathcal{P}_X \smallsetminus I(z_1^\vee)$ and $d \in \mathbb{Z}$. By Lemma 7.12 and (4), this is equivalent to $1 - d_X(2) \leq d \leq 0$ and $I(\xi(\alpha)(d)) \neq \emptyset$. We deduce that $(z_1 s_\gamma)^e.\alpha \in \mathcal{P}_X \smallsetminus I(z_1)$ for $0 \leq e < -d$, and $q^d[X^{\xi(\alpha)}] = \tau((z_1 s_\gamma)^{-d}.\alpha) \in \tau(\mathcal{P}_X)$. This proves part (c).

Given $\alpha \in \mathcal{P}_X$, Proposition 7.10 implies that $\tau(\alpha) \cup 1 = [X^u]$ for some $u \in W^X$. Since $\tau(\alpha') \not\leq 1$ for each $\alpha' \in \mathcal{P}_X$ by Lemma 7.12, another application of Lemma 7.12 shows that $\alpha' \in I(u)$ holds if and only if $\tau(\alpha') \leq \tau(\alpha)$, so it follows from part (c) that $u = \xi(\alpha)$. This proves part (a).

Let $I \subset \widehat{\mathcal{P}}_X$ be any non-empty, proper, lower order ideal. Since I has finitely many maximal elements, say $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_\ell$, it follows from Proposition 7.10 that I has a well-defined least upper bound $q^d[X^u] = \widehat{\alpha}_1 \cup \cdots \cup \widehat{\alpha}_\ell$ in $\mathcal{B}$. For any element $\widehat{\beta} \in I(q^d[X^u])$, we obtain

$$\widehat{\beta} = \widehat{\beta} \cap q^d[X^u] = (\widehat{\beta} \cap \widehat{\alpha}_1) \cup \cdots \cup (\widehat{\beta} \cap \widehat{\alpha}_\ell)\,.$$

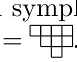
Since $\widehat{\beta}$ is join-irreducible, this implies $\widehat{\beta} = \widehat{\beta} \cap \widehat{\alpha}_i$ for some i, so $\widehat{\beta} \in I$. We deduce that $I = I(q^d[X^u])$. Part (d) follows from this, noting that any element $q^d[X^u] \in \mathcal{B}$ is the least upper bound of the finite set $\{q^d\} \cup \{q^d \tau(\alpha) \mid \alpha \in I(u)\}$ by part (a). □

Remark 7.14. By Theorem 7.13(c), we may identify $\mathcal{P}_X$ with the subset $\tau(\mathcal{P}_X)$ of $\widehat{\mathcal{P}}_X$. Lemma 7.12 shows that the shift operations on W^X can be expressed as

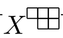
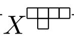
$$I(u(d)) \;=\; I(q^d[X^u]) \cap \mathcal{P}_X \;=\; q^d I([X^u]) \cap \mathcal{P}_X$$

for all $u \in W^X$ and $d \in \mathbb{Z}$. Theorem 7.8 and Theorem 7.13(d) show that q^d occurs in $[X_u] \star [X^v]$ if and only if the order ideal of $q^d[X^u]$ lies between the order ideals of $[X^v]$ and $[\text{point}] \star [X^v]$. When X is a Grassmannian of type A, $\widehat{\mathcal{P}}_X$ is Postnikov's *cylinder* from [Pos05, §3]. The analogue of Postnikov's *torus* is the quotient of $\widehat{\mathcal{P}}_X$ by the group $\{[\text{point}]^d \mid d \in \mathbb{Z}\}$ of powers of a point. Pictures of $\widehat{\mathcal{P}}_X$ for cominuscule varieties of types other than A can be found in Example 7.15 and Figure 1.

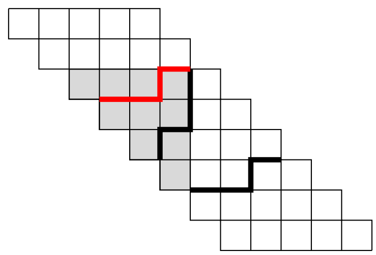
The partially ordered sets $\widehat{\mathcal{P}}_X$ are isomorphic to certain *full heaps* of affine Dynkin diagrams that were defined in [Gre13, Ch. 6] based on a type-by-type construction, and used to study minuscule representations. Postnikov's cylinder was constructed in [Hag04, §8] from a similar viewpoint.

Example 7.15. Let $X = \mathrm{LG}(4,8)$ be the Lagrangian Grassmannian of maximal isotropic subspaces in an 8-dimensional symplectic vector space, and define $u, v \in W^X$ by the shapes $I(u) =$ and $I(v) =$ . We have

$$[X_u] \star [X^v] = q^2 [X^{\square}] + q^2 [X^{\square}] + q^3$$

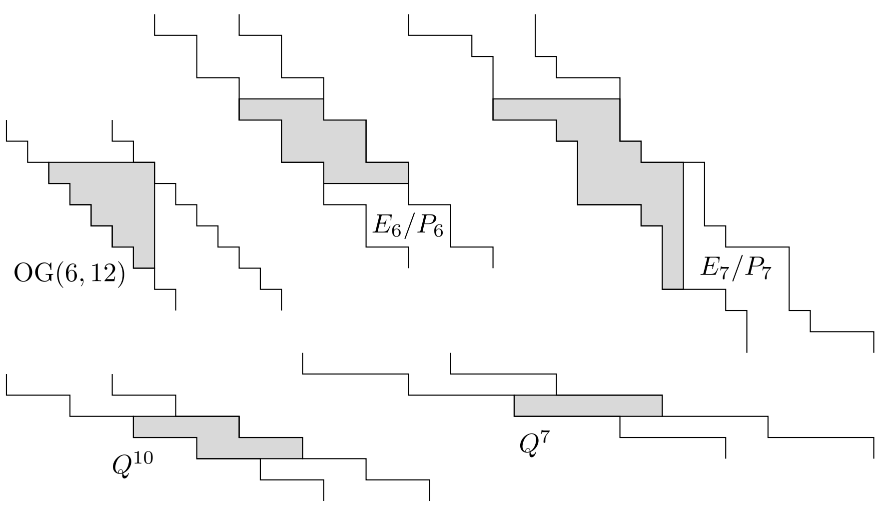
 

in $\mathrm{QH}(X)$, so $d_{\min}(u^\vee, v) = 2$ and $d_{\max}(u^\vee, v) = 3$. The following picture shows a section of the partially ordered set $\widehat{\mathcal{P}}_X$, with the boxes of $\mathcal{P}_X$ colored gray.



The south-east borders of the order ideals of $[X^v]$ and $[\mathrm{point}] \star [X^v]$ are colored black, and the south-east border of the order ideal of $[X^u]$ is colored red. The order ideal of $[\mathrm{point}] \star [X^v]$ is obtained by reflecting $I(v)$ in a diagonal line and attaching the result to the right side of $\mathcal{P}_X$; this follows from [BS16, Lemma 2.9], by observing that multiplication by a point preserves the partial order on $\mathcal{B}$. Notice that the red border will fit between the two black borders if it is shifted south-east by 2 or 3 steps.

FIGURE 1. The partially ordered sets $\widehat{\mathcal{P}}_X$ for a collection of cominuscule flag varieties, with the boxes of $\mathcal{P}_X$ colored gray.



Remark 7.16. Let α_0 be the added simple root of the affine root system corresponding to G. For any affine root $\theta = n_0\alpha_0 + \sum_{\beta\in\Delta} n_\beta \beta$, let $\lambda(\theta) = n_\gamma - n_0$. Since γ is cominuscule, we have $\lambda(\theta) \in \{-1, 0, 1\}$. We plan to prove in a follow-up paper with Nicolas Ressayre that $\widehat{\mathcal{P}}_X$ is isomorphic to the partially ordered set of affine roots θ for which $\lambda(\theta) = 1$, where the order on this set is defined by the covering relation $\theta_1 \lessdot \theta_2$ if and only if $\theta_2 - \theta_1$ is a positive affine root.

7.3. **Proof of the main lemma.** Our proof of Lemma 7.2 utilizes a relationship between all the cominuscule flag varieties $F = G/P_F$ of the same group G. Let $W^{\text{comin}} \subset W$ denote the set of representatives of single points in these varieties, together with the identity element:

$$W^{\text{comin}} \ = \ \{w_0^F \mid F \text{ is a cominuscule flag variety of } G\ \} \cup \{1\}\,.$$

For each cominuscule root $\gamma \in \Delta$ we let $F_\gamma = G/P_\gamma$ denote the corresponding cominuscule flag variety. The following result was used to determine the Seidel representation on the quantum cohomology ring of any flag variety in [CMP09], see also [Bou81, Prop. VI.2.6].

Proposition 7.17. *The set W^{comin} is a subgroup of W isomorphic to the coweight lattice of Φ modulo the coroot lattice. The isomorphism maps $w_0^{F_\gamma}$ to the class of the fundamental coweight $\omega_\gamma^\vee$ corresponding to γ.*

We mostly need this result when G has Lie type A. Let $w_0^{\text{Gr}(d,n)} \in S_n$ denote the permutation representing the point class on $\text{Gr}(d,n)$. This permutation is determined by $w_0^{\text{Gr}(d,n)}(p) \equiv p - d \pmod{n}$ for $p \in [1,n]$. The following consequence of Proposition 7.17 is also immediate from this description.

Corollary 7.18. *The assignment $d \mapsto w_0^{\text{Gr}(d,n)}$ defines an isomorphism of groups $\mathbb{Z}/n\mathbb{Z} \to S_n^{\text{comin}}$.*

Corollary 7.19. *For $0 \leq d \leq d_X(2)$ we have $z_d/\kappa_d = (z_1/\kappa_1)^d = (z_1 s_\gamma)^d$.*

Proof. Since $z_d/\kappa_d = w_{0,X}^{Z_d}$ represents a point in $F_d = P_X/P_{Z_d}$ by Lemma 6.1, we can prove the identity by applying Proposition 7.17 to the Weyl group W_X of P_X. If X is a Grassmannian of type A, a Lagrangian Grassmannian, or a maximal orthogonal Grassmannian, then the Levi subgroup of P_X is a group of type A (or a product of two such groups), and the identity follows from Corollary 7.18 and Table 3. If X is a quadric hypersurface, then F_1 is also a quadric, F_2 is a point, and the identity follows from Proposition 5.16(4) because F_1 is primitive. Finally, if X is the Cayley plane E_6/P_6 or the Freudenthal variety E_7/P_7, the identity follows from Table 3 together with the isomorphisms $W_{D_5}^{\text{comin}} \cong \mathbb{Z}/4\mathbb{Z}$ and $W_{E_6}^{\text{comin}} \cong \mathbb{Z}/3\mathbb{Z}$. $\square$

Lemma 7.20. *For $1 \leq d \leq d_X(2)$ we have $(z_1 s_\gamma)^{d-1}.(I(z_d) \smallsetminus I(z_{d-1})) = I(z_1 \cap z_{d-1}^\vee)$ and $I(\kappa_d) \cup w_{0,X} z_1 s_\gamma.I(z_1 \cap z_{d-1}^\vee) = I(z_1 \cup \kappa_d)$.*

Proof. Recall the definition of the set $\mathcal{S}_d$ before Proposition 5.13. Noting that $I(z_d) \smallsetminus I(z_{d-1}) = \mathcal{S}_d \cap (\mathcal{P}_X \smallsetminus I(z_{d-1}))$ and $I(z_1 \cap z_{d-1}^\vee) = \mathcal{S}_1 \cap I(z_{d-1}^\vee)$, the first identity of the lemma follows from Proposition 5.13, Proposition 5.9(a), and Corollary 5.11. Proposition 5.10 implies that

$$\begin{aligned} w_{0,X} z_1 s_\gamma.I(z_1 \cap z_{d-1}^\vee) \ &= \ w_{0,X} z_1 s_\gamma.I(z_1) \cap (z_1 s_\gamma)^{-1} w_{0,X}.I(z_{d-1}^\vee) \\ &= \ I(z_1) \smallsetminus (z_1 s_\gamma)^{-1}.I(z_{d-1})\,, \end{aligned}$$

so the second identity is equivalent to

$$I(z_1) \cap (z_1 s_\gamma)^{-1}.I(z_{d-1}) \subset I(\kappa_d)\,.$$

Since $(z_1 s_\gamma)^{-1}.I(z_{d-1}) = \bigcup_{e=2}^{d} \mathcal{S}_e$ by Proposition 5.13, it suffices to show that $I(z_1) \cap \mathcal{S}_d \subset I(\kappa_d)$ for $d \geq 2$. This follows from the definition of $\mathcal{S}_d$, as $I(z_1) \cap (I(z_d) \smallsetminus I(z_{d-1})) = \emptyset$. □

Proof of Lemma 7.2. The definition of v^d is equivalent to $v^d \kappa_d = (v \cap z_d) \cup \kappa_d$, which specializes to $v^d \kappa_d = v \cap z_d$ for $d \leq d_{\max}(v)$. Part (b) follows from this. For part (a), let $v = u^\vee$ be dual to u in W^X. Then Proposition 6.7(c) shows that u_d is dual to v^d in W^{F_d}, and u_{d-1} is dual to v^{d-1} in $W^{F_{d-1}}$. Since Corollary 7.19 and Lemma 6.1 show that $w_{0,X}(z_1 s_\gamma)^d = w_{0,Z_d} = \kappa_d w_{0,Z_d} \kappa_d$, it follows from Proposition 6.2(a) that

$$\begin{aligned} w_{0,X}(z_1 s_\gamma)^d.(I(z_d) \smallsetminus I(v^d \kappa_d)) &= \kappa_d w_{0,Z_d} \kappa_d.(I(z_d) \smallsetminus I(v^d \kappa_d)) \\ &= \kappa_d w_{0,Z_d}.(\mathcal{P}_{F_d} \smallsetminus I(v^d)) = \kappa_d.I(u_d) \\ &= I(u_d \kappa_d) \smallsetminus I(\kappa_d)\,, \end{aligned}$$

and part (b) implies that

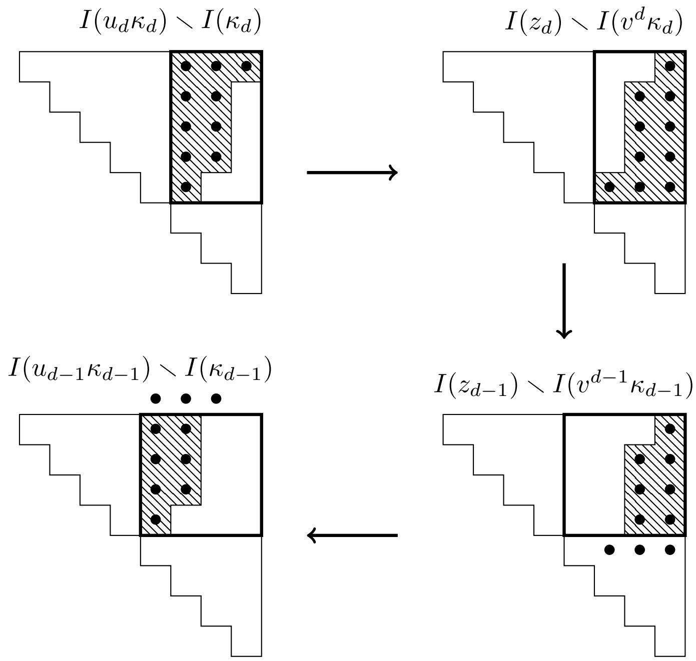
$$\begin{aligned} I(z_{d-1}) \smallsetminus I(v^{d-1} \kappa_{d-1}) &= (I(z_d) \smallsetminus I(v^d \kappa_d)) \cap I(z_{d-1}) \\ &= (I(z_d) \smallsetminus I(v^d \kappa_d)) \smallsetminus (I(z_d) \smallsetminus I(z_{d-1}))\,. \end{aligned}$$

By combining these identities and using Lemma 7.20 and Lemma 6.6, we obtain

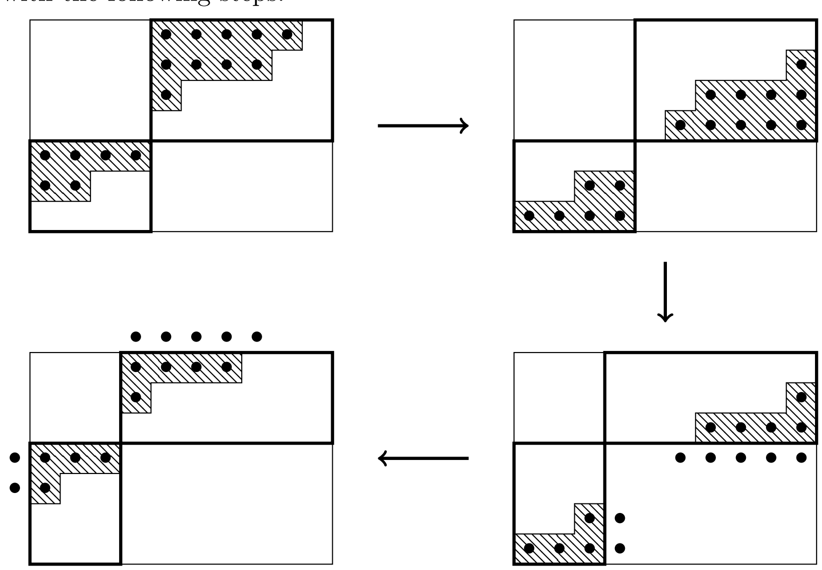
$$\begin{aligned} I(u_{d-1}\kappa_{d-1}) \smallsetminus I(\kappa_{d-1}) &= w_{0,X}(z_1 s_\gamma)^{d-1}.(I(z_{d-1}) \smallsetminus I(v^{d-1}\kappa_{d-1})) \\ &= z_1 s_\gamma w_{0,X}(z_1 s_\gamma)^d.((I(z_d) \smallsetminus I(v^d \kappa_d)) \smallsetminus (I(z_d) \smallsetminus I(z_{d-1}))) \\ &= z_1 s_\gamma. \left((I(u_d \kappa_d) \smallsetminus I(\kappa_d)) \smallsetminus w_{0,X} z_1 s_\gamma.I(z_1 \cap z_{d-1}^\vee)\right) \\ &= z_1 s_\gamma. (I(u_d \kappa_d) \smallsetminus I(z_1 \cup \kappa_d)) \\ &= z_1 s_\gamma. (I(u_d \kappa_d) \smallsetminus I(z_1)) \smallsetminus z_1 s_\gamma. (I(\kappa_d) \smallsetminus I(z_1)) \\ &= I((u_d \kappa_d)(-1)) \smallsetminus I(\kappa_{d-1})\,. \end{aligned}$$

This proves part (a). □

Example 7.21. Let $X = \mathrm{LG}(8, 16)$, $u = (8, 6, 2) \in W^X$, and set $d = 5$. Then $u_d = (5, 4, 1)$ is obtained by intersecting $I(u)$ with the rectangle $z_d.\mathcal{P}_{F_d}$, see Example 6.4 and Example 6.8. The following pictures illustrate how the skew shape $I(u_{d-1}\kappa_{d-1}) \smallsetminus I(\kappa_{d-1})$ is obtained from $I(u_d \kappa_d) \smallsetminus I(\kappa_d)$. Notice that, since the bijection $z_d.\mathcal{P}_{F_d} \cong \kappa_d.\mathcal{P}_{F_d}$ is given by a transposition, the partition of u_d is conjugated when we form $I(u_d \kappa_d) \smallsetminus I(\kappa_d)$.



Example 7.22. Let $X = \mathrm{Gr}(7, 17)$, $u = (10, 8, 5, 5, 4, 1, 0) \in W^X$, and $d = 4$. Then $F_d = \mathrm{Gr}(3, 7) \times \mathrm{Gr}(4, 10)$, so elements of W^{F_d} can be represented by pairs of partitions. We find $u_d = ((4, 2, 0), (5, 4, 1, 0))$ by intersecting $I(u)$ with $z_d.\mathcal{P}_{F_d}$, see Example 6.3. The skew shape $I(u_{d-1}\kappa_{d-1}) \smallsetminus I(\kappa_{d-1})$ is obtained from $I(u_d\kappa_d) \smallsetminus I(\kappa_d)$ with the following steps.



8. Results about quantum K-theory

8.1. **The small quantum K-theory ring.** Let $X = G/P_X$ be a cominuscule flag variety. The (small) quantum K-theory ring $\mathrm{QK}(X)$ of Givental and Lee

[Giv00, Lee01] is an algebra over the ring $\mathbb{Z}[\![q]\!]$ of formal power series in a single variable q called the *deformation parameter*. As a $\mathbb{Z}[\![q]\!]$-module we have $\mathrm{QK}(X) = K(X) \otimes \mathbb{Z}[\![q]\!]$. The associative product $\star$ of $\mathrm{QK}(X)$ is defined in terms of K-theoretic Gromov-Witten invariants. We recall a construction of this product from [BCMP18a].

Let $\psi : K(X) \to K(X)$ be the linear map defined by $\psi(\mathcal{O}^w) = \mathcal{O}^{w(-1)}$. This map can also be defined by $\psi = (\mathrm{ev}_2)_*(\mathrm{ev}_1)^*$, where ev_1 and ev_2 are the evaluation maps from $\overline{\mathcal{M}}_{0,2}(X,1)$. Corollary 5.23 implies that $\psi = (p_1)_*(q_1)^*(q_1)_*(p_1)^*$. Given $u, v \in W^X$ and $d \geq 1$, we define the class

$$(\mathcal{O}_u \star \mathcal{O}^v)_d \;=\; [\mathcal{O}_{\Gamma_d(X_u,X^v)}] - \psi([\mathcal{O}_{\Gamma_{d-1}(X_u,X^v)}]) \tag{5}$$

in $K(X)$. Let $(\mathcal{O}_u \star \mathcal{O}^v)_0 = \mathcal{O}_u \cdot \mathcal{O}^v$ be the product in the K-theory ring. It then follows from [BCMP18a, Prop. 3.2] that Givental's product in $\mathrm{QK}(X)$ is given by

$$\mathcal{O}_u \star \mathcal{O}^v \;=\; \sum_{d \geq 0} (\mathcal{O}_u \star \mathcal{O}^v)_d \, q^d \,. \tag{6}$$

The proof in [BCMP18a] showing that (6) agrees with Givental's definition relies on a version of the quantum-to-classical principle for large degrees that was established in [BM11, CP11, BCMP18b].

The definition (5) implies that $(\mathcal{O}_u \star \mathcal{O}^v)_d = 0$ for all sufficiently large degrees d, since eventually we have $\Gamma_{d-1}(X_u, X^v) = X$. As a consequence, the product $\mathcal{O}_u \star \mathcal{O}^v$ contains only finitely many non-zero terms. A similar finiteness result is known for the quantum K-theory of arbitrary flag varieties [BCMP13, BCMP16, Kat, ACT22]. In the cominuscule case we have $(\mathcal{O}_u \star \mathcal{O}^v)_d = 0$ whenever $d > d_X(2)$ by [BCMP13, Thm. 1]. Using this explicit bound, we can focus on the terms $(\mathcal{O}_u \star \mathcal{O}^v)_d$ of small degrees, which will be studied using the tools developed in the previous sections.

The Schubert structure constants of $\mathrm{QK}(X)$ are the integers $N_{u,v}^{w,d}$ defined by

$$\mathcal{O}^u \star \mathcal{O}^v \;=\; \sum_{w, d \geq 0} N_{u,v}^{w,d} \, q^d \, \mathcal{O}^w \,. \tag{7}$$

Equivalently, we have $(\mathcal{O}_u \star \mathcal{O}^v)_d = \sum_w N_{u^\vee,v}^{w,d} \mathcal{O}^w$ for each degree d. These structure constants are expected to have alternating signs in the following sense [LM06, BM11].

Conjecture 8.1. *We have* $(-1)^{\ell(uvw)+\deg(q^d)} N_{u,v}^{w,d} \geq 0$.

Here $\deg(q^d) = d \int_{X_{s_\gamma}} c_1(T_X) = d\,(\ell(z_1) + 1)$ denotes the degree of q^d in the quantum cohomology ring $\mathrm{QH}(X)$. This conjecture generalizes the fact that the structure constants $N_{u,v}^{w,0}$ of the ordinary K-theory ring of X have alternating signs [Buc02, Bri02]. A more general version of Conjecture 8.1 for the equivariant quantum K-theory of arbitrary flag varieties is discussed in [BCMP18a, §2.4].

Recall from Corollary 6.9(d) that the general fibers of the map $p_d : Z_d(X_u, X^v) \to \Gamma_d(X_u, X^v)$ are translates of $(F_d)_{u_d}^{u_d \cap v^d}$. The quotient $(u_d \cup v^d)/v^d$ of Weyl group elements will be called a *short rook strip* if it is a product of commuting reflections associated to short simple roots of Φ. We emphasize that the lengths of the simple roots should be measured relative to the root system Φ of G, as opposed to the root system Φ_X of the group P_X acting on F_d. For example, when $X = \mathrm{LG}(n, 2n)$ is a Lagrangian Grassmannian, Φ_X is a root system of type A, but its roots are considered short since they are short in Φ.

Definition 8.2. Let $u, v \in W^X$. An integer d is an *exceptional degree* of the product $\mathcal{O}_u \star \mathcal{O}^v$ if $d \leq \min(d_{\max}(u^\vee), d_{\max}(v))$ and $(u_d \cup v^d)/v^d$ is a non-empty short rook strip.

Notice that exceptional degrees do not occur when X is minuscule, as all roots of a simply laced root system are considered long. In addition, most products $\mathcal{O}_u \star \mathcal{O}^v$ on odd quadrics Q^{2n-1} and Lagrangian Grassmannians $\mathrm{LG}(n, 2n)$ have no exceptional degrees, see Example 8.5 and Table 4. Notice also that, if d is an exceptional degree of $\mathcal{O}_u \star \mathcal{O}^v$, then we must have $d = d_{\max}(u^\vee, v) + 1$ by Lemma 7.3(b). We proceed to state our main results about quantum K-theory of cominuscule flag varieties.

Theorem 8.3. *Let $u, v \in W^X$. The quantum K-theory product $\mathcal{O}_u \star \mathcal{O}^v$ in $\mathrm{QK}(X)$ contains the same powers of q as the quantum cohomology product $[X_u] \star [X^v]$ in $\mathrm{QH}(X)$, with the exception that q^d may also occur in $\mathcal{O}_u \star \mathcal{O}^v$ if $d = d_{\max}(u^\vee, v) + 1$ is an exceptional degree. In particular, the powers q^d occurring in $\mathcal{O}_u \star \mathcal{O}^v$ form an integer interval.*

We conjecture that $(\mathcal{O}_u \star \mathcal{O}^v)_d \neq 0$ whenever d is an exceptional degree. This is true for quadrics and has been verified for Lagrangian Grassmannians $\mathrm{LG}(n, 2n)$ with $n \leq 6$. See Conjecture 8.28 for a more detailed statement.

Theorem 8.4. *Conjecture 8.1 is true whenever X is minuscule or any quadric hypersurface. It is also true whenever d is not an exceptional degree of the product $\mathcal{O}_{u^\vee} \star \mathcal{O}^v$.*

Example 8.5. Let $X = Q^{2n-1} = \mathrm{OG}(1, 2n+1)$ be a quadric hypersurface of type B_n. We have $d_X(2) = 2$, $\deg(q) = 2n - 1$, and

$$\mathcal{P}_X = \boxed{1}\boxed{2}\boxed{\cdots}\boxed{n{-}1}\boxed{n}\boxed{n{-}1}\boxed{\cdots}\boxed{2}\boxed{1}\,.$$

All boxes of $\mathcal{P}_X$ are long except the middle box. Notice that $\kappa_1.\mathcal{P}_{F_1} = z_1.\mathcal{P}_{F_1}$ consists of the middle $2n-3$ boxes of $\mathcal{P}_X$. It follows that $\mathcal{O}_u \star \mathcal{O}^v$ has an exceptional degree if and only if $I(u) \smallsetminus I(v)$ consists of the middle box. In other words, the only exceptional product is $\mathcal{O}^{n-1} \star \mathcal{O}^{n-1} = \mathcal{O}_n \star \mathcal{O}^{n-1}$; here we denote each element $u \in W^X$ by its length $\ell(u)$. Using the Chevalley formula from [BCMP18a] together with the associativity of the quantum K-theory product, we obtain

$$\mathcal{O}^{n-1} \star \mathcal{O}^{n-1} = \mathcal{O}^{n-1} \star (\mathcal{O}^1)^{n-1} = 2\mathcal{O}^{2n-2} - \mathcal{O}^{2n-1} - q + q\mathcal{O}^1\,.$$

This product has alternating signs and exceptional degree 1. The corresponding product in $\mathrm{QH}(X)$ is $[X^{n-1}] \star [X^{n-1}] = 2[X^{2n-2}]$.

Example 8.6. Let $\mathcal{I}^u = [\mathcal{I}_{\partial X^u}] \in K(X)$ denote the class of the ideal sheaf $\mathcal{I}_{\partial X^u} \subset \mathcal{O}_{X^u}$ of the boundary $\partial X^u = X^u \smallsetminus \mathring{X}^u$, for $u \in W^X$. These classes are dual to the Schubert structure sheaves in the sense that $\chi_X(\mathcal{O}_u \cdot \mathcal{I}^v) = \delta_{u,v}$ [Bri02, Cor. 2], and the structure constants of $K(X)$ with respect to the dual basis $\{\mathcal{I}^u\}$ have alternating signs by [Bri02, Thm. 1] and [GK08, Remark 3.7]. More precisely, if we write $\mathcal{I}^u \cdot \mathcal{I}^v = \sum_w \widetilde{C}_{u,v}^w \mathcal{I}^w$ in $K(X)$, with $u, v, w \in W^X$, then $(-1)^{\ell(uvw)} \widetilde{C}_{u,v}^w \geq 0$. An equivariant generalization can be found in [GK08, Conj. 3.1] and [AGM11, Cor. 5.2]. However, this version of positivity does not extend to quantum K-theory. In fact, the dual basis of $K(\mathbb{P}^1)$ consists of the classes

$$\mathcal{I}^0 = 1 - [\mathcal{O}_{\text{point}}] \qquad \text{and} \qquad \mathcal{I}^1 = [\mathcal{O}_{\text{point}}]\,,$$

TABLE 4. The number of products $\mathcal{O}_u \star \mathcal{O}^v$ with exceptional degrees on Lagrangian Grassmannians $\mathrm{LG}(n, 2n)$.

n	Total products	Exceptional degrees
2	10	1 (10%)
3	36	3 (8.3%)
4	136	17 (12.5%)
5	528	70 (13.3%)
6	2080	313 (15.0%)
7	8256	1317 (16.0%)
8	32896	5590 (17.0%)
9	131328	23310 (17.7%)
10	524800	96932 (18.5%)

and in $\mathrm{QK}(\mathbb{P}^1)$ we have

$$\mathcal{I}^0 \star \mathcal{I}^0 = 1 - 2[\mathcal{O}_{\text{point}}] + q = \mathcal{I}^0 - \mathcal{I}^1 + q\mathcal{I}^0 + q\mathcal{I}^1 \,.$$

8.2. **A geometric construction of the quantum product.** We give a geometric construction of the classes $(\mathcal{O}_u \star \mathcal{O}^v)_d \in K(X)$ that is better suited for determining the signs of the structure constants of $\mathrm{QK}(X)$.

Lemma 8.7. *Let $1 \le d \le d_X(2)$. The diagonal action of G on the set $\{(\eta, \omega) \in Y_{d-1} \times Y_d \mid \Gamma_\eta \subset \Gamma_\omega\}$ is transitive.*

Proof. Let $(\eta, \omega) \in Y_{d-1} \times Y_d$ be such that $\Gamma_\eta \subset \Gamma_\omega$. We must show that (η, ω) is in the orbit $G.(1.P_{Y_{d-1}}, 1.P_{Y_d})$. Since G acts transitively on Y_d, we may assume that $\omega = 1.P_{Y_d}$. Choose $x, y \in \Gamma_\eta$ such that $\mathrm{dist}(x, y) = d-1$. Then $\Gamma_{d-1}(x, y) = \Gamma_\eta$ by Corollary 5.20, and Lemma 5.6 applied to Γ_ω shows that we can find $g \in P_{Y_d}$ such that $g.(x, y) = (1.P_X, \kappa_{d-1}.P_X)$. It follows that $g.\Gamma_\eta = X_{\kappa_{d-1}}$, as required. □

For $1 \le d \le d_X(2)$ we set $Y_{d-1,d} = G/(P_{Y_{d-1}} \cap P_{Y_d})$. By Lemma 8.7 we can make the identification

$$Y_{d-1,d} \;=\; \{(\eta, \omega) \in Y_{d-1} \times Y_d \mid \Gamma_\eta \subset \Gamma_\omega\} \,.$$

Let $\phi_{d-1} : Y_{d-1,d} \to Y_{d-1}$ and $\phi_d : Y_{d-1,d} \to Y_d$ be the projections. Given $u, v \in W^X$ we define the varieties

$$\begin{aligned} Y_{d-1,1}(X_u, X^v) \;&=\; \phi_d(\phi_{d-1}^{-1}(Y_{d-1}(X_u, X^v))) \\ &=\; \{\omega \in Y_d \mid \exists\, \eta \in Y_{d-1}(X_u, X^v) : \Gamma_\eta \subset \Gamma_\omega\} \,, \\ Z_{d-1,1}(X_u, X^v) \;&=\; q_d^{-1}(Y_{d-1,1}(X_u, X^v)) \,, \text{ and} \\ \Gamma_{d-1,1}(X_u, X^v) \;&=\; p_d(Z_{d-1,1}(X_u, X^v)) \,. \end{aligned}$$

Notice that $\phi_{d-1}^{-1}(Y_{d-1}(X_u, X^v))$ is a Richardson variety and $Y_{d-1,1}(X_u, X^v)$ is a projected Richardson variety, so Theorem 2.13 implies that $Y_{d-1,1}(X_u, X^v)$ has rational singularities and $[\mathcal{O}_{Y_{d-1,1}(X_u,X^v)}] = (\phi_d)_*(\phi_{d-1})^*[\mathcal{O}_{Y_{d-1}(X_u,X^v)}]$. Since the map $q_d : Z_{d-1,1}(X_u, X^v) \to Y_{d-1,1}(X_u, X^v)$ is a locally trivial fibration with non-singular fibers, it follows that $Z_{d-1,1}(X_u, X^v)$ has rational singularities as well. On the other hand, $\Gamma_{d-1,1}(X_u, X^v)$ is not in general a (projected) Richardson variety. It would be interesting to understand the singularities of this variety. In Example 8.31 we give an example where $\Gamma_{d-1,1}(X_u, X^v)$ has rational singularities and fails to be a projected Richardson variety.

Question 8.8. Does $\Gamma_{d-1,1}(X_u, X^v)$ always have rational singularities?

The following lemma applied to $Y_{d-1}(X_u, X^v)$ shows that $\Gamma_{d-1,1}(X_u, X^v) = \Gamma_1(\Gamma_{d-1}(X_u, X^v))$, that is, $\Gamma_{d-1,1}(X_u, X^v)$ is the set of all points in X that are connected by a line to a stable curve of degree $d-1$ meeting X_u and X^v.

Lemma 8.9. *For any subset $\Omega \subset Y_{d-1}$ we have*

$$p_d\, q_d^{-1}\, \phi_d\, \phi_{d-1}^{-1}(\Omega) = p_1\, q_1^{-1}\, q_1\, p_1^{-1}\, p_{d-1}\, q_{d-1}^{-1}(\Omega)\,. \tag{8}$$

Proof. A point $z \in X$ belongs to the left hand side of (8) if and only if there exists $\eta \in \Omega$ and $\omega \in Y_d$ such that $z \in \Gamma_\omega$ and $\Gamma_\eta \subset \Gamma_\omega$. Since $p_1 q_1^{-1} q_1 p_1^{-1}(x) = \Gamma_1(x)$ for all $x \in X$ by Corollary 5.23, the point $z \in X$ belongs to the right hand side of (8) if and only if there exists $\eta \in \Omega$ such that $\Gamma_1(z) \cap \Gamma_\eta \neq \emptyset$. Assume that z belongs to the left hand side of (8) and choose $(\eta, \omega) \in \Omega \times Y_d$ such that $z \in \Gamma_\omega$ and $\Gamma_\eta \subset \Gamma_\omega$. Since $\Gamma_1(z) \cap \Gamma_\omega$ and Γ_η represent dual classes in $H^*(\Gamma_\omega; \mathbb{Z})$ by Lemma 5.18, we have $\Gamma_1(z) \cap \Gamma_\eta \neq \emptyset$, so z belongs to the right hand side of (8). On the other hand, if z belongs to the right hand side of (8), then choose $(\eta, x) \in \Omega \times X$ such that $x \in \Gamma_1(z) \cap \Gamma_\eta$. Then choose $y \in \Gamma_\eta$ such that $\operatorname{dist}(x, y) = d-1$. Since there exists a (possibly reducible) rational curve of degree d through x, y, and z, it follows from Proposition 5.21 that we may choose $\omega \in Y_d$ such that $x, y, z \in \Gamma_\omega$. Since $x, y \in \Gamma_\omega$ and $\operatorname{dist}(x, y) = d-1$, it follows from Corollary 5.20 that $\Gamma_\eta = \Gamma_{d-1}(x, y) \subset \Gamma_\omega$. This shows that z belongs to the left hand side of (8). □

Theorem 8.10. *We have $(\mathcal{O}_u \star \mathcal{O}^v)_d = [\mathcal{O}_{\Gamma_d(X_u, X^v)}] - (p_d)_*[\mathcal{O}_{Z_{d-1,1}(X_u, X^v)}]$.*

Proof. By equation (5) and Corollary 5.23 we have

$$(\mathcal{O}_u \star \mathcal{O}^v)_d = [\mathcal{O}_{\Gamma_d(X_u, X^v)}] - (p_1)_*(q_1)^*(q_1)_*(p_1)^*[\mathcal{O}_{\Gamma_{d-1}(X_u, X^v)}]\,.$$

It is therefore enough to show that

$$(p_d)_*(q_d)^*(\phi_d)_*(\phi_{d-1})^*[\mathcal{O}_{Y_{d-1}(X_u, X^v)}] = (p_1)_*(q_1)^*(q_1)_*(p_1)^*(p_{d-1})_*(q_{d-1})^*[\mathcal{O}_{Y_{d-1}(X_u, X^v)}]\,.$$

More generally, the linear operators

$$(p_d)_*(q_d)^*(\phi_d)_*(\phi_{d-1})^* \qquad \text{and} \qquad (p_1)_*(q_1)^*(q_1)_*(p_1)^*(p_{d-1})_*(q_{d-1})^*$$

define the same map $K(Y_{d-1}) \to K(X)$. In fact, using that $K(Y_{d-1})$ has a basis of Schubert classes $[\mathcal{O}_\Omega]$, this follows from Lemma 8.9. □

8.3. **Proof of our main theorems.** In [BCMP18b] we proved that $\Gamma_d(X_u, X^v)$ has rational singularities and that

$$(p_d)_*[\mathcal{O}_{Z_d(X_u, X^v)}] = [\mathcal{O}_{\Gamma_d(X_u, X^v)}]\,.$$

In fact, this follows from Corollary 5.23 and Theorem 2.13.

Let Ω be an irreducible variety defined over $\mathbb{C}$ and let $\rho : \widetilde{\Omega} \to \Omega$ be a resolution of singularities. Define the *resolution class* of Ω to be the image $\rho_*[\mathcal{O}_{\widetilde{\Omega}}] = \sum_{i \geq 0} (-1)^i [R^i \rho_* \mathcal{O}_{\widetilde{\Omega}}]$ in the Grothendieck group $K(\Omega)$ of coherent sheaves on Ω. This class is independent of the chosen desingularization and will be denoted simply by $[\mathcal{O}_{\widetilde{\Omega}}]$. When $\Omega \subset X$ is a closed subvariety, we also write $[\mathcal{O}_{\widetilde{\Omega}}]$ for the image of the resolution class in $K(X)$. If Ω has rational singularities, then $[\mathcal{O}_{\widetilde{\Omega}}] = [\mathcal{O}_\Omega]$. We need the following result [Bri02, §4, Remark].

Theorem 8.11 (Brion)**.** *Let $M = G/P_M$ be a flag variety over $\mathbb{C}$ and let $\Omega \subset M$ be an irreducible closed subvariety. Then the resolution class $[\mathcal{O}_{\widetilde{\Omega}}]$ is an alternating linear combination of Schubert classes, that is, we have*

$$[\mathcal{O}_{\widetilde{\Omega}}] = \sum_{w \in W^M} c_w(\Omega)\, \mathcal{O}^w$$

in $K(M)$, where $(-1)^{\ell(w)-\operatorname{codim}(\Omega,M)}\, c_w(\Omega) \geq 0$ for all $w \in W^M$.

Theorem 8.12. *Let $f : \Omega' \to \Omega$ be a surjective morphism between complex projective varieties with rational singularities. Then f is cohomologically trivial if and only if the general fibers of f are cohomologically trivial.*

Proof. The implication *'if'* follows from [Kol86, Thm 7.1] (see the proof of [BM11, Thm. 3.2]), and *'only if'* follows from [Har77, III.12.8 and III.12.9]. □

We will use the following consequence of Theorem 8.12 when condition (b) is satisfied. The condition that Ω' has rational singularities is necessary in this case, see Example 8.14.

Corollary 8.13. *Let $f : \Omega' \to \Omega$ be a surjective morphism of irreducible projective varieties over $\mathbb{C}$. Assume that either* (a) *the general fibers of f are rationally connected, or* (b) *Ω' has rational singularities and the general fibers of f are cohomologically trivial. Then, $f_*[\mathcal{O}_{\widetilde{\Omega}'}] = [\mathcal{O}_{\widetilde{\Omega}}]$.*

Proof. Let $\widetilde{\Omega}$ be a desingularization of Ω, and let $\widetilde{\Omega}'$ be a desingularization of the unique irreducible component of $\Omega' \times_\Omega \widetilde{\Omega}$ that maps birationally onto Ω'. We obtain a commutative diagram where the vertical maps are resolutions of singularities.

$$\begin{array}{ccc} \widetilde{\Omega}' & \xrightarrow{\widetilde{f}} & \widetilde{\Omega} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Omega' & \xrightarrow{f} & \Omega \end{array}$$

Let $U' \subset \Omega'$ be a dense open subset such that $\pi' : {\pi'}^{-1}(U') \to U'$ is an isomorphism, and set $Z = \widetilde{\Omega}' \smallsetminus {\pi'}^{-1}(U')$. For $x \in \Omega$ we let $\Omega'_x \subset \Omega'$, $\widetilde{\Omega}'_x \subset \widetilde{\Omega}'$, and $Z_x \subset Z$ denote the fibers over x. Set $r = \dim(\Omega') - \dim(\Omega)$. Choose a dense open subset $U \subset \Omega$ such that $f\pi' : (f\pi')^{-1}(U) \to U$ is smooth, $\dim(Z_x) < r$ for all $x \in U$, and Ω'_x is rationally connected for $x \in U$ in case (a), or cohomologically trivial with rational singularities in case (b). Here we use that the general fibers of f have rational singularities when Ω' has rational singularities by [Bri02, Lemma 3].

Let $x \in U$. Then $\widetilde{\Omega}'_x$ is a disjoint union of non-singular varieties of dimension r, and Ω'_x is irreducible. Since $\widetilde{\Omega}'_x \cap {\pi'}^{-1}(U') \subset \widetilde{\Omega}'_x$ is a dense open subset isomorphic to $\Omega'_x \cap U'$, it follows that $\widetilde{\Omega}'_x$ is birational to Ω'_x. We deduce that $\widetilde{\Omega}'_x$ is cohomologically trivial; this follows from [Deb01, Cor. 4.18(a)] if Ω'_x is rationally connected, and from the Leray spectral sequence if Ω'_x is cohomologically trivial with rational singularities. Theorem 8.12 now shows that $\widetilde{f}$ is cohomologically trivial, which completes the proof. □

The alternating signs conjecture for $\mathrm{QK}(X)$ would be a consequence of the following two statements.

TABLE 5. Divisors in $Y_d(X_u, X^v)$ and $\Gamma_d(X_u, X^v)$.

Range of degrees	$Y_{d-1,1}(X_u, X^v)$	$\Gamma_{d-1,1}(X_u, X^v)$
$1 \le d \le d_{\min}(u^\vee, v)$	$= \emptyset$	$= \emptyset$
$d_{\min}(u^\vee, v) < d \le d_{\max}(u^\vee, v)$	$\subsetneq Y_d(X_u, X^v)$	$\subsetneq \Gamma_d(X_u, X^v)$
$d_{\max}(u^\vee, v) < d \le \min(d_{\max}(u^\vee), d_{\max}(v))$	$\subsetneq Y_d(X_u, X^v)$	$= \Gamma_d(X_u, X^v)$
$\min(d_{\max}(u^\vee), d_{\max}(v)) < d \le d_X(2)$	$= Y_d(X_u, X^v)$	$= \Gamma_d(X_u, X^v)$

(I) The general fibers of the map $p_d : Z_{d-1,1}(X_u, X^v) \to \Gamma_{d-1,1}(X_u, X^v)$ are cohomologically trivial.

(II) The variety $\Gamma_{d-1,1}(X_u, X^v)$ is either equal to $\Gamma_d(X_u, X^v)$ or a divisor in $\Gamma_d(X_u, X^v)$.

In fact, the class $[\mathcal{O}_{\Gamma_d(X_u,X^v)}]$ has alternating signs by Theorem 8.11, and these signs are compatible with Conjecture 8.1 for $d_{\min}(u^\vee, v) \le d \le d_{\max}(u^\vee, v)$, as Proposition 7.1 shows that $\operatorname{codim}(\Gamma_d(X_u, X^v), X) = \ell(u^\vee) + \ell(v) - \deg(q^d)$. Property (I) implies that $(p_d)_*[\mathcal{O}_{Z_{d-1,1}(X_u,X^v)}]$ is the resolution class of $\Gamma_{d-1,1}(X_u, X^v)$ by Corollary 8.13, which also has alternating signs by Theorem 8.11. The point of (II) is that, if $\Gamma_{d-1,1}(X_u, X^v)$ is a divisor in $\Gamma_d(X_u, X^v)$, then the alternating signs of the two terms in Theorem 8.10 enhance each other to yield the alternating signs of $(\mathcal{O}_u \star \mathcal{O}^v)_d$. Properties (I) and (II) also imply that $(\mathcal{O}_u \star \mathcal{O}^v)_d$ is non-zero if and only if $\Gamma_{d-1,1}(X_u, X^v) \neq \Gamma_d(X_u, X^v)$. This determines whether the power q^d occurs in $\mathcal{O}_u \star \mathcal{O}^v$. We will show that (II) is true in all cases, whereas (I) holds if and only if d is not an exceptional degree of $\mathcal{O}_u \star \mathcal{O}^v$. These results are sufficient to establish Theorem 8.3 and Theorem 8.4. Table 4 illustrates that most products $\mathcal{O}_u \star \mathcal{O}^v$ on Lagrangian Grassmannians are fully described by these results.

Proofs of Theorem 8.3 and Theorem 8.4. Table 5 shows the range of degrees where $Y_{d-1,1}(X_u, X^v)$ is a divisor in $Y_d(X_u, X^v)$, and where $\Gamma_{d-1,1}(X_u, X^v)$ is a divisor in $\Gamma_d(X_u, X^v)$. The codimension of $Y_{d-1,1}(X_u, X^v)$ is determined by Proposition 8.18 and Proposition 8.21, after which the codimension of $\Gamma_{d-1,1}(X_u, X^v)$ is determined by Proposition 8.20 and Corollary 8.24. The results now follow from Corollary 8.25 using the strategy discussed above. □

Example 8.14. Let $E \subset \mathbb{P}^2$ be an elliptic curve, let $\Omega' \subset \mathbb{P}^3$ be the cone over E, set $\Omega = \operatorname{Spec}(\mathbb{C})$, and let $f : \Omega' \to \Omega$ be the structure morphism. Using the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\Omega'} \to 0$, it follows that the fibers of f are cohomologically trivial. Let $\widetilde{\Omega}'$ be the blow-up of Ω' at its vertex, and let $g : \widetilde{\Omega}' \to E$ be the map induced by the projection $\Omega' \dashrightarrow E$. Since the fibers of g are projective lines, it follows from Corollary 8.13(a) that $g_*[\mathcal{O}_{\widetilde{\Omega}'}] = [\mathcal{O}_E]$. Since $\chi(E, \mathcal{O}_E) = 0$, we deduce that $f_*[\mathcal{O}_{\widetilde{\Omega}'}] = 0 \neq [\mathcal{O}_\Omega]$. This shows that Corollary 8.13(b) may fail without the assumption that Ω' has rational singularities.

Remark 8.15. The equivariant quantum K-theory ring $\mathrm{QK}_T(X)$ is defined by (5) and (6), except that all structure sheaves are endowed with their natural T-equivariant structure, see [BCMP18a] for details. Since Corollary 8.13 remains true in equivariant K-theory, our results imply that the identity

$$(\mathcal{O}_u \star \mathcal{O}^v)_d = [\mathcal{O}_{\Gamma_d(X_u,X^v)}] - [\mathcal{O}_{\widetilde{\Gamma_{d-1,1}(X_u,X^v)}}]$$

holds in $\mathrm{QK}_T(X)$ whenever d is not an exceptional degree. In particular, Theorem 8.3 holds for $\mathrm{QK}_T(X)$.

Let $\widetilde{N}_{u,v}^{w,d} \in K_T(\text{point})$, for $u, v, w \in W^X$, denote the structure constants describing the action of the B^--stable Schubert basis $\{\mathcal{O}^v\}$ on the B-stable basis $\{\mathcal{O}_{u^\vee}\}$ of $\mathrm{QK}_T(X)$:

$$\mathcal{O}_{u^\vee} \star \mathcal{O}^v = \sum_{w,d\geq 0} \widetilde{N}_{u,v}^{w,d}\, q^d\, \mathcal{O}_w$$

If Theorem 8.11 is upgraded to the equivariant setting of [AGM11, Thm. 4.1], then these constants would satisfy the positivity property

$$(-1)^{\ell(uvw)+\deg(q^d)}\, \widetilde{N}_{u,v}^{w,d} \;\in\; \mathbb{N}\big[[\mathbb{C}_{-\beta}] - 1 : \beta \in \Delta\big]\,.$$

We thank D. Anderson [And] for sending us an outline of a proof of the equivariant version of Theorem 8.11, with some details left to check. We hope to address this elsewhere, and possibly prove a slight generalization of Theorem 8.11.

The classes $\mathcal{O}_{u^\vee}$ and $\mathcal{O}^u$ are distinct in $K_T(X)$, so it is not clear how to apply our results to products $\mathcal{O}^u \star \mathcal{O}^v$ of two B^--stable Schubert classes in $\mathrm{QK}_T(X)$. A positivity conjecture for the structure constants of such products is discussed in [BCMP18a, Conj. 2.2], generalizing [GR04, Conj. 5.1] and [AGM11, Cor. 5.3].

Conjecture 8.16. *The power q^d occurs in the equivariant quantum product $\mathcal{O}^u \star \mathcal{O}^v \in \mathrm{QK}_T(X)$ if and only if $0 \leq d \leq d_{\max}(u,v)$ or $d = d_{\max}(u,v) + 1$ is an exceptional degree of $\mathcal{O}_{u^\vee} \star \mathcal{O}^v$.*

8.4. **Proofs and counterexamples to (I) and (II).** Fix elements $u, v \in W^X$ and a degree $1 \leq d \leq d_X(2)$. We proceed to establish the required properties of the map $p_d : Z_{d-1,1}(X_u, X^v) \to \Gamma_{d-1,1}(X_u, X^v)$. Notice that $\Gamma_d(X_u, X^v)$ is empty for $d < d_{\min}(u^\vee, v)$, and $\Gamma_{d-1,1}(X_u, X^v)$ is empty for $d \leq d_{\min}(u^\vee, v)$.

Lemma 8.17. *We have $w_{0,Y_d}^{Y_{d-1,d}} = s_{\widetilde{\alpha}_d}$.*

Proof. The element $w_{0,Y_d}^{Y_{d-1,d}}$ describes the fiber of the map $\phi_d : Y_{d-1,d} \to Y_d$ over $1.P_{Y_d}$, that is, $\phi_d^{-1}(1.P_{Y_d}) = (Y_{d-1,d})_{w_{0,Y_d}^{Y_{d-1,d}}}$. Since this fiber does not change if the cominuscule variety X is replaced with X_{κ_d}, we may assume that X is a primitive cominuscule variety of diameter d. In this case Y_d is a point and $Y_{d-1,d} = Y_{d-1}$, so Lemma 6.1 and Lemma 5.18 imply that $w_{0,Y_d}^{Y_{d-1,d}} = w_0 w_{0,Y_{d-1}} = w_0 w_{0,X} z_{d-1} = w_0^X z_{d-1} = s_{\widetilde{\alpha}_d}$, as required. □

We first consider degrees in the range $d_{\min}(u^\vee, v) < d \leq \min(d_{\max}(u^\vee), d_{\max}(v))$. In this case the maps $q_d : p_d^{-1}(X_u) \to Y_d(X_u)$ and $q_d : p_d^{-1}(X^v) \to Y_d(X^v)$ are birational by Corollary 6.9(a). Since the fibers of $p_d : Z_d \to X$ and $q_d : Z_d \to Y_d$ are described by $w_{0,X}^{Z_d} = z_d/\kappa_d$ and $w_{0,Y_d}^{Z_d} = \kappa_d$ by Lemma 6.1, we deduce that $uz_d\kappa_d = u(z_d/\kappa_d)$ and $v\kappa_d = v/\kappa_d$ belong to W^{Y_d}, and $uz_d \in W^{Z_d}$. With the notation for projected Richardson varieties from Section 3, we obtain $Y_d(X_u, X^v) = (Y_d)_{uz_d\kappa_d}^{v\kappa_d} = \Pi_{uz_d\kappa_d}^{v\kappa_d}(Y_d)$ and $Z_d(X_u, X^v) = (Z_d)_{uz_d}^{v\kappa_d} = \Pi_{uz_d}^{v\kappa_d}(Z_d)$.

Proposition 8.18. *Assume that $d_{\min}(u^\vee, v) < d \leq \min(d_{\max}(u^\vee), d_{\max}(v))$. Then $v\kappa_d s_\gamma \leq_{Y_d} uz_d\kappa_d$, and $Y_{d-1,1}(X_u, X^v) = \Pi_{uz_d\kappa_d}^{v\kappa_d s_\gamma}(Y_d)$ is a divisor in $Y_d(X_u, X^v)$.*

Proof. Define $\eta \in W_{Z_d}$ by $s_\gamma \eta = \kappa_d/\kappa_{d-1}$. We have $Y_{d-1}(X^v) = (Y_{d-1})^{v/\kappa_{d-1}} = (Y_{d-1})^{v\kappa_d s_\gamma \eta}$, and Proposition 6.7(a) shows that $v\kappa_d s_\gamma \eta = v/\kappa_{d-1} \in W^{Y_{d-1}}$. We also have $\kappa_d/\kappa_{d-1} \in W_{Y_d} \cap W^{Y_{d-1}}$, hence $uz_d\kappa_d\eta \in W^{Y_{d-1,d}}$. Define

$$Z_{d-1,d} = G/(P_{Y_{d-1}} \cap P_{Y_d} \cap P_X) = \{(\eta, \omega, x) \in Y_{d-1} \times Y_d \times X \mid x \in \Gamma_\eta \subset \Gamma_\omega\}\,,$$

with projections $p : Z_{d-1,d} \to Z_d$ and $q : Z_{d-1,d} \to Y_{d-1,d}$.

$$\begin{array}{ccccccccc} X & \xleftarrow{p_{d-1}} & Z_{d-1} & \longleftarrow & Z_{d-1,d} & \xrightarrow{p} & Z_d & \xrightarrow{p_d} & X \\ & & \downarrow{\scriptstyle q_{d-1}} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_d} & & \\ & & Y_{d-1} & \xleftarrow{\phi_{d-1}} & Y_{d-1,d} & \xrightarrow{\phi_d} & Y_d & & \end{array}$$

Using that $\kappa_d w_{0,Z_d}^{Z_{d-1,d}} = w_{0,Y_d}^{Y_{d-1,d}} w_{0,Y_{d-1,d}}^{Z_{d-1,d}} = s_{\tilde{\alpha}_d}\kappa_{d-1}$ by Lemma 6.1 and Lemma 8.17, we obtain $w_{0,Z_d}^{Z_{d-1,d}} = \kappa_d s_{\tilde{\alpha}_d}\kappa_{d-1} = \eta$. This implies

$$\phi_{d-1}^{-1}(Y_{d-1}(X_u)) = qp^{-1}p_d^{-1}(X_u) = (Y_{d-1,d})_{uz_d\kappa_d\eta}\,.$$

We obtain

$$\begin{aligned} Y_{d-1,1}(X_u, X^v) &= \phi_d(\phi_{d-1}^{-1}(Y_{d-1}(X_u, X^v))) \\ &= \phi_d\left(\phi_{d-1}^{-1}(Y_{d-1}(X_u)) \cap \phi_{d-1}^{-1}(Y_{d-1}(X^v))\right) \\ &= \phi_d\left((Y_{d-1,d})_{uz_d\kappa_d\eta}^{v\kappa_d s_\gamma \eta}\right) = \Pi_{uz_d\kappa_d\eta}^{v\kappa_d s_\gamma \eta}(Y_d) = \Pi_{uz_d\kappa_d}^{v\kappa_d s_\gamma}(Y_d)\,, \end{aligned}$$

where the last two equalities follow from Corollary 3.6 and Theorem 3.4(b), or [KLS14, Prop. 3.3]. The inequality $v\kappa_d s_\gamma \leq_{Y_d} uz_d\kappa_d$ holds because $Y_{d-1}(X_u, X^v) \neq \emptyset$ and $uz_d\kappa_d \in W^{Y_d}$. Finally, it follows from Proposition 3.2 that $Y_{d-1,1}(X_u, X^v)$ is a divisor in $Y_d(X_u, X^v) = \Pi_{uz_d\kappa_d}^{v\kappa_d}(Y_d)$. □

Lemma 8.19. *Assume that $d_{\min}(u^\vee, v) < d \leq \min(d_{\max}(u^\vee), d_{\max}(v))$. Then $Z_{d-1,1}(X_u, X^v) \cap \mathring{\Pi}_{uz_d}^{v\kappa_d}(Z_d)$ is a dense open subset of $Z_{d-1,1}(X_u, X^v)$.*

Proof. By Proposition 8.18, $Z_{d-1,1}(X_u, X^v)$ is a divisor in $Z_d(X_u, X^v) = \Pi_{uz_d}^{v\kappa_d}(Z_d)$. If the claim is false, then Theorem 3.5 implies that $Z_{d-1,1}(X_u, X^v) = \Pi_a^b(Z_d)$, where $v\kappa_d \leq b \leq_{Z_d} a \leq uz_d$. Since $\Pi_a^b(Y_d) = \Pi_{uz_d\kappa_d}^{v\kappa_d s_\gamma}(Y_d)$ by Proposition 8.18 and $uz_d\kappa_d \in W^{Y_d}$, it follows from Theorem 3.4 that $a \geq uz_d\kappa_d$ and $b \geq v\kappa_d s_\gamma$. Since we also have $\ell(a) - \ell(b) = \ell(uz_d) - \ell(v\kappa_d s_\gamma)$, it follows that $a = uz_d$ and $b = v\kappa_d s_\gamma$. But Theorem 3.4 also implies that $\Pi_{uz_d}^{v\kappa_d s_\gamma}(Y_d) = \Pi_{uz_d\kappa_d}^{v\kappa_d}(Y_d)$, a contradiction. □

Proposition 8.20. *Assume that $d_{\min}(u^\vee, v) < d \leq d_{\max}(u^\vee, v)$. Then the map $p_d : Z_{d-1,1}(X_u, X^v) \to \Gamma_{d-1,1}(X_u, X^v)$ is birational.*

Proof. Since $p_d : Z_d(X_u, X^v) \to \Gamma_d(X_u, X^v)$ is birational by Proposition 7.1 and Corollary 6.9(d), it follows from Proposition 3.2 that the restriction of p_d to the open projected Richardson variety $\mathring{\Pi}_{uz_d}^{v\kappa_d}(Z_d)$ is injective. The result therefore follows from Lemma 8.19. □

Our next result shows that $Z_{d-1,1}(X_u, X^v) = Z_d(X_u, X^v)$ and $\Gamma_{d-1,1}(X_u, X^v) = \Gamma_d(X_u, X^v)$ whenever $d > \min(d_{\max}(u^\vee), d_{\max}(v))$.

Proposition 8.21. *Assume $d > \min(d_{\max}(u^\vee), d_{\max}(v))$. Then, $Y_{d-1,1}(X_u, X^v) = Y_d(X_u, X^v)$.*

Proof. Let $\omega \in Y_d(X_u, X^v)$ and assume that $d > d_{\max}(v)$. Then $\Gamma_\omega \cap X^v$ has positive dimension by Corollary 6.9(a). Choose any point $x \in \Gamma_\omega \cap X_u$. Then $\Gamma_{d-1}(x) \cap \Gamma_\omega$ is a divisor in Γ_ω by Lemma 5.18. It follows that $\Gamma_{d-1}(x) \cap \Gamma_\omega \cap X^v \neq \emptyset$.

Choose any point $y \in \Gamma_{d-1}(x) \cap \Gamma_\omega \cap X^v$. Since $\operatorname{dist}(x,y) \leq d-1$, there exists $\eta \in Y_{d-1}$ such that $x, y \in \Gamma_\eta \subset \Gamma_\omega$ by Corollary 5.20(b) applied to Γ_ω. This proves that $\omega \in Y_{d-1,1}(X_u, X^v)$. A symmetric argument works if $d > d_{\max}(u^\vee)$. □

We finally discuss the remaining range $d_{\max}(u^\vee, v) < d \leq \min(d_{\max}(u^\vee), d_{\max}(v))$. For degrees in this range, the map $p_d : Z_d(X_u, X^v) \to \Gamma_d(X_u, X^v)$ has fibers of positive dimension by Proposition 7.1 and Corollary 6.9(d).

Notation 8.22. Given a fixed cominuscule variety X, we let ϵ denote the constant defined by $\epsilon = 1$ if X is minuscule or an odd quadric of dimension at least five, while $\epsilon = 2$ if X is a Lagrangian Grassmannian. The three-dimensional quadric $Q^3 = \mathrm{LG}(2,4)$ is considered a Lagrangian Grassmannian.

The proof of the following result is postponed to Section 9, where we also justify the definition of ϵ. Let $[\partial Y_d] = \sum_{\beta \in \Delta \smallsetminus \Delta_{Y_d}} [Y_d^{s_\beta}]$ denote the (ample) sum of the Schubert divisors in Y_d.

Proposition 8.23. *Let $d_{\max}(u^\vee, v) < d \leq \min(d_{\max}(u^\vee), d_{\max}(v))$. For all points z in a dense open subset of $\Gamma_d(X_u, X^v)$, the fiber $D = p_d^{-1}(z) \cap Z_{d-1,1}(X_u, X^v)$ is a Cartier divisor of class $\epsilon\, q_d^*[\partial Y_d]$ in the Richardson variety $R = p_d^{-1}(z) \cap Z_d(X_u, X^v)$.*

Corollary 8.24. *For $d > d_{\max}(u^\vee, v)$ we have $\Gamma_{d-1,1}(X_u, X^v) = \Gamma_d(X_u, X^v)$.*

Proof. This follows from Proposition 8.21 and Proposition 8.23. □

Corollary 8.25. *Let $d_{\min}(u^\vee, v) < d \leq d_X(2)$. The general fibers of the map $p_d : Z_{d-1,1}(X_u, X^v) \to \Gamma_{d-1,1}(X_u, X^v)$ are cohomologically trivial if and only if d is not an exceptional degree of $\mathcal{O}_u \star \mathcal{O}^v$.*

Proof. This follows from Proposition 8.20 if $d \leq d_{\max}(u^\vee, v)$, and it follows from Proposition 8.21 and Corollary 2.11 for $d > \min(d_{\max}(u^\vee), d_{\max}(v))$. Assume that $d_{\max}(u^\vee, v) < d \leq \min(d_{\max}(u^\vee), d_{\max}(v))$, and let R and D be as in Proposition 8.23. Then R is a translate of the Richardson variety $(F_d)_{u_d}^{u_d \cap v^d}$, and we have $[D] = \epsilon[\partial F_d]$ in $\operatorname{Pic}(R)$. Notice that R has positive dimension by Proposition 7.1. We use Corollary 4.12 to argue that D is cohomologically trivial if and only if $(u_d \cup v^d)/v^d$ is not a short rook strip. If X is minuscule, then F_d is a product of minuscule varieties, $\epsilon = 1$, and D is cohomologically trivial because there are no tableaux of shape $I(u_d) \smallsetminus I(v^d)$ with integer values from the interval $[\frac{1}{2}, 1)$. If $X = Q^{2n-1}$ is an odd quadric with $n \geq 3$, then $\epsilon = 1$, and the assumptions imply that $d = 1$, hence $F_d = Q^{2n-3}$. This time D is cohomologically trivial if and only if there are no decreasing primed tableau of shape $I(u_d) \smallsetminus I(v^d)$ using only the label $\frac{1}{2}$, that is, $(u_d \cup v^d)/v_d$ is not a short rook strip. Finally, if $X = \mathrm{LG}(n, 2n)$ is a Lagrangian Grassmannian, then $\epsilon = 2$, $F_d = \mathrm{Gr}(n-d, n)$ is a Grassmannian of type A, and D is cohomologically trivial if and only if there are no decreasing primed tableaux of shape $I(u_d) \smallsetminus I(v^d)$ using only the label 1, that is, $(u_d \cup v^d)/v^d$ is not a rook strip. In this case a rook strip is the same as a short rook strip, since all boxes of $\mathcal{P}_{F_d}$ are short by convention. The result follows from these observations. □

Since the general fibers of $p_d : Z_{d-1,1}(X_u, X^v) \to \Gamma_{d-1,1}(X_u, X^v)$ have rational singularities by [Bri02, Lemma 3], it follows from Corollary 8.25 that these fibers are irreducible projective varieties of arithmetic genus zero for any non-exceptional degree in the range $d_{\min}(u^\vee, v) < d \leq d_X(2)$. The following result describes the fibers for exceptional degrees.

Theorem 8.26. *Let $d = d_{\max}(u^\vee, v)+1$ be an exceptional degree of $\mathcal{O}_u \star \mathcal{O}^v$. Then the general fibers of $p_d : Z_{d-1,1}(X_u, X^v) \to \Gamma_d(X_u, X^v)$ have rational singularities and arithmetic genus one. They are irreducible projective varieties if they have positive dimension.*

Proof. The general fibers $D = Z_{d-1,1}(X_u, X^v) \cap p_d^{-1}(z)$ have rational singularities by [Bri02, Lemma 3], and it follows from Theorem 4.9 that D has arithmetic genus one. In the positive dimensional case, the general fibers are connected by Proposition 8.23 and the Fulton-Hansen theorem [FH79]. □

Remark 8.27. When $d = d_{\max}(u^\vee, v) + 1$ is an exceptional degree of $\mathcal{O}_u \star \mathcal{O}^v$, the general fibers of $p_d : Z_{d-1,1}(X_u, X^v) \to \Gamma_d(X_u, X^v)$ can be described more explicitly as follows. Since $(u_d \cup v^d)/v^d$ is a rook strip, it follows from Corollary 6.9(d) and [BR12, Lemma 3.2(b)] that the Richardson variety $R = Z_d(X_u, X^v) \cap p_d^{-1}(z)$ is a product of projective lines for general $z \in \Gamma_d(X_u, X^v)$. Proposition 8.23 shows that $D = Z_{d-1,1}(X_u, X^v) \cap p_d^{-1}(z)$ has multidegree $(2, 2, \dots, 2)$ in R. The arithmetic genus of D can also be computed from this description.

Given a non-zero K-theory class $\mathcal{F} \in K(X)$, the *initial term* $\operatorname{lead}(\mathcal{F})$ is defined as the homogeneous component of lowest degree in the Chern character $\operatorname{ch}(\mathcal{F}) \in H^*(X, \mathbb{Q})$. Equivalently, $\operatorname{lead}(\mathcal{F})$ is the leading term of $\mathcal{F}$ modulo the topological filtration of $K(X)$ (see [Ful98, Ex. 15.2.16]). Let $\operatorname{codim}(\mathcal{F})$ denote the complex degree of $\operatorname{lead}(\mathcal{F})$, so that $\operatorname{lead}(\mathcal{F}) \in H^{2\operatorname{codim}(\mathcal{F})}(X, \mathbb{Z})$, and let

$$\mathcal{F} = \sum_{w \in W^X} c_w(\mathcal{F})\, \mathcal{O}^w$$

be the expansion of $\mathcal{F}$ in the Schubert basis of $K(X)$. Then $\operatorname{codim}(\mathcal{F})$ is the minimal length $\ell(w)$ for which $c_w(\mathcal{F}) \neq 0$. The class $\mathcal{F}$ has *alternating signs* if $(-1)^{\ell(w)-\operatorname{codim}(\mathcal{F})} c_w(\mathcal{F}) \geq 0$ holds for all $w \in W^X$.

Part (a) of the following conjecture might point towards a generalization of Brion's positivity theorem. Parts (b) and (c) imply that $(\mathcal{O}_u \star \mathcal{O}^v)_d \neq 0$ whenever d is an exceptional degree.

Conjecture 8.28. *Assume that $d = d_{\max}(u^\vee, v) + 1$ is an exceptional degree of $\mathcal{O}_u \star \mathcal{O}^v$.*

(a) *The class $(p_d)_*[\mathcal{O}_{Z_{d-1,1}}(X_u, X^v)] \in K(X)$ has alternating signs.*
(b) *If $\dim \Gamma_d(X_u, X^v) \not\equiv \dim Z_d(X_u, X^v)$ (mod 2), then the initial term of $(p_d)_*[\mathcal{O}_{Z_{d-1,1}}(X_u, X^v)]$ is equal to $2\,[\Gamma_d(X_u, X^v)]$.*
(c) *If $\dim \Gamma_d(X_u, X^v) \equiv \dim Z_d(X_u, X^v)$ (mod 2), then the initial term of $(p_d)_*[\mathcal{O}_{Z_{d-1,1}}(X_u, X^v)]$ has complex degree $\operatorname{codim}(\Gamma_d(X_u, X^v), X) + 1$.*

Example 8.29. Let $X = Q^{2n-1}$ be a quadric of odd dimension. By Example 8.5, the only exceptional product in $\mathrm{QK}(X)$ is $\mathcal{O}_n \star \mathcal{O}^{n-1}$, with corresponding exceptional degree $d = 1$. Since $(\mathcal{O}_n \star \mathcal{O}^{n-1})_1 = -1 + \mathcal{O}^1$, we obtain from Theorem 8.10 that $\Gamma_1(X_n, X^{n-1}) = X$ and

$$(p_1)_*[\mathcal{O}_{Z_{0,1}(X_n, X^{n-1})}] = [\mathcal{O}_{\Gamma_1(X_n, X^{n-1})}] - (\mathcal{O}_n \star \mathcal{O}^{n-1})_1 = 2 - \mathcal{O}^1\,.$$

The general fibers of $p_1 : Z_1(X_n, X^{n-1}) \to X$ are projective lines by Corollary 6.9(d), and the general fibers of $p_1 : Z_{0,1}(X_n, X^{n-1}) \to X$ consist of two reduced points by Proposition 8.23. This proves Conjecture 8.28 for odd quadrics.

Example 8.30. Let $X = \mathrm{LG}(4,8)$ and define $u, v \in W^X$ by $I(u) = (4,2)$ and $I(v) = (3,1)$. The corresponding products in $\mathrm{QH}(X)$ and $\mathrm{QK}(X)$ are given by

$$[X_u] \star [X^v] = 4[X^{(4,3,1)}] + 4q[X^{(3)}] + 2q[X^{(2,1)}]$$

and

$$\begin{aligned}\mathcal{O}_u \star \mathcal{O}^v = {} & 4\mathcal{O}^{(4,3,1)} - 4\mathcal{O}^{(4,3,2)} + \mathcal{O}^{(4,3,2,1)} \\ & + 4q\mathcal{O}^{(3)} + 2q\mathcal{O}^{(2,1)} - 4q\mathcal{O}^{(4)} - 11q\mathcal{O}^{(3,1)} + 7q\mathcal{O}^{(3,2)} + 7q\mathcal{O}^{(4,1)} \\ & - 5q\mathcal{O}^{(4,2)} - 2q\mathcal{O}^{(3,2,1)} + q\mathcal{O}^{(4,3)} + 2q\mathcal{O}^{(4,2,1)} - q\mathcal{O}^{(4,3,1)} \\ & + q^2 - 2q^2\mathcal{O}^{(1)} + 2q^2\mathcal{O}^{(2)} - q^2\mathcal{O}^{(3)} - q^2\mathcal{O}^{(2,1)} + q^2\mathcal{O}^{(3,1)} .\end{aligned}$$

The product $\mathcal{O}_u \star \mathcal{O}^v$ has exceptional degree $d = 2$, and we have $\Gamma_d(X_u, X^v) = X$, $F_d = \mathrm{Gr}(2,4)$, $u_d = (2,1)$, and $v^d = (1)$. The general fibers of $p_d : Z_{d-1,1}(X_u, X^v) \to X$ are elliptic curves by Theorem 8.26. The identity

$$(p_d)_*[\mathcal{O}_{Z_{d-1,1}(X_u,X^v)}] = 1 - (\mathcal{O}_u \star \mathcal{O}^v)_d = 2\mathcal{O}^{(1)} - 2\mathcal{O}^{(2)} + \mathcal{O}^{(3)} + \mathcal{O}^{(2,1)} - \mathcal{O}^{(3,1)}$$

shows that Conjecture 8.28 holds for the product $\mathcal{O}_u \star \mathcal{O}^v$.

In [BCMP18b, Ex. 5.4] we gave an example of a projected Richardson variety in the Grassmannian $\mathrm{Gr}(2,6)$ that is not of the form $\Gamma_d(X_u, X^v)$. On the other hand, the following example shows that not all varieties of the form $\Gamma_{d-1,1}(X_u, X^v)$ are projected Richardson varieties. The studied variety $\Gamma_{d-1,1}(X_u, X^v)$ has rational singularities and satisfies $(p_d)_*[\mathcal{O}_{Z_{d-1,1}(X_u,X^v)}] = [\mathcal{O}_{\Gamma_{d-1,1}(X_u,X^v)}]$.

Example 8.31. Let $X = \mathrm{Gr}(3,6)$ be the Grassmannian of 3-planes in $\mathbb{C}^6$ and set $v = s_2 s_4 s_3$ and $u = v^\vee$. Then v corresponds to the partition $I(v) = (2,1)$. A calculation in $\mathrm{QH}(X)$ gives $([X_u] \star [X^v])_1 = 1$, so we have $\Gamma_1(X_u, X^v) = X$, and it follows from Proposition 8.20 that $\Gamma_{0,1}(X_u, X^v)$ is a divisor in X. Let $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ be the standard basis of $\mathbb{C}^6$ and set $A_1 = \mathrm{Span}\{e_1, e_2\}$, $A_2 = \mathrm{Span}\{e_3, e_4\}$, and $A_3 = \mathrm{Span}\{e_5, e_6\}$. The Richardson variety X_u^v is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and consists of all 3-planes $V = \mathrm{Span}\{a_1, a_2, a_3\}$ for which $a_i \in A_i$. The variety $\Gamma_{0,1}(X_u, X^v) = \Gamma_1(X_u^v)$ is the union of all lines through X_u^v. For any point $V' \in X$ we have $V' \in \Gamma_1(X_u^v)$ if and only if there exists a point $V \in X_u^v$ such that $\dim(V + V') \leq 4$. Consider the open affine subset $U \subset X$ corresponding to matrices of the form:

$$\begin{bmatrix} 1 & x_{11} & 0 & x_{12} & 0 & x_{13} \\ 0 & x_{21} & 1 & x_{22} & 0 & x_{23} \\ 0 & x_{31} & 0 & x_{32} & 1 & x_{33} \end{bmatrix} \tag{9}$$

The row space of such a matrix belongs to X_u^v if and only if has the form:

$$\begin{bmatrix} 1 & t_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t_3 \end{bmatrix} \tag{10}$$

The span of the 6 row vectors in (9) and (10) has rank 4 or less if and only if the matrix

$$\begin{bmatrix} x_{11}-t_1 & x_{12} & x_{13} \\ x_{21} & x_{22}-t_2 & x_{23} \\ x_{31} & x_{32} & x_{33}-t_3 \end{bmatrix}$$

has rank at most one, which implies that $x_{12}x_{23}x_{31} = x_{32}x_{21}x_{13}$. Since the divisor defined by this equation is irreducible, it coincides with $\Gamma_1(X_u^v) \cap U$.

Let p_{ijk} for $1 \leq i < j < k \leq 6$ denote the Plücker coordinates on X. Then $\Gamma_1(X_u^v)$ is defined by the equation $p_{123}\,p_{456} = p_{124}\,p_{356}$. It follows that $\Gamma_1(X_u^v)$ is a divisor of degree 2 in X, so it is not a projected Richardson variety. In fact, it follows from Theorem 3.4 that there are 6 projected Richardson divisors in X, namely $\Pi^1_{s_3 w_0^X}(X)$ and $\Pi^{s_k}_{w_0^X}(X)$ for $1 \leq k \leq 5$, and since their union is anticanonical by [KLS14, Lemma 5.4], each of these divisors has degree 1. Moreover, we obtain

$$[\mathcal{O}_{\Gamma_1(X_u^v)}] = 2\mathcal{O}^{(1)} - \mathcal{O}^{(1)} \cdot \mathcal{O}^{(1)} = 2\mathcal{O}^{(1)} - \mathcal{O}^{(2)} - \mathcal{O}^{(1,1)} + \mathcal{O}^{(2,1)}\,.$$

Using the Pieri formula [BM11, Thm. 5.4] we obtain

$$(\mathcal{O}_u \star \mathcal{O}^v)_1 = 1 - 2\mathcal{O}^{(1)} + \mathcal{O}^{(2)} + \mathcal{O}^{(1,1)} - \mathcal{O}^{(2,1)}$$

Using Theorem 8.10, we obtain

$$(p_1)_*[\mathcal{O}_{Z_{0,1}(X_u,X^v)}] = [\mathcal{O}_{\Gamma_1(X_u,X^v)}] - (\mathcal{O}_u \star \mathcal{O}^v)_1 = [\mathcal{O}_{\Gamma_1(X_u^v)}]\,.$$

This identity also follows from Proposition 8.20, granted that $\Gamma_1(X_u^v)$ has rational singularities. In fact, Chenyang Xu has shown us a proof [Xu] that the local equation $x_{12}x_{23}x_{31} = x_{32}x_{21}x_{13}$ is a canonical singularity, which implies that $\Gamma_1(X_u^v)\cap U$ has rational singularities. One can check that the local neighborhood of $\Gamma_1(X_u^v)$ defined by the non-vanishing of any Plücker coordinate p_{ijk} is a deformation of $\Gamma_1(X_u^v) \cap U$. It therefore follows from [Kaw99] that $\Gamma_1(X_u^v)$ has canonical singularities globally, or from [Elk78] that $\Gamma_1(X_u^v)$ has rational singularities globally. As mentioned earlier, it would be interesting to know if all varieties of the form $\Gamma_{d-1,1}(X_u, X^v)$ have rational singularities.

9. Divisors of the quantum-to-classical construction

Let $X = G/P_X$ be cominuscule and fix a degree $1 \leq d \leq d_X(2)$. Define the variety

$$Z_d^{(2)} \;=\; Z_d \times_{Y_d} Z_d \;=\; \{(\omega, x, y) \in Y_d \times X^2 \mid x, y \in \Gamma_\omega\}\,,$$

with projections $e_i : Z_d^{(2)} \to Z_d$ for $i = 1, 2$. Recall from Notation 8.22 that we set $\epsilon = 2$ if X is a Lagrangian Grassmannian and $\epsilon = 1$ otherwise. This means that the roots of $\Delta \smallsetminus \Delta_{Y_d}$ are long if $\epsilon = 1$ and short if $\epsilon = 2$. In particular, we have $(\alpha^\vee, \omega_\gamma) = \epsilon$ for any $\alpha \in \mathcal{P}_X$ satisfying $\delta(\alpha) \in \Delta \smallsetminus \Delta_{Y_d}$.

Proposition 9.1. *The set $\mathcal{D} = \{(\omega, x, y) \in Z_d^{(2)} \mid \operatorname{dist}(x, y) \leq d - 1\}$ is a divisor in $Z_d^{(2)}$ with rational singularities. The class of $\mathcal{D}$ in $\operatorname{Pic} Z_d^{(2)}$ is given by*

$$[\mathcal{D}] \;=\; (p_d e_1)^*[X^{s_\gamma}] + (p_d e_2)^*[X^{s_\gamma}] - \epsilon\,(q_d e_1)^*[\partial Y_d]\,.$$

Proof. The projection $e_2 : \mathcal{D} \to Z_d$ is G-equivariant and therefore a locally trivial fibration [BCMP13, Prop. 2.3], with fibers given by $\mathcal{D} \cap e_2^{-1}(\omega, x) \cong \Gamma_{d-1}(x) \cap \Gamma_\omega$. Lemma 5.18 and Lemma 5.4 therefore imply that $\mathcal{D}$ is a divisor in $Z_d^{(2)}$ with rational singularities.

The group $H^2(Z_d^{(2)};\mathbb{Z})$ is a free abelian group generated by the basis elements $(p_d e_i)^*[X^{s_\gamma}]$ for $i=1,2$ and $(q_d e_1)^*[Y_d^{s_\beta}]$ for $\beta \in \Delta \smallsetminus \Delta_{Y_d}$. Set $\omega_0 = 1.P_{Y_d} \in Y_d$ and $x_0 = 1.P_X \in X$, and define an embedding $\zeta : \Gamma_{\omega_0} \to Z_d^{(2)}$ by $\zeta(x) = (\omega_0, x, x_0)$. Since $\zeta(\Gamma_{\omega_0}) = e_2^{-1}(\omega_0, x_0)$, it follows from the local triviality of $\mathcal{D}$ that $\zeta^{-1}(\mathcal{D})$ is reduced. The identity $\zeta^*[\mathcal{D}] = [\Gamma_{\omega_0} \cap \Gamma_{d-1}(x_0)] = \zeta^*(p_d e_1)^*[X^{s_\gamma}]$ then implies that the coefficient of $(p_d e_1)^*[X^{s_\gamma}]$ in $[\mathcal{D}]$ is one. A symmetric argument shows that the coefficient of $(p_d e_2)^*[X^{s_\gamma}]$ in $[\mathcal{D}]$ is one.

Given $\beta \in \Delta \smallsetminus \Delta_{Y_d}$, let $\alpha \in \mathcal{P}_X$ be the minimal root for which $\delta(\alpha) = \beta$. This root α can be constructed as the sum of all simple roots in the interval $[\gamma, \beta]$ from γ to β in the Dynkin diagram. Then $I(\kappa_d) \cup \{\alpha\}$ is a straight shape in $\mathcal{P}_X$, and $\beta = \delta(\alpha) = \kappa_d.\alpha$. Let $C \subset Z_d$ be the T-stable curve through the points $\kappa_d.(\omega_0, x_0)$ and $\kappa_d s_\alpha.(\omega_0, x_0)$. Since $\kappa_d^{-1} = \kappa_d$, we obtain $s_\alpha \kappa_d.x_0 = \kappa_d s_\beta.x_0 = \kappa_d.x_0 \in \Gamma_{\omega_0}$. It follows that $x_0 \in \Gamma_\omega$ for each $\omega \in q_d(C)$, so $C' = \{(\omega, x, x_0) \mid (\omega, x) \in C\}$ is a curve in $Z_d^{(2)}$. Since $\kappa_d.x_0$ and $\kappa_d s_\alpha.x_0$ are points in $\Gamma_d(x_0) \smallsetminus \Gamma_{d-1}(x_0)$ by Theorem 5.1, we obtain $\operatorname{dist}(x, x_0) = d$ for all $x \in p_d(C)$, hence $C' \cap \mathcal{D} = \emptyset$ and $\int_{C'}[\mathcal{D}] = 0$. Finally, since $\int_{C'}(p_d e_1)^*[X^{s_\gamma}] = (\alpha^\vee, \omega_\gamma) = \epsilon$ and $\int_{C'}(q_d e_1)^*[Y_d^{s_\beta}] = (\alpha^\vee, \omega_\beta) = 1$, we deduce that the coefficient of $(q_d e_1)^*[Y_d^{s_\beta}]$ in $[\mathcal{D}]$ is $-\epsilon$, as required. □

Given any closed subset $\Omega \subset X$ we set $\mathring{\Gamma}_d(\Omega) = \Gamma_d(\Omega) \smallsetminus \Gamma_{d-1}(\Omega)$. We have

$$Y_d(\Omega, \mathring{\Gamma}_d(\Omega)) \;=\; \{\omega \in Y_d(\Omega) \mid \Gamma_\omega \cap \mathring{\Gamma}_d(\Omega) \neq \emptyset\}\,.$$

For any point $\omega \in Y_d(\Omega)$ we have $\Gamma_\omega \cap \Omega \neq \emptyset$, and since Γ_ω has diameter d, we obtain $\Gamma_\omega \subset \Gamma_d(\Omega)$. It follows that $Y_d(\Omega, \mathring{\Gamma}_d(\Omega)) = Y_d(\Omega, X \smallsetminus \Gamma_{d-1}(\Omega))$. Since q_d is an open map, this shows that $Y_d(\Omega, \mathring{\Gamma}_d(\Omega))$ is a relatively open subset of $Y_d(\Omega)$. Notice also that for $v \in W^X$, we have $Y_d(X^v, \mathring{\Gamma}_d(X^v)) \neq \emptyset$ if and only if $d \le d_{\max}(v)$.

Proposition 9.2. *Assume that $\Omega \subset X$ is a Schubert variety.*

(a) *For each $\omega \in Y_d(\Omega, \mathring{\Gamma}_d(\Omega))$, $\Gamma_\omega \cap \Omega$ is a (reduced) single point.*

(b) *The map $\sigma : Y_d(\Omega, \mathring{\Gamma}_d(\Omega)) \to \Omega$ defined by $\{\sigma(\omega)\} = \Gamma_\omega \cap \Omega$ is a morphism of varieties.*

Proof. Given any point $\omega \in Y_d(\Omega, \mathring{\Gamma}_d(\Omega))$, the intersection $\Gamma_\omega \cap \Omega$ is a Schubert variety in Γ_ω by Theorem 2.8. If it has positive dimension, then it meets the Schubert divisor $\Gamma_\omega \cap \Gamma_{d-1}(z)$ for every point $z \in \Gamma_\omega$ by Lemma 5.18. This implies that $\Gamma_\omega \subset \Gamma_{d-1}(\Omega)$, a contradiction. This proves part (a).

Since $q_d : p_d^{-1}(\Omega) \to Y_d(\Omega)$ is a projective morphism, so is the restriction $q_d : p_d^{-1}(\Omega) \cap Z_d(\mathring{\Gamma}_d(\Omega)) \to Y_d(\Omega, \mathring{\Gamma}_d(\Omega))$, and part (a) implies that this restricted map is bijective. Since the target is normal, the map is an isomorphism by Zariski's main theorem. Part (b) follows from this because σ is the composition of the inverse map with p_d. □

Given $u, v \in W^X$ we define the varieties

$$\begin{aligned}\mathring{Y}_d(X_u, X^v) \;&=\; Y_d(X_u, \mathring{\Gamma}_d(X_u)) \cap Y_d(X^v, \mathring{\Gamma}_d(X^v))\,, \text{ and}\\ \mathring{Y}_{d-1,1}(X_u, X^v) \;&=\; \mathring{Y}_d(X_u, X^v) \cap Y_{d-1,1}(X_u, X^v)\,.\end{aligned}$$

It follows from Kleiman's transversality theorem [Kle74] that $\mathring{Y}_d(X_u, X^v)$ is a dense open subset of $Y_d(X_u, X^v)$ whenever $d \le \min(d_{\max}(u^\vee), d_{\max}(v))$. By Proposition 9.2 there are morphisms $\sigma_1 : \mathring{Y}_d(X_u, X^v) \to X_u$ and $\sigma_2 : \mathring{Y}_d(X_u, X^v) \to X^v$

defined by $\{\sigma_1(\omega)\} = \Gamma_\omega \cap X_u$ and $\{\sigma_2(\omega)\} = \Gamma_\omega \cap X^v$. By Corollary 5.20(b) we have

$$\mathring{Y}_{d-1,1}(X_u, X^v) \;=\; \left\{\omega \in \mathring{Y}_d(X_u, X^v) \mid \operatorname{dist}(\sigma_1(\omega), \sigma_2(\omega)) \le d-1\right\}. \tag{11}$$

Proposition 9.3. *Assume* $1 \le d \le \min(d_{\max}(u^\vee), d_{\max}(v))$. *Then* $\mathring{Y}_{d-1,1}(X_u, X^v)$ *is a Cartier divisor in* $\mathring{Y}_d(X_u, X^v)$*, with class in* $\operatorname{Pic} \mathring{Y}_d(X_u, X^v)$ *given by*

$$[\mathring{Y}_{d-1,1}(X_u, X^v)] \;=\; \sigma_1^*[X^{s_\gamma}] + \sigma_2^*[X^{s_\gamma}] - \epsilon\,[\partial Y_d]\,.$$

Proof. Define the variety

$$\mathring{Z}_d^{(2)}(X_u, X^v) = e_1^{-1}(\Omega_1) \cap e_2^{-1}(\Omega_2)\,,$$

where $\Omega_1 = p_d^{-1}(X_u) \cap Z_d(\mathring{\Gamma}_d(X_u))$ and $\Omega_2 = p_d^{-1}(X^v) \cap Z_d(\mathring{\Gamma}_d(X^v))$. Since $\Omega_1 \subset p_d^{-1}(X_u)$ and $\Omega_2 \subset p_d^{-1}(X^v)$ are open subsets of opposite Schubert varieties in Z_d, it follows from Kleiman's transversality theorem [Kle74] that $\mathring{Z}_d^{(2)}(X_u, X^v)$ and $\mathcal{D} \cap \mathring{Z}_d^{(2)}(X_u, X^v)$ are reduced, where $\mathcal{D}$ is the divisor of Proposition 9.1. Proposition 9.2 shows that the map $\varphi : \mathring{Y}_d(X_u, X^v) \to \mathring{Z}_d^{(2)}(X_u, X^v)$ defined by $\varphi(\omega) = (\omega, \sigma_1(\omega), \sigma_2(\omega))$ is an isomorphism with inverse morphism $q_d e_1$, and (11) shows that $\mathring{Y}_{d-1,1}(X_u, X^v) = \varphi^{-1}(\mathcal{D})$ holds as (reduced) subschemes of $\mathring{Y}_d(X_u, X^v)$. The result therefore follows from Proposition 9.1. □

Proposition 9.4. *Let* $u \in W^X$, $1 \le d \le d_{\max}(u^\vee)$, *and* $z \in \mathring{X}_{u(d)}$. *Then the morphism* $\sigma : Y_d(X_u, z) \to X_u$ *defined by* $\{\sigma(\omega)\} = \Gamma_\omega \cap X_u$ *is injective, and we have* $\sigma^*[X^{s_\gamma}] = \epsilon[\partial Y_d]$ *in* $\operatorname{Pic} Y_d(X_u, z)$.

Proof. The assumptions imply that $z \in \mathring{\Gamma}_d(X_u)$, so we must have $\operatorname{dist}(\sigma(\omega), z) \ge d$ for any point $\omega \in Y_d(X_u, z)$. We deduce from Corollary 5.20(b) that $\Gamma_\omega = \Gamma_d(\sigma(\omega), z)$. This shows that σ is injective.

The projection $q_d : p_d^{-1}(z) \to Y_d(z)$ is an isomorphism, and the inverse image of $Y_d(X_u, z)$ is $p_d^{-1}(z) \cap Z_d(X_u)$, which is a translate of the Schubert variety $(F_d)_{u_d}$ by Corollary 6.9(c). This shows that the restriction map $\operatorname{Pic} Y_d(z) \to \operatorname{Pic} Y_d(X_u, z)$ is surjective. Since the restriction map $\operatorname{Pic} Y_d \to \operatorname{Pic} Y_d(z)$ is also surjective, it follows that $\operatorname{Pic} Y_d(X_u, z)$ is generated by (the restrictions of) the divisors $[Y_d^{s_\beta}]$ for $\beta \in \Delta \smallsetminus \Delta_{Y_d}$. The class $[Y_d^{s_\beta}]$ is non-zero if and only if $\beta \in I(u_d)$, which by Proposition 6.7(b) is equivalent to $z_d.\beta \in I(u)$. Notice also that $z_d.\beta$ is a minimal box of $I(\kappa_d^\vee) \smallsetminus I(z_d^\vee)$ by Proposition 6.2(b).

To compute $\sigma^*[X^{s_\gamma}]$, we may assume that $z = u(d).P_X$, since the maps p_d and q_d are equivariant. Let $\beta \in \Delta \smallsetminus \Delta_{Y_d}$ and assume that $\alpha = z_d.\beta \in I(u)$. Set $\overline{u} = u \cap z_d^\vee$. Then $u(d) = \overline{u} z_d$, $\overline{u} s_\alpha \in W^X$, and $I(\overline{u} s_\alpha) = I(\overline{u}) \cup \{\alpha\}$. We claim that the points $u(d).P_{Y_d}$ and $u(d)s_\beta.P_{Y_d}$ belong to $Y_d(X_u, z)$. Indeed these points are in $Y_d(z)$, since they are the images of $u(d).P_{Z_d}$ and $u(d)s_\beta.P_{Z_d}$. Since $\kappa_d.P_X \in X_{\kappa_d} = p_d q_d^{-1}(1.P_{Y_d})$, we have $1.P_{Y_d} \in Y_d(\kappa_d.P_X)$, hence $u(d).P_{Y_d} \in Y_d(\overline{u} z_d \kappa_d.P_X) = Y_d(\overline{u}.P_X)$ and $u(d)s_\beta.P_{Y_d} = \overline{u} s_\alpha z_d.P_{Y_d} \in Y_d(\overline{u} s_\alpha.P_X)$. This proves the claim, and also shows that $\sigma(u(d).P_{Y_d}) = \overline{u}.P_X$ and $\sigma(u(d)s_\beta.P_{Y_d}) = \overline{u} s_\alpha.P_X$. We deduce that $Y_d(X_u, z)$ contains the T-stable curve $C \subset Y_d$ through $u(d).P_{Y_d}$ and $u(d)s_\beta.P_{Y_d}$, and that $\sigma(C) \subset X_u$ is the T-stable curve through $\overline{u}.P_X$ and $\overline{u} s_\alpha.P_X$. This implies that $\sigma_*[(Y_d)_{s_\beta}] = \sigma_*[C] = [\sigma(C)] = (\alpha^\vee, \omega_\gamma)[X_{s_\gamma}]$, so it follows from Poincaré duality that the coefficient of $[Y_d^{s_\beta}]$ in $\sigma^*[X^{s_\gamma}]$ is equal to $\epsilon = (\alpha^\vee, \omega_\gamma)$, as required. □

Proof of Proposition 8.23. Let $z \in \Gamma_d(X_u, X^v)$ be a general point, and set $R = p_d^{-1}(z) \cap Z_d(X_u, X^v)$ and $D = p_d^{-1}(z) \cap Z_{d-1,1}(X_u, X^v)$. Then R is a Richardson variety by Theorem 2.10, and q_d restricts to an isomorphism of R onto $R' = q_d p_d^{-1}(z) \cap Y_d(X_u, X^v) = Y_d(X_u, z) \cap Y_d(X^v, z)$, under which D is pulled back from $D' = R' \cap Y_{d-1,1}(X_u, X^v)$. By the choice of z and the bound $d \leq \min(d_{\max}(u^\vee), d_{\max}(v))$, we may assume that $z \in \mathring{X}_{u(d)} \cap \mathring{X}^{v(-d)} \subset \mathring{\Gamma}(X_u) \cap \mathring{\Gamma}(X^v)$. This implies that R' is contained in $\mathring{Y}_d(X_u, X^v)$, so it follows from Proposition 9.3 and Proposition 9.4 that D' is a Cartier divisor in R' of class $[D'] = \sigma_1^*[X^{s_\gamma}] + \sigma_2^*[X^{s_\gamma}] - \epsilon[\partial Y_d] = \epsilon[\partial Y_d]$. The result follows from this. $\square$

Remark 9.5. We demonstrate in Example 9.6 that the identity $\sigma^*[X^{s_\gamma}] = \epsilon[\partial Y_d]$ may fail to hold in $\operatorname{Pic} Y_d(X_u, \mathring{\Gamma}_d(X_u))$, with σ as in Proposition 9.2. However, the proof of Proposition 9.4 shows that this identity holds whenever $\Delta \smallsetminus \Delta_{Y_d} \subset I(u_d)$, as in this case we have $\operatorname{Pic} Y_d(X_u, z) = \operatorname{Pic} Y_d(X_u, \mathring{\Gamma}_d(X_u)) = \operatorname{Pic} Y_d$.

Example 9.6. Let $X = \operatorname{Gr}(m,n)$ be a Grassmannian of diameter $d_X(2) \geq 3$, and set $d = 2$ and $u = s_\gamma$. Let $E_k = \langle e_1, e_2, \dots, e_k \rangle \subset \mathbb{C}^n$ be the subspace spanned by the first k basis vectors, for $0 \leq k \leq n$. Then $X_u = \mathbb{P}(E_{m+1}/E_{m-1}) = \{V \in X \mid E_{m-1} \subset V \subset E_{m+1}\}$. Set $N_0 = \langle e_{m+2}, e_{m+3} \rangle$, $S_0 = E_m \oplus N_0$, and let $C \subset Y_d = \operatorname{Fl}(m-2, m+2; n)$ be the curve given by $C = \{(K, S_0) \mid K \in \mathbb{P}(E_{m-1}/E_{m-3})\}$. Since $E_m \in \Gamma_\omega \cap X_u$ for each $\omega \in C$, we have $C \subset Y_d(X_u)$. Define $z : C \to X$ by $z((K, S_0)) = K \oplus N_0$. For $V \in X_u$ and $(K, S_0) \in C$ we have $V \cap (K \oplus N_0) = K$. This implies that $\operatorname{dist}(V, z(\omega)) = 2$ for each $V \in X_u$ and $\omega \in C$, so $z(\omega) \in \mathring{\Gamma}_d(X_u) \cap \Gamma_\omega$. In particular, we have $\omega \in Y_d(X_u, z(\omega))$ and $C \subset Y_d(X_u, \mathring{\Gamma}_d(X_u))$. However, since the restriction $\sigma : C \to X_u$ of the morphism of Proposition 9.2 is the constant function $\sigma(\omega) = E_m$, we obtain $\int_C \sigma^*[X^{s_\gamma}] = 0 \neq 1 = \int_C [\partial Y_d]$. More generally, our construction shows that $\sigma^*[X^{s_\gamma}] = 0 \in \operatorname{Pic} Y_d(X_u, \mathring{\Gamma}_d(X_u)) = \operatorname{Pic} Y_d$.

Proposition 8.23 shows that the restriction of the divisor $Z_{d-1,1}(X_u, X^v)$ to $R = Z_d(X_u, X^v) \cap p_d^{-1}(z)$ is a Cartier divisor that can be pulled back from Z_d. The following example shows that $Z_{d-1,1}(X_u, X^v)$ may not itself be a Cartier divisor pulled back from Z_d.

Example 9.7. Let $X = \operatorname{LG}(3,6)$ and define $u, v \in W^X$ by $I(u) = (3,2)$ and $I(v) = (2,1)$. The corresponding products in $\operatorname{QH}(X)$ and $\operatorname{QK}(X)$ are given by

$$[X_u] \star [X^v] = 2[X^{(3,1)}] \quad \text{and}$$
$$\mathcal{O}_u \star \mathcal{O}^v = 2\mathcal{O}^{(3,1)} - \mathcal{O}^{(3,2)} + q\mathcal{O}^{(1)} - q\mathcal{O}^{(2)}\,.$$

Let $d = 1$. We have $Y_d = \operatorname{IG}(2,6) = C_3/P_2$ and $Y_d(X_u, X^v) = q_d p_d^{-1}(X^{s_3 s_2 s_3}) = Y_d^{s_3 s_2}$. The general fibers of $p_d : Z_d(X_u, X^v) \to \Gamma_d(X_u, X^v)$ are projective lines, and $p_d : Z_{d-1,1}(X_u, X^v) \to \Gamma_d(X_u, X^v)$ is a morphism of degree 2. We also have $Y_{d-1} = X$, $Y_{d-1,d} = Z_d$, and $Y_{d-1,1}(X_u, X^v) = q_d p_d^{-1}(X_u \cap X^v)$. It follows that

$$[Z_{d-1,1}(X_u, X^v)] = (q_d)^*(q_d)_*(p_d)^*([X_u] \cdot [X^v]) = 2[Z_d^{s_3 s_1 s_2}]\,.$$

Assume that $Z_{d-1,1}(X_u, X^v)$ is the intersection of $Z_d(X_u, X^v)$ with an effective Cartier divisor $D \subset Z_d$. Then we must have $[D] \cdot [Z_d(X_u, X^v)] = [Z_{d-1,1}(X_u, X^v)]$ in $H^*(Z_d)$. But we have $[Z_d(X_u, X^v)] = [Z_d^{s_3 s_2}]$ and $[D] = a[Z_d^{s_2}] + b[Z_d^{s_3}]$ for some integers a and b. Now compute the products

$$[Z_d^{s_2}] \cdot [Z_d^{s_3 s_2}] = [Z_d^{s_3 s_1 s_2}] + [Z_d^{s_2 s_3 s_2}]$$

and

$$[Z_d^{s_3}] \cdot [Z_d^{s_3 s_2}] = [Z_d^{s_3 s_2 s_3}] + 2[Z_d^{s_2 s_3 s_2}]$$

It follows that the coefficient of $[Z_d^{s_3 s_2 s_3}]$ in $[D]\cdot[Z_d(X_u, X^v)]$ is b, and the coefficient of $[Z_d^{s_2 s_3 s_2}]$ is $a+2b$. Since these Schubert classes do not appear in $[Z_{d-1,1}(X_u, X^v)]$, we obtain $a = b = 0$, a contradiction.

10. Fibers of Gromov-Witten varieties

Let X be a cominuscule flag variety and fix $u, v \in W^X$ and $1 \leq d \leq d_X(2)$. We finish this paper by proving that completions of the general fibers of the rational maps $M_d(X_u, X^v) \dashrightarrow Z_d(X_u, X^v)$ and $M_{d-1,1}(X_u, X^v) \dashrightarrow Z_{d-1,1}(X_u, X^v)$ are cohomologically trivial. While this assertion from the introduction is not required for the proofs of our main results, it provides additional details of the relationship between the geometry of Gromov-Witten varieties and analogous varieties obtained from the quantum-to-classical construction.

Recall the maps of the diagram (3), and define the varieties

$$\begin{aligned} \mathrm{B}\ell_{d-1,1} &= \pi^{-1}(M_{d-1,1}) \subset \mathrm{B}\ell_d\,, \\ \mathrm{B}\ell_d(X_u, X^v) &= \pi^{-1}(M_d(X_u, X^v))\,, \text{ and} \\ \mathrm{B}\ell_{d-1,1}(X_u, X^v) &= \mathrm{B}\ell_d(X_u, X^v) \cap \mathrm{B}\ell_{d-1,1}\,. \end{aligned}$$

Since the birational map $M_d \dashrightarrow Z_d$ is defined as a morphism exactly on the open subset of M_d over which $\pi : \mathrm{B}\ell_d \to M_d$ is an isomorphism, our assertion is justified by the following result. (We consider a map between empty varieties to have cohomologically trivial fibers.)

Theorem 10.1. *The general fibers of the maps $e_3\phi : \mathrm{B}\ell_d(X_u, X^v) \to Z_d(X_u, X^v)$ and $e_3\phi : \mathrm{B}\ell_{d-1,1}(X_u, X^v) \to Z_{d-1,1}(X_u, X^v)$ are cohomologically trivial.*

The proof requires some additional results, starting with the following consequence of Theorem 8.12.

Corollary 10.2. *Let $f : M \to N$ and $g : N \to P$ be morphisms of complex projective varieties with rational singularities. Assume that the general fibers of f are cohomologically trivial. Then the general fibers of g are cohomologically trivial if and only if the general fibers of gf are cohomologically trivial.*

Proof. This follows from Theorem 8.12, as the Grothendieck spectral sequence shows that $R^i g_* \mathcal{O}_N = R^i(gf)_* \mathcal{O}_M$. □

Lemma 10.3. *Let $f : M \to N$ be a birational morphism of irreducible varieties, with N normal. Let $D \subset M$ be an irreducible subvariety of codimension 1, and assume that $f(D)$ has codimension 1 in N. Then the restricted map $f : D \to \overline{f(D)}$ is birational.*

Proof. The assumptions imply that $f(D)$ meets the non-singular locus of N, so we may assume that N is non-singular. Let $Z \subset N$ be the closed subset of points where the rational map f^{-1} is not defined as a morphism. Then $f^{-1}(Z)$ is a proper closed subset of M, and the fibers of $f^{-1}(Z) \to Z$ have positive dimension by [Sha94, 4.4, Thm. 2]. It follows that Z has codimension at least 2 in N, so f^{-1} is defined on a dense open subset of $f(D)$. □

Recall the variety $Y_{d-1,d} = G/(P_{Y_{d-1}} \cap P_{Y_d})$ from Section 8.2 and define

$$\begin{aligned}
\widehat{Z}_{d-1,1} &= Z_d \times_{Y_d} Y_{d-1,d} \\
&= \{(\eta,\omega,z) \in Y_{d-1} \times Y_d \times X \mid \Gamma_\eta \subset \Gamma_\omega \text{ and } z \in \Gamma_\omega\}, \\
\widehat{Z}^{(3)}_{d-1,1} &= Z_{d-1} \times_{Y_{d-1}} Z_{d-1} \times_{Y_{d-1}} \widehat{Z}_{d-1,1} \\
&= \{(\eta,\omega,x,y,z) \in Y_{d-1} \times Y_d \times X^3 \mid x,y \in \Gamma_\eta \subset \Gamma_\omega \text{ and } z \in \Gamma_\omega\}, \text{ and} \\
Z^{(3)}_{d-1,1} &= \{(\omega,x,y,z) \in Z^{(3)}_d \mid \operatorname{dist}(x,y) \le d-1\} \ \subset Z^{(3)}_d .
\end{aligned}$$

Lemma 10.4. *The restricted morphism $\phi : \mathrm{B}\ell_{d-1,1} \to Z^{(3)}_{d-1,1}$ and the projection $p' : \widehat{Z}^{(3)}_{d-1,1} \to Z^{(3)}_{d-1,1}$ are birational.*

Proof. Let $(\omega,x,y,z) \in Z^{(3)}_{d-1,1}$. By Corollary 5.20(b) there exists $\eta \in Y_{d-1}$ such that $x,y \in \Gamma_\eta \subset \Gamma_\omega$, and η is unique when $\operatorname{dist}(x,y) = d-1$. This shows that p' is birational. It follows from Lemma 5.18 that $\Gamma_\eta \cap \Gamma_1(z)$ contains at least one point t. There exists a stable curve in Γ_η of degree $d-1$ through x, y, and t by Theorem 5.17, and t is connected to z by a line. This shows that (ω,x,y,z) belongs to $\phi(\mathrm{B}\ell_{d-1,1})$. Since $\phi : \mathrm{B}\ell_d \to Z^{(3)}_d$ is birational by Proposition 5.21, it follows from Proposition 9.1 and Lemma 10.3 that $\phi : \mathrm{B}\ell_{d-1,1} \to Z^{(3)}_{d-1,1}$ is birational. □

The proof of Theorem 10.1 uses the following varieties:

$$\begin{aligned}
Z^{(3)}_d(X_u,X^v) &= (p_d e_1)^{-1}(X_u) \cap (p_d e_2)^{-1}(X^v) \ \subset Z^{(3)}_d, \\
Z^{(3)}_{d-1,1}(X_u,X^v) &= Z^{(3)}_d(X_u,X^v) \cap Z^{(3)}_{d-1,1}, \\
\widehat{Z}^{(3)}_{d-1,1}(X_u,X^v) &= p^{-1}_{d-1}(X_u) \times_{Y_{d-1}} p^{-1}_{d-1}(X^v) \times_{Y_{d-1}} \widehat{Z}_{d-1,1}, \\
Y_{d-1,d}(X_u,X^v) &= \phi^{-1}_{d-1}(Y_{d-1}(X_u,X^v)), \text{ and} \\
\widehat{Z}_{d-1,1}(X_u,X^v) &= Y_{d-1,d}(X_u,X^v) \times_{Y_d} Z_d .
\end{aligned}$$

The first three spaces are the subvarieties of $Z^{(3)}_d$, $Z^{(3)}_{d-1,1}$, $\widehat{Z}^{(3)}_{d-1,1}$ defined by $x \in X_u$ and $y \in X^v$. The last variety $\widehat{Z}_{d-1,1}(X_u,X^v)$ consists of all triples $(\eta,\omega,z) \in \widehat{Z}_{d-1,1}$ for which $X_u \cap \Gamma_\eta \neq \emptyset$ and $X^v \cap \Gamma_\eta \neq \emptyset$.

Proof of Theorem 10.1. It follows from Proposition 5.21 and Kleiman's transversality theorem [Kle74] that $\phi : \mathrm{B}\ell_d(X_u,X^v) \to Z^{(3)}_d(X_u,X^v)$ is birational, and the fiber of $e_3 : Z^{(3)}_d(X_u,X^v) \to Z_d(X_u,X^v)$ over (ω,z) is isomorphic to $(\Gamma_\omega \cap X_u) \times (\Gamma_\omega \cap X^v)$, which is a product of Schubert varieties by Theorem 2.8. Using that $\mathrm{B}\ell_d(X_u,X^v)$, $Z^{(3)}_d(X_u,X^v)$, and $Z_d(X_u,X^v)$ have rational singularities, it follows from Corollary 10.2 that the general fibers of $e_3\phi : \mathrm{B}\ell_d(X_u,X^v) \to Z_d(X_u,X^v)$ are cohomologically trivial.

Consider the commutative diagram:

$$
\begin{array}{ccccccc}
Z^{(3)}_{d-1,1}(X_u, X^v) & \xleftarrow[\phi]{\approx} & & & \mathrm{B}\ell_{d-1,1}(X_u, X^v) & & \\
\approx \big\uparrow p' & & \searrow^{e_3} & & \big\downarrow e_3\phi & & \\
\widehat{Z}^{(3)}_{d-1,1}(X_u, X^v) & \xrightarrow[\widehat{p}]{} & \widehat{Z}_{d-1,1}(X_u, X^v) & \xrightarrow[\phi'_d]{} & Z_{d-1,1}(X_u, X^v) & \xrightarrow[\subset]{} & Z_d \\
 & & \big\downarrow & & \big\downarrow q_d & & \big\downarrow q_d \\
 & & Y_{d-1,d}(X_u, X^v) & \xrightarrow[\phi_d]{} & Y_{d-1,1}(X_u, X^v) & \xrightarrow[\subset]{} & Y_d
\end{array} \tag{12}
$$

Here $\widehat{p}$ is the projection that forgets x and y, and ϕ'_d is the base change of ϕ_d along q_d. It follows from [BCMP13, Thm. 2.5 and Prop. 3.7] together with Theorem 2.13 and Proposition 9.1 that all varieties in the diagram (12) have rational singularities. The maps p' and ϕ with target $Z^{(3)}_{d-1,1}(X_u, X^v)$ are birational by Lemma 10.4 and Kleiman's transversality theorem [Kle74]. The fiber of $\widehat{p}$ over $(\eta, \omega, z) \in \widehat{Z}_{d-1,1}(X_u, X^v)$ is the product $(\Gamma_\eta \cap X_u) \times (\Gamma_\eta \cap X^v)$ of Schubert varieties by Theorem 2.8. The fibers of ϕ'_d coincide with the fibers of ϕ_d, and the general such fibers are Richardson varieties by Theorem 2.10. We deduce from Corollary 10.2 that the general fibers of the maps e_3 and $e_3\phi$ with target $Z_{d-1,1}(X_u, X^v)$ are cohomologically trivial. This completes the proof. □

Corollary 10.5. *The restricted maps* $e_3\phi : \mathrm{B}\ell_d(X_u, X^v) \to Z_d(X_u, X^v)$ *and* $e_3\phi : \mathrm{B}\ell_{d-1,1}(X_u, X^v) \to Z_{d-1,1}(X_u, X^v)$ *are birational for* $d \leq \min(d_{\max}(u^\vee), d_{\max}(v))$.

Proof. It follows from Corollary 5.14(a) and Lemma 6.1 that $\dim \mathrm{B}\ell_d(X_u, X^v) = \dim Z_d(X_u, X^v) = \ell(u) - \ell(v) + \int_d c_1(T_X)$ (when these varieties are not empty), and from Proposition 8.18 that $\dim \mathrm{B}\ell_{d-1,1}(X_u, X^v) = \dim Z_{d-1,1}(X_u, X^v) = \ell(u) - \ell(v) + \int_d c_1(T_X) - 1$. □

Remark 10.6. The proof of Theorem 10.1 shows more generally that the general fibers of $e_3\phi : \mathrm{B}\ell_d(X_u, X^v) \to Z_d(X_u, X^v)$ are rational, and the general fibers of $e_3\phi : \mathrm{B}\ell_{d-1,1}(X_u, X^v) \to Z_{d-1,1}(X_u, X^v)$ are rationally connected. The last statement uses [GHS03, Cor. 1.3].

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