EULER CHARACTERISTICS OF COMINUSCULE QUANTUM K-THEORY

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Abstract. We prove an identity relating the product of two opposite Schubert varieties in the (equivariant) quantum K-theory ring of a cominuscule flag variety to the minimal degree of a rational curve connecting the Schubert varieties. We deduce that the sum of the structure constants associated to any product of Schubert classes is equal to one. Equivalently, the sheaf Euler characteristic map extends to a ring homomorphism defined on the quantum K-theory ring.

1. Introduction

Let $X = G/P$ be a flag variety defined by a semisimple complex Lie group G and a parabolic subgroup P. The (small) equivariant quantum K-theory ring $\mathrm{QK}_T(X)$ of Givental [12] is a formal deformation of the Grothendieck ring $K_T(X)$ of T-equivariant algebraic vector bundles on X, where T is a maximal torus in G. This ring encodes geometric information about families of rational curves meeting triples of general Schubert varieties in X, including the arithmetic genera of such families. In this note we prove three results that present aspects of this information in a concrete form when X is a cominuscule flag variety, that is, a Grassmannian $\mathrm{Gr}(m,n)$ of type A, a Lagrangian Grassmannian $\mathrm{LG}(n,2n)$, a maximal orthogonal Grassmannian $\mathrm{OG}(n,2n)$, a quadric hypersurface Q^n, or one of two exceptional spaces known as the Cayley plane E_6/P_6 and the Freudenthal variety E_7/P_7.

The ring $K_T(X)$ has a basis of Schubert structure sheaves $\mathcal{O}^w = [\mathcal{O}_{X^w}]$ over the ring $\Gamma = K_T(\mathrm{pt})$ of virtual representations of T. The quantum K-theory ring $\mathrm{QK}_T(X)$ consists of all formal power series with coefficients in $K_T(X)$. The product of two Schubert classes in this ring has the form

$$\mathcal{O}^u \star \mathcal{O}^v = \sum_{w,d\geq 0} N_{u,v}^{w,d}\, q^d\, \mathcal{O}^w \,, \tag{1}$$

where the sum is over all Schubert classes $\mathcal{O}^w$ and effective degrees $d \in H_2(X;\mathbb{Z})$. Givental defined the structure constants $N_{u,v}^{w,d} \in \Gamma$ as polynomial expressions in the K-theoretic Gromov-Witten invariants of X and proved that the resulting product is associative [12]. The structure of the ring

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$\mathrm{QK}_T(X)$ has been studied in the cominuscule case in a series of papers by Chaput, Mihalcea, Perrin, and the first author [7, 8, 5, 4, 3]. In particular, it has been proved that only finitely many of the coefficients $N_{u,v}^{w,d}$ are non-zero [5, 6]. The quantum K-theory of Grassmannians of type A has been related to integrable systems by Gorbounov and Korff [13]. Conjectures for the ring structure of $\mathrm{QK}_T(X)$ have also been given by Lenart and Maeno [17] and Lenart and Postnikov [18] when $X = G/B$ is defined by a Borel subgroup of G. In general the structure constants $N_{u,v}^{w,d}$ are conjectured to satisfy Griffeth-Ram positivity [14], that is, up to a sign these constants are polynomials with non-negative coefficients in the classes $[\mathbb{C}_{-\alpha}]-1 \in \Gamma$, where $\mathbb{C}_{-\alpha}$ is any one-dimensional representation of T defined by a negative root (see e.g. [3]). This conjecture has been proved for the structure constants $N_{u,v}^{w,0}$ of the equivariant K-theory ring $K_T(X)$ by Anderson, Griffeth, and Miller [1], and for the equivariant quantum cohomology ring $\mathrm{QH}_T(X)$ by Mihalcea [19].

Assume now that X is a cominuscule flag variety. Our work started with the experimental observation that the sum of the structure constants defining any product $\mathcal{O}^u \star \mathcal{O}^v$ of Schubert classes in $\mathrm{QK}_T(X)$ is equal to 1. This is our first result.

Theorem 1. *For fixed u, v we have* $\displaystyle\sum_{w,d\geq 0} N_{u,v}^{w,d} = 1$ *in* Γ.

Let $\chi_X : K_T(X) \to \Gamma$ be the sheaf Euler characteristic map, defined as the equivariant pushforward along the structure morphism $X \to \{\mathrm{pt}\}$. Equivalently, χ_X is the unique Γ-linear map defined by $\chi_X(\mathcal{O}^w) = 1$ for all w. While this map is not a ring homomorphism unless X is a single point, Theorem 1 is equivalent to the following statement.

Theorem 2. *Let $QK_T^{\mathrm{poly}}(X) \subset QK_T(X)$ be the subring of all finite power series. There exists a unique ring homomorphism $\widehat{\chi} : QK_T^{\mathrm{poly}}(X) \to \Gamma$ defined by $\widehat{\chi}(q) = 1$ and $\widehat{\chi}(\mathcal{O}^w) = 1$ for all w.*

Given two opposite Schubert varieties X^u and X_v in the cominuscule flag variety X, let $\mathrm{dist}(X^u, X_v)$ denote the minimal degree of a rational curve connecting these subvarieties. This degree is the smallest power of the deformation parameter q that occurs in the product $\mathcal{O}^u \star \mathcal{O}_v$, where $\mathcal{O}_v = [\mathcal{O}_{X_v}]$. Let $\chi : \mathrm{QK}_T(X) \to \Gamma[\![q]\!]$ denote the $\Gamma[\![q]\!]$-linear extension of the sheaf Euler characteristic map, defined by $\chi(\mathcal{O}^w) = 1$ for all w. Both of the above theorems are consequences of the following identity.

Theorem 3. *We have $\chi(\mathcal{O}^u \star \mathcal{O}_v) = q^{\mathrm{dist}(X^u, X_v)}$.*

The proof of Theorem 3 is based on a construction of the ring $\mathrm{QK}_T(X)$ using projected Gromov-Witten varieties [3], together with a relation between such varieties and K-theoretic Gromov-Witten invariants [15, 4].

Our results have been utilized by the second author to give an explicit formula for the Schubert structure constants of the equivariant quantum K-theory of projective space $\mathrm{QK}_T(\mathbb{P}^n)$. This formula establishes Griffeth-Ram positivity in this case [9].

Theorem 1 was observed independently by Changzheng Li and Leonardo Mihalcea, who also obtained proofs in some cases. We thank Li and Mihalcea for helpful discussions on this subject.

2. Quantum K-theory

In this section we briefly recall the definitions used in the statements of our results, as well as the background required to prove them. A more detailed introduction to quantum K-theory can be found in [3, §2].

Let G be a semisimple complex linear algebraic group and fix a maximal torus T, a Borel subgroup B, and a parabolic subgroup P such that $T \subset B \subset P \subset G$. Let $W = N_G(T)/T$ be the Weyl group of G, let $W_P = N_P(T)/T$ be the Weyl group of P, and let $W^P \subset W$ be the subset of minimal representatives for the cosets in W/W_P. Each element $w \in W^P$ defines the *Schubert varieties* $X_w = \overline{Bw.P}$ and $X^w = \overline{B^- w.P}$ in the flag variety $X = G/P$, where $B^- \subset G$ denotes the opposite Borel subgroup defined by $B \cap B^- = T$. We have $\dim(X_w) = \operatorname{codim}(X^w, X) = \ell(w)$, where $\ell(w)$ is the length of w. A simple root γ of G is called *cominuscule* if the coefficient of γ is one when the highest root is written as a linear combination of simple roots. The flag variety X is *cominuscule* if W^P contains a single simple reflection s_γ defined by a cominuscule simple root γ. We will assume this in what follows. In particular, we can identify $H_2(X;\mathbb{Z}) = \mathbb{Z}\,[X_{s_\gamma}]$ with the group of integers $\mathbb{Z}$.

Given a non-negative degree $d \in H_2(X;\mathbb{Z})$ we let $\overline{\mathcal{M}}_{0,n}(X,d)$ denote the Kontsevich moduli space of n-pointed stable maps to X of degree d and genus zero, see [11]. This space is equipped with evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X,d) \to X$ for $1 \leq i \leq n$. Given any closed subvariety $Z \subset X$, the *curve neighborhood* $\Gamma_d(Z) = \mathrm{ev}_2(\mathrm{ev}_1^{-1}(Z))$ is the union of all connected rational curves of degree d in X that meet Z. It was proved in [5] that, if Z is a Schubert variety in X, then so is $\Gamma_d(Z)$. For $w \in W^P$ we let $w(-d) \in W^P$ denote the unique element for which $\Gamma_d(X^w) = X^{w(-d)}$. Given two opposite Schubert varieties X^u and X_v, the corresponding *projected Gromov-Witten variety* is defined by $\Gamma_d(X^u, X_v) = \mathrm{ev}_3(\mathrm{ev}_1^{-1}(X^u) \cap \mathrm{ev}_2^{-1}(X_v))$. This is the union of all connected rational curves of degree d that meet both X^u and X_v. It was shown in [4] that projected Gromov-Witten varieties are also projected Richardson varieties as studied in [16], hence non-empty projected Gromov-Witten varieties are unirational with rational singularities. This generalizes the fact that any non-empty Richardson variety $X^u \cap X_v$ is rational with rational singularities [22, 20, 21, 2]. We let $\operatorname{dist}(X^u, X_v)$ denote the smallest degree d for which $\Gamma_d(X^u, X_v) \neq \emptyset$.

Let $K_T(X)$ denote the Grothendieck ring of T-equivariant algebraic vector bundles on X. Every T-stable closed subvariety $Z \subset X$ defines a class $[\mathcal{O}_Z] \in K_T(X)$. If Z is unirational with rational singularities, then we have $\chi_X([\mathcal{O}_Z]) = 1 \in \Gamma$, see [10, Cor. 4.18(a)]. The *Schubert classes* $\mathcal{O}^w = [\mathcal{O}_{X^w}]$ for $w \in W^P$ form a basis of $K_T(X)$ as a module over the subring $\Gamma = K_T(\mathrm{pt})$. An alternative basis is provided by the B-stable Schubert classes $\mathcal{O}_w = [\mathcal{O}_{X_w}]$. Let $\Gamma[\![q]\!]$ denote the ring of formal power series in a single variable q with coefficients in Γ. The *equivariant quantum K-theory ring* $\mathrm{QK}_T(X)$ is a $\Gamma[\![q]\!]$-algebra, which as a module over $\Gamma[\![q]\!]$ is defined by $\mathrm{QK}_T(X) = K_T(X) \otimes_\Gamma \Gamma[\![q]\!]$. Givental defined the product in $\mathrm{QK}_T(X)$ in terms of structure constants obtained as polynomial expressions of Gromov-Witten invariants [12]. In this paper we will use an alternative construction from [4, 3]. For $u, v \in W^P$, define a power series in $\mathrm{QK}_T(X)$ by

$$\mathcal{O}^u \odot \mathcal{O}_v = \sum_{d \geq 0} [\mathcal{O}_{\Gamma_d(X^u, X_v)}]\, q^d \,.$$

Let $\psi : \mathrm{QK}_T(X) \to \mathrm{QK}_T(X)$ be the unique $\Gamma[\![q]\!]$-linear map defined by $\psi(\mathcal{O}^w) = \mathcal{O}^{w(-1)}$. The product in $\mathrm{QK}_T(X)$ is the unique $\Gamma[\![q]\!]$-bilinear operator $\star$ defined by [3, Prop. 3.2]

$$\mathcal{O}^u \star \mathcal{O}_v = (1 - q\psi)(\mathcal{O}^u \odot \mathcal{O}_v)\,.$$

3. Proof of Theorems 1, 2, and 3

Let $\chi : \mathrm{QK}_T(X) \to \Gamma[\![q]\!]$ be the $\Gamma[\![q]\!]$-linear extension of the Euler characteristic map. Since we have $\chi\psi = \chi$, Theorem 3 follows from the calculation

$$\begin{aligned} \chi(\mathcal{O}^u \star \mathcal{O}_v) \;&=\; \chi(1 - q\psi)(\mathcal{O}^u \odot \mathcal{O}_v) \;=\; (1-q)\chi(\mathcal{O}^u \odot \mathcal{O}_v) \\ &=\; (1-q) \sum_{d \geq \mathrm{dist}(X^u, X_v)} q^d \;=\; q^{\mathrm{dist}(X^u, X_v)}\,. \end{aligned}$$

It follows from [5, Thm. 1] that the group $\mathrm{QK}_T^{\mathrm{poly}}(X) = K_T(X) \otimes_\Gamma \Gamma[q]$ of finite power series is a subring of $\mathrm{QK}_T(X)$. Let $\mu : \Gamma[q] \to \Gamma$ be the ring homomorphism defined by $\mu(q) = 1$ and $\mu(\alpha) = \alpha$ for $\alpha \in \Gamma$. If we consider Γ as a module over $\Gamma[q]$ through this map, then the composition $\widehat{\chi} = \mu\chi : \mathrm{QK}_T^{\mathrm{poly}}(X) \to \Gamma$ is a $\Gamma[q]$-linear map. Since both of the sets $\{\mathcal{O}^u : u \in W^P\}$ and $\{\mathcal{O}_v : v \in W^P\}$ are bases for $\mathrm{QK}_T^{\mathrm{poly}}(X)$ over $\Gamma[q]$, it follows from the identity $\widehat{\chi}(\mathcal{O}^u \star \mathcal{O}_v) = 1 = \widehat{\chi}(\mathcal{O}^u) \cdot \widehat{\chi}(\mathcal{O}_v)$ that $\widehat{\chi}$ is a homomorphism of $\Gamma[q]$-algebras. This proves Theorem 2. Finally, Theorem 1 follows from applying $\widehat{\chi}$ to both sides of (1).

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