POSITIVITY DETERMINES THE QUANTUM COHOMOLOGY OF GRASSMANNIANS

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Abstract. We prove that if X is a Grassmannian of type A, then the Schubert basis of the (small) quantum cohomology ring $\mathrm{QH}(X)$ is the only homogeneous deformation of the Schubert basis of the ordinary cohomology ring $H^*(X)$ that multiplies with non-negative structure constants. This implies that the (three point, genus zero) Gromov-Witten invariants of X are uniquely determined by Witten's presentation of $\mathrm{QH}(X)$ and the fact that they are non-negative. We conjecture that the same is true for any flag variety $X = G/P$ of simply laced Lie type. For the variety of complete flags in $\mathbb{C}^n$, this conjecture is equivalent to Fomin, Gelfand, and Postnikov's conjecture that the quantum Schubert polynomials of type A are uniquely determined by positivity properties. Our proof for Grassmannians answers a question of Fulton.

1. Introduction

In this paper we give a proof of Bertram's structure theorems [4] for the (small) quantum cohomology ring of a Grassmannian that uses only that this ring is a graded and associative deformation of the singular cohomology ring [52, 38], satisfies Witten's presentation [55], and the fact that the defining Gromov-Witten invariants are non-negative. We expect that similar results hold more generally, so we will start by discussing the quantum cohomology of flag manifolds in general.

Let $X = G/P$ be a flag variety defined by a complex semisimple linear algebraic group G and a parabolic subgroup P. Fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset P \subset G$. The opposite Borel subgroup $B^- \subset G$ is defined by $B \cap B^- = T$. Let $W = N_G(T)/T$ be the Weyl group of G, $W_P = N_P(T)/T$ the Weyl group of P, and let $W^P \subset W$ be the subset of minimal representatives of the cosets in W/W_P. Each element $w \in W^P$ determines a B-stable Schubert variety $X_w = \overline{Bw.P}$ and an (opposite) B^--stable Schubert variety $X^w = \overline{B^-w.P}$, such that $\dim(X_w) = \operatorname{codim}(X^w, X) = \ell(w)$. The Schubert classes $[X^w]$ form a basis of the cohomology ring $H^*(X, \mathbb{Z})$.

Let Φ be the root system of G, with positive roots Φ^+ and simple roots $\Delta \subset \Phi^+$. The parabolic subgroup P is determined by the subset $\Delta_P = \{\beta \in \Delta \mid s_\beta \in W_P\}$. The group $H_2(X, \mathbb{Z})$ has a basis consisting of curve classes $[X_{s_\beta}]$ indexed by the simple roots $\beta \in \Delta \smallsetminus \Delta_P$. The elements of $H_2(X, \mathbb{Z})$ will be called *degrees*, and a degree $d = \sum_\beta d_\beta [X_{s_\beta}]$ is *effective*, written $d \geq 0$, if $d_\beta \geq 0$ for each β.

Given an effective degree $d \in H_2(X, \mathbb{Z})$, we let $\overline{\mathcal{M}}_{0,3}(X, d)$ denote the Kontsevich moduli space of 3-pointed stable maps $f : C \to X$ of genus zero and degree d. See

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[25] for a construction of this moduli space. The dimension is given by

$$\dim \overline{\mathcal{M}}_{0,3}(X,d) = \dim X + \int_d c_1(T_X)\,.$$

Let $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,3}(X,d) \to X$ be the evaluation map that sends a stable map f to the image of the i-th marked point in its domain, for $1 \leq i \leq 3$. Given $u, v, w \in W^P$ and $d \in H_2(X,Z)$ effective, the corresponding (3 point, genus zero) *Gromov-Witten invariant* of X is defined by

$$\langle [X^u], [X^v], [X_w] \rangle_d = \int_{\overline{\mathcal{M}}_{0,3}(X,d)} \mathrm{ev}_1^*[X^u] \cdot \mathrm{ev}_2^*[X^v] \cdot \mathrm{ev}_3^*[X_w]\,.$$

This invariant is non-zero only if $\ell(u) + \ell(v) = \ell(w) + \int_d c_1(T_X)$, in which case it is the number of parametrized curves $\mathbb{P}^1 \to X$ of degree d that map three points in $\mathbb{P}^1$ to fixed general translates of the Schubert varieties X^u, X^v, and X_w (see [25]).

To simplify the statements of our results and conjectures, we will work with quantum cohomology over the field of rational numbers. Let $\mathbb{Q}[q] = \mathbb{Q}[q_\beta : \beta \in \Delta \smallsetminus \Delta_P]$ denote a polynomial ring in one variable q_β for each Schubert curve X_{s_β}. These variables are called *deformation parameters*. We equip this ring with the grading defined by $\deg(q_\beta) = \int_{X_{s_\beta}} c_1(T_X)$. Given an effective degree $d = \sum_\beta d_\beta [X_{s_\beta}]$ in $H_2(X,\mathbb{Z})$, set $q^d = \prod q_\beta^{d_\beta}$. The (small) *quantum cohomology ring* $\mathrm{QH}(X)$ is a graded $\mathbb{Q}[q]$-algebra, which as a $\mathbb{Q}[q]$-module is defined by $\mathrm{QH}(X) = H^*(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[q]$. In other words, the Schubert classes $[X^w]$ for $w \in W^P$ form a $\mathbb{Q}[q]$-basis of this ring. The grading is given by $\deg\,[X^w] = \ell(w)$, and the multiplicative structure is defined by

$$[X^u] \star [X^v] = \sum_{w,d \geq 0} \langle [X^u], [X^v], [X_w] \rangle_d \, q^d \, [X^w]$$

where the sum is over all $w \in W^P$ and effective $d \in H_2(X,\mathbb{Z})$. It was proved by Ruan and Tian [52] and by Kontsevich and Manin [38] that this product is associative.

The definition of the quantum cohomology ring came from physics, see [55]. In mathematics the quantum ring $\mathrm{QH}(X)$ provides an effective tool for computing the Gromov-Witten invariants of X, as these invariants can be calculated if enough is known about the structure of the ring $\mathrm{QH}(X)$. For example, the computation of Gromov-Witten invariants is reduced to combinatorics if one knows all invariants $\langle [X^u], [X^v], [X_w] \rangle_d$ for which $[X^u]$ belongs to a set of generators of the cohomology ring $H^*(X,\mathbb{Q})$. Formulas for Gromov-Witten invariants of this type are frequently called *quantum Pieri formulas* or *quantum Chevalley formulas*, see [4, 22, 21, 26, 39, 40, 49, 14, 31] for examples.

It is in general not enough to know a presentation of the ring $\mathrm{QH}(X)$ by generators and relations for the Gromov-Witten invariants of X to be determined (see Example 6). However, a presentation together with a *quantum Giambelli formula* that expresses each Schubert class $[X^w]$ as a polynomial in the chosen generators of $\mathrm{QH}(X)$ is sufficient. Examples of presentations and Giambelli formulas in quantum cohomology can be found in [55, 54, 4, 28, 35, 20, 37, 22, 39, 40, 50, 16, 15, 1].

It should be noted that quantum cohomology is not functorial, so there is in general no ring homomorphism $\mathrm{QH}(G/P) \to \mathrm{QH}(G/B)$ that extends the pullback map $H^*(G/P,\mathbb{Q}) \to H^*(G/B,\mathbb{Q})$ along the projection $G/B \to G/P$. For this reason, the study of quantum cohomology must be carried out separately for each flag variety X. Relations between the Gromov-Witten invariants of G/B and G/P do exist in the form of the Peterson comparison formula [56], but it requires work to extract algebraic properties of the quantum ring from this formula (see e.g. [31]).

The first examples of structure theorems for the quantum cohomology ring $\mathrm{QH}(X)$ were provided by Witten [55] and Bertram [4] when $X = \mathrm{Gr}(m,n)$ is the Grassmannian of m-planes in $\mathbb{C}^n$. Bertram's results, a Pieri formula and a Giambelli formula for $\mathrm{QH}(X)$, were proved by applying results from intersection theory to the quot scheme compactification of the moduli space $\mathcal{M}_{0,3}(X,d)$ of parametrized curves. More elementary proofs were later given in [6] by studying the kernels and spans of the curves counted by Gromov-Witten invariants. The idea is that if a Gromov-Witten invariant of a Grassmannian is non-zero, then the *span* of any counted curve, defined as the linear span of the m-planes given by points of the curve, will produce a point in a related intersection of Schubert varieties in a different Grassmannian, and this intersection can be studied with classical Schubert calculus. More generally, it was shown in [13] that the map that sends a curve to its kernel-span pair provides a bijection between the curves counted by a Gromov-Witten invariant with the points in an intersection of Schubert varieties in a two-step flag variety. Further generalizations of this *quantum equals classical* phenomenon can be found in [18, 14, 17, 10] and the references therein.

The quantum cohomology ring of the variety $X = \mathrm{GL}(n)/B$ of complete flags in $\mathbb{C}^n$ is described by structure theorems of Fomin, Gelfand, and Postnikov [22]. The main result of [22] states that the *quantum Schubert polynomials* provide Giambelli formulas for the Schubert classes in $\mathrm{QH}(X)$. In addition to a known presentation of the ring $\mathrm{QH}(X)$ [28, 35, 20] and a special case of the Giambelli formula [20], the proof uses the geometric fact that the structure constants of $\mathrm{QH}(X)$ are non-negative integers. Moreover, it was conjectured in [22] that the quantum Schubert polynomials are uniquely determined by the non-negativity of their structure constants, in addition to the presentation, grading, and deformation properties of $\mathrm{QH}(X)$. Positivity properties have received much attention in the study of quantum cohomology [48, 2, 12, 7], but usually as a benchmark of our understanding of the combinatorial aspects of quantum cohomology rather than a tool for proving other results.

Fulton asked the related question, whether the quantum cohomology ring of a Grassmannian is uniquely determined by positivity conditions [24]. The immediate answer is no; it is always possible to rescale the deformation parameter q by a positive factor $\alpha > 0$, or equivalently, multiply each Gromov-Witten invariant $\langle [X^u], [X^v], [X_w] \rangle_d$ by α^{-d}. However, this modification will change the relations among the Schubert class generators of $\mathrm{QH}(X)$. In this paper we show that, for Grassmannians, rescaling the Gromov-Witten invariants is the only change to the quantum cohomology ring that preserves its formal properties as well as positivity of its structure constants (Corollary 19). By adding the condition that the quantum cohomology ring satisfies Witten's presentation [55], this ring is uniquely determined, and Bertram's structure theorems can be proved with purely combinatorial methods.

Our results establish the Grassmannian case of the following conjecture, which also generalizes the conjectured uniqueness of quantum Schubert polynomials [22].

Conjecture 1. *Let $X = G/P$ be any flag variety of simply laced Lie type. Then the Schubert basis of $\mathrm{QH}(X)$ is the only homogeneous $\mathbb{Q}[q]$-basis that deforms the Schubert basis of $H^*(X,\mathbb{Q})$ and multiplies with non-negative structure constants.*

In other words, if $\{\tau_w \mid w \in W^P\}$ is any $\mathbb{Q}[q]$-basis of $\mathrm{QH}(X)$ such that τ_w is homogeneous of degree $\ell(w)$, $\tau_w - [X^w]$ belongs to the ideal $\langle q \rangle$ generated by the deformation parameters q_β, and each product $\tau_u \star \tau_v$ is a non-negative linear combination of the $\mathbb{Q}$-basis $\{q^d \tau_w\}$, then we have $\tau_w = [X^w]$ for all $w \in W^P$.

In addition to Grassmannians of type A, we have verified Conjecture 1 when $X = \mathrm{OG}(n, 2n)$ is a maximal orthogonal Grassmannian of type D_n, with $n \leq 6$, and when $X = \mathrm{OG}(1, 2n)$ is any quadric hypersurface of even dimension. The condition that the root system of G is simply laced is necessary, since the conjecture is false for the Lagrangian Grassmannian $\mathrm{LG}(2, 4)$ of type C_2, a 3-dimensional quadric. These examples are treated in Section 2. We have also obtained computational verification of Conjecture 1 for all partial flag varieties $\mathrm{GL}(n)/P$ of dimension at most 10, except $\mathrm{Fl}(1, 2, 3; \mathbb{C}^5)$, $\mathrm{Fl}(1, 2, 4; \mathbb{C}^5)$, and $\mathrm{Fl}(1, 2, 3, 4; \mathbb{C}^5)$. We note that, since the number of monomials in $\mathbb{Q}[q]$ grows very fast with respect to total degree when X has large Picard rank, so does the computational complexity of this problem.

Our proof of Conjecture 1 for Grassmannians of type A uses that quantum multiplication with the top Chern class of a tautological vector bundle maps any Schubert class to a power of q times a different Schubert class. This follows from Bertram's quantum Pieri formula [4] and is a special case of the Seidel representation of the group $\pi_1(\mathrm{Aut}(X))$ on $\mathrm{QH}(X)/\langle q - 1\rangle$, which was introduced in [53] and computed explicitly for flag varieties in [19]. Here $\langle q-1 \rangle$ is the ideal generated by all differences $q_\beta - 1$. For any flag variety X, the action of this representation is given by quantum multiplication by certain Schubert classes $[X^u]$ of finite order in the group of units $(\mathrm{QH}(X)/\langle q-1\rangle)^\times$. We show that basis elements of finite order in any non-negative quantum deformation must behave in a similar way (Lemma 3), which gives a way to compare different basis elements. For the Grassmannian $X = \mathrm{Gr}(m, n)$ we then use that the sum of the degrees of the Schubert classes in any Seidel orbit is always the same number, namely $\frac{1}{2}mn(n - m)$ ([51, Thm. 19], see also Lemma 16). This is a special phenomenon for Grassmannians of type A; a counterexample for the variety of complete flags in $\mathbb{C}^6$ is provided in Example 21. The Grassmannian case of Conjecture 1 is proved by combining these observations.

The quantum cohomology ring of a flag variety is generalized by the quantum K-theory ring [27, 42], which has received much attention in recent years, see e.g. [17, 8, 29, 32, 9, 41, 33, 34, 3, 11]. In contrast to quantum cohomology, structure theorems for the quantum K-theory ring are known only for cominuscule spaces [17, 29, 9], while conjectures exist in other cases [43, 44]. It would be very interesting to find a way to uniquely describe the quantum K-theory ring in terms of combinatorial properties. The same question can be asked about the equivariant quantum cohomology ring [36, 37, 49, 48, 2].

Another interesting open problem is to give a combinatorial proof that the quantum cohomology ring of a Grassmannian, as defined by Bertram's structure theorems [4], has non-negative structure constants (see [5]). While positive formulas for these Gromov-Witten invariants do exist [12, 7], their proofs utilize the quantum equals classical theorem [13, 17] rather than studying the quantum cohomology ring directly. The methods presented in this paper do not solve this problem; we simply prove that any quantum deformation of a Grassmannian with non-negative structure constants must obey Bertram's structure theorems.

This paper is organized as follows. In Section 2 we formulate the notion of a *quantum deformation*, which captures the properties of the quantum cohomology ring that are automatic from its definition. We also provide some initial examples of Conjecture 1. In Section 3 we prove that any non-negative quantum deformation of a Grassmannian is obtained from the quantum cohomology ring by rescaling the defining Gromov-Witten invariants. This section can also be read as a combinatorial introduction to the quantum cohomology of Grassmannians. We thank W. Fulton for his inspiring question, and O. Pechenik for making us aware of [51]. Finally, we thank an anonymous referee for helpful suggestions.

2. Quantum deformations

2.1. **Quantum deformations.** The (small) quantum cohomology ring $\mathrm{QH}(X)$ of a flag variety $X = G/P$ is a quantum deformation of the singular cohomology ring $H^*(X, \mathbb{Q})$ in the following sense.

Definition 2. A *quantum deformation* of $H^*(X, \mathbb{Q})$ is a graded associative $\mathbb{Q}[q]$-algebra QH with unit, together with a $\mathbb{Q}[q]$-basis $\{\tau_w : w \in W^P\}$ of QH indexed by W^P, such that each basis element τ_w is homogeneous of degree $\ell(w)$, and the assignments $\tau_w \mapsto [X^w]$ and $q_\beta \mapsto 0$ define a ring homomorphism $\mathrm{QH} \to H^*(X, \mathbb{Q})$.

We will often denote a quantum deformation $(\mathrm{QH}, \{\tau_w\})$ simply by QH, and we will occasionally misuse terminology and call it a quantum deformation of X. Notice that $\mathrm{QH}/\langle q\rangle \cong H^*(X, \mathbb{Q})$, where $\langle q\rangle \subset \mathrm{QH}$ is the ideal generated by the deformation parameters q_β. In addition, the set $\mathcal{T} = \{q^d \tau_w \mid w \in W^P, d \geq 0\}$ is a basis of QH as a vector space over $\mathbb{Q}$. The structure constants of QH with respect to this basis are the numbers $N_{u,v}^{w,d} \in \mathbb{Q}$, indexed by $u, v, w \in W^P$ and $d \geq 0$, defined by the identity

$$\tau_u \tau_v = \sum_{w,d} N_{u,v}^{w,d}\, q^d\, \tau_w \,.$$

We will say that the quantum deformation QH is *non-negative* if $N_{u,v}^{w,d} \geq 0$ for all $u, v, w \in W^P$ and $d \geq 0$.

The quantum deformation of main interest is the quantum cohomology ring $\mathrm{QH}(X)$, equipped with its basis of Schubert classes. Since the Gromov-Witten invariants of X are enumerative [25], we know a priori that $\mathrm{QH}(X)$ is a non-negative quantum deformation. It is natural to ask which non-negative quantum deformations exist.

One way to produce new quantum deformations is to replace the Schubert basis of $\mathrm{QH}(X)$ with any $\mathbb{Q}[q]$-basis $\{\tau_w : w \in W^P\}$ for which each element τ_w is a homogeneous deformation of the Schubert class $[X^w]$, i.e. $\tau_w - [X^w] \in \langle q\rangle$. Such deformations will be called *change-of-basis* quantum deformations of $H^*(X, \mathbb{Q})$.

We remark that historically, a presentation of the quantum ring $\mathrm{QH}(X)$ was known early on for many flag varieties [55, 28, 20, 37]. For example, Witten's paper [55] contains a presentation of $\mathrm{QH}(X)$ when X is a Grassmannian (see also [54]). When a presentation is known, a determination of the quantum ring $\mathrm{QH}(X)$ and the associated Gromov-Witten invariants of X is equivalent to finding the correct non-negative change-of-basis quantum deformation of X, so it is also natural to ask for a classification of these deformations. Conjecture 1 states that, if X is of simply laced Lie type, then $\mathrm{QH}(X)$ is the only non-negative change-of-basis quantum deformation up to isomorphism.[1]

2.2. **General properties.** Let $(\mathrm{QH}, \{\tau_w\})$ be a quantum deformation of $H^*(X, \mathbb{Q})$ and let $\phi : \mathrm{QH} \to H^*(X, \mathbb{Q})$ be the ring homomorphism of Definition 2. Assume that $\phi(\eta) = [X^u]$ for some $\eta \in \mathrm{QH}$ and $u \in W^P$. Then $\eta - \tau_u \in \langle q \rangle$. If η is homogeneous with $\deg(\eta) < \deg(q_\beta)$ for all $\beta \in \Delta \smallsetminus \Delta_P$, then this implies that $\eta = \tau_u$. For example we have $\tau_e = 1$, where $e \in W$ is the identity element.

Lemma 3. *Let* $(\mathrm{QH}, \{\tau_w\})$ *be any non-negative quantum deformation of* $H^*(X, \mathbb{Q})$ *and let* $u_1, u_2, \dots, u_\ell \in W^P$. *Assume that the product* $\tau_{u_\ell} \tau_{u_{\ell-1}} \cdots \tau_{u_1}$ *is a non-zero multiple of a monomial from* $\mathbb{Q}[q]$, *that is,* $\tau_{u_\ell} \tau_{u_{\ell-1}} \cdots \tau_{u_1} = c\, q^d$ *for some rational number* $c > 0$ *and effective degree* d. (a) *For every* $v \in W^P$, *the product* $\tau_{u_1} \tau_v$ *is a positive multiple of a single element from* $\mathcal{T} = \{q^d \tau_w\}$. (b) *If* $[X^{u_1}] \cdot [X^v] = [X^w]$ *holds in* $H^*(X, \mathbb{Q})$, *then* $\tau_{u_1} \tau_v = \tau_w$ *holds in* QH.

Proof. If part (a) is false, then since $\tau_{u_\ell} \cdots \tau_{u_1} \tau_v = c\, q^d \tau_v$, we can choose $p > 0$ such that the expansion of $\tau_{u_p} \cdots \tau_{u_1} \tau_v$ involves more than one basis element from $\mathcal{T}$, whereas $\tau_{u_{p+1}} \tau_{u_p} \cdots \tau_{u_1} \tau_v$ is a multiple of a single basis element $q^e \tau_w$. Write $\tau_{u_p} \cdots \tau_{u_1} \tau_v = b_1\, q^{d_1} \tau_{v_1} + \cdots + b_r\, q^{d_r} \tau_{v_r}$ where $r \geq 2$ and $b_i > 0$ for $1 \leq i \leq r$. Since $\tau_{u_{p+1}} q^{d_i} \tau_{v_i}$ is a non-negative linear combination of $\mathcal{T}$ for each i, and since $\tau_{u_{p+1}} \sum b_i\, q^{d_i} \tau_{v_i}$ is a multiple of $q^e \tau_w$, we deduce that $\tau_{u_{p+1}}\, q^{d_i} \tau_{v_i}$ is a multiple of $q^e \tau_w$ for each i. This contradicts that multiplication by $\tau_{u_{p+1}}$ is an injective operation on QH. Finally, part (b) follows from part (a). $\square$

2.3. **Examples.** We finish this section by examining Conjecture 1 for a selection of cominuscule spaces $X = G/P$. The quantum cohomology rings of these varieties contain only one deformation parameter which we denote by q.

Example 4. Let $X = \mathrm{OG}(1, 2n)$ be the quadric hypersurface of dimension $2n-2$. A description of the Schubert classes and quantum cohomology ring of this variety can be found in [18, 14]. Let $\{\tau_w : w \in W^P\}$ be any homogeneous deformation of the Schubert basis of $\mathrm{QH}(X)$ that multiplies with non-negative structure constants. Since $\deg(q) = \dim(X) = 2n-2$, it follows that $\tau_w = [X^w]$ for all $w \in W^P$, except for the element w_0^P representing a point. Let $u \in W^P$ have length $n-1$, and choose $u^\vee \in W^P$ such that $[X^u] \cdot [X^{u^\vee}] = [X^{w_0^P}]$ in $H^*(X; \mathbb{Q})$. It follows from [19, Thm. 1] or [14, Thm. 3.4] that $\tau_u^4 = [X^u]^4 = q^2$, so we deduce from Lemma 3 that $\tau_{w_0^P} = \tau_u \star \tau_{u^\vee} = [X^u] \star [X^{u^\vee}] = [X^{w_0^P}]$. This shows that Conjecture 1 holds for X. Similar arguments can be used to show that any non-negative quantum deformation of X is obtained from $\mathrm{QH}(X)$ by rescaling the defining Gromov-Witten invariants.

Example 5. Let $X = \mathrm{OG}(n, 2n)$ be the maximal orthogonal Grassmannian of type D_n. The Schubert classes and quantum cohomology of this space are described in [40]. We have $\dim(X) = \binom{n}{2}$ and $\deg(q) = 2n-2$. The elements of W^P can be identified with strict partitions $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$ with $\lambda_1 \leq n-1$, so that $\operatorname{codim}(X^\lambda, X) = |\lambda| = \sum \lambda_i$. Let $\{\tau_\lambda\}$ be any homogeneous deformation of the Schubert basis of $\mathrm{QH}(X)$ that multiplies with non-negative structure constants. Then we have $\tau_\lambda = [X^\lambda]$ for $|\lambda| < 2n-2$. The classical Pieri formula [30] implies that $[X^{n-1}] \cdot [X^\lambda] = [X^{n-1,\lambda}]$ holds in $H^*(X, \mathbb{Q})$ whenever $\lambda_1 < n-1$, and [40,

[1]An (iso)morphism $(\mathrm{QH}, \{\tau_w\}) \to (\mathrm{QH}', \{\tau'_w\})$ of quantum deformations is a $\mathbb{Q}[q]$-algebra homomorphism that maps τ_w to τ'_w for each $w \in W^P$.

Cor. 5] gives $\tau_{n-1}^2 = [X^{n-1}]^2 = q$. Lemma 3 therefore implies that $\tau_{n-1,\lambda} = \tau_{n-1} \star \tau_\lambda = [X^{n-1}] \star [X^\lambda] = [X^{n-1,\lambda}]$ for all strict partitions λ such that $\lambda_1 < n-1$ and $|\lambda| < 2n-2$. This proves Conjecture 1 when $n \leq 5$, and for $n = 6$ we obtain $\tau_\lambda = [X^\lambda]$ for all strict partitions except $\nu = (4,3,2,1)$ and $(5,\nu)$. We must have $\tau_\nu = [X^\nu] + cq$ and $\tau_{5,\nu} = \tau_5 \star \tau_\nu = [X^{5,\nu}] + cq\tau_5$ for some constant $c \in \mathbb{Q}$. Since $\tau_1 \star \tau_{4,3,2} = [X^1] \star [X^{4,3,2}] = [X^{5,3,2}] + [X^\nu] = \tau_{5,3,2} + \tau_\nu - cq$, it follows that $c \leq 0$. Since $\tau_1 \star \tau_\nu = [X^1] \star [X^\nu] + cq[X^1] = [X^{5,3,2,1}] + cq[X^1] = \tau_{5,3,2,1} + cq\tau_1$, it follows that $c \geq 0$. This shows that Conjecture 1 also holds for $\mathrm{OG}(6,12)$.

The following example shows that Conjecture 1 does not hold for arbitrary flag varieties. It would be interesting to know if the non-negative quantum deformations of any flag variety correspond to points in a rational polyhedral cone.

Example 6. Let $X = \mathrm{LG}(2,4)$ be the Lagrangian Grassmannian of type C_2. Equivalently, $X \subset \mathbb{P}^4$ is a quadric of dimension 3. Then $H^*(X,\mathbb{Q})$ has a basis of Schubert classes $\{[X^0] = 1, [X^1], [X^2], [X^3] = [\mathrm{pt}]\}$, where $[X^p] \in H^{2p}(X)$. The multiplicative structure of $H^*(X,\mathbb{Q})$ is determined by $[X^1]^2 = 2[X^2]$ and $[X^1] \cdot [X^2] = [X^3]$. One can check that any quantum deformation $(\mathrm{QH}, \{\tau_0 = 1, \tau_1, \tau_2, \tau_3\})$ of $H^*(X,\mathbb{Q})$ is given by

$$\begin{aligned}
\tau_1^2 &= 2\,\tau_2 \\
\tau_1\,\tau_2 &= \tau_3 + (2a-b)\,q \\
\tau_1\,\tau_3 &= b\,q\,\tau_1 \\
\tau_2^2 &= a\,q\,\tau_1 \\
\tau_2\,\tau_3 &= b\,q\,\tau_2 \\
\tau_3^2 &= (2b-2a)\,q\,\tau_3 + (2ab-b^2)\,q^2
\end{aligned}$$

where $a, b \in \mathbb{Q}$ are constants. It is non-negative if and only if $a \leq b \leq 2a$, and it is a change-of-basis deformation if and only if $a = 1$. In particular, we obtain a non-negative change-of-basis deformation of $H^*(X,\mathbb{Q})$ whenever $a = 1$ and $1 \leq b \leq 2$. The quantum cohomology ring $\mathrm{QH}(X)$ corresponds to $a = b = 1$ [39].

3. Grassmannians

Let $X = \mathrm{Gr}(m,n)$ be the Grassmannian of m-dimensional vector subspaces in $\mathbb{C}^n$ and set $k = n - m$. In this section we classify the non-negative quantum deformations of the cohomology ring $H^*(X,\mathbb{Q})$. We start by recalling some basic definitions and facts about symmetric functions.

3.1. **Partitions.** We will identify any *partition* $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ with its *Young diagram* of boxes, which has λ_i boxes in row i. Rows are numbered from top to bottom and are left aligned. The *length* $\ell(\lambda)$ is the number of non-empty rows, and the *weight* $|\lambda| = \sum \lambda_i$ is the total number of boxes. We write $\lambda_i = 0$ for $i > \ell(\lambda)$. The *conjugate* Young diagram λ' is obtained by mirroring λ in the diagonal, so that rows and columns are interchanged.

Given an additional partition μ, we write $\mu \subset \lambda$ if and only if $\mu_i \leq \lambda_i$ for each i. In this case λ/μ denotes the *skew diagram* of boxes in λ that are not in μ. This skew diagram is a *horizontal strip* if it has at most one box in each column, and a *vertical strip* if it has at most one box in each row. The *outer rim* of a λ is the skew diagram of boxes in λ that are not strictly north and strictly west of other boxes in λ. In other words, the box in row i and column j belongs to the outer rim of λ if and only if $\lambda_{i+1} \leq j \leq \lambda_i$.

For integers $r, c \geq 0$ we let $(c^r) = (c, c, \dots, c)$ denote the rectangular Young diagram with r rows and c columns. Since the Schubert varieties in $X = \mathrm{Gr}(m, n)$ can be indexed by partitions λ contained in the rectangle (k^m) (see Section 3.3), we also denote this rectangle as $m \times k$.

Recall also that the *lexicographic order* on integer sequences is defined by $\mu <_{\mathrm{lex}} \lambda$ if and only if for some index p we have $\mu_i = \lambda_i$ for $1 \leq i < p$ whereas $\mu_p < \lambda_p$. The sequence μ is smaller than λ in the *degree-lexicographic order* if and only if $|\mu| < |\lambda|$, or $|\mu| = |\lambda|$ and $\mu <_{\mathrm{lex}} \mu$.

3.2. **Symmetric functions.** Given a sequence $h = (h_1, h_2, \dots)$ of elements in a commutative ring R and a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0)$, define

$$\Delta_\lambda(h) = \det(h_{\lambda_i + j - i})_{\ell \times \ell}\,.$$

Here we set $h_0 = 1$ and $h_i = 0$ for $i < 0$. Let $e = (e_1, e_2, \dots)$ be the dual sequence defined by $e_p = \Delta_{(1^p)}(h) = \det(h_{1+j-i})_{p \times p}$. Equivalently, we can define e_p recursively by the formula

$$e_p = \sum_{i=1}^{p} (-1)^{i-1} h_i e_{p-i}\,.$$

We need the following facts about Schur polynomials. Proofs can be found in e.g. [45, Ch. I] or [23, §2.2, §6.1].

Fact 7. *We have $e_p \, \Delta_\lambda(h) = \sum_\mu \Delta_\mu(h)$, where the sum is over all partitions μ obtained by adding a vertical strip of p boxes to λ.*

Fact 8. *We have $\Delta_\lambda(e) = \Delta_{\lambda'}(h)$, where λ' is the conjugate partition of λ.*

Define the polynomial ring $S = \mathbb{Q}[c_1, \dots, c_m]$, with the grading defined by $\deg(c_i) = i$. Let $c = (c_1, c_2, \dots c_m, 0, 0, \dots)$ be the sequence of its variables, with $c_i = 0$ for $i > m$. Let $\sigma = (\sigma_1, \sigma_2, \dots)$ be the dual sequence defined by $\sigma_p = \Delta_{(1^p)}(c) = \det(c_{1+j-i})_{p \times p}$. For each partition λ we set $\sigma_\lambda = \Delta_\lambda(\sigma)$. This is a homogeneous element of S of degree $|\lambda|$.

Lemma 9. *The set $\{\sigma_\lambda \mid \ell(\lambda) \leq m\}$ is a $\mathbb{Q}$-basis of S, and we have $\sigma_\lambda = 0$ whenever $\ell(\lambda) > m$.*

Proof. By Fact 8 we have $\sigma_\lambda = \Delta_{\lambda'}(c)$. If $\ell(\lambda) > m$, then all entries in the top row of the determinant defining $\Delta_{\lambda'}(c)$ are zero. Otherwise the initial term of $\Delta_{\lambda'}(c)$ is $c_1^{\lambda_1 - \lambda_2} c_2^{\lambda_2 - \lambda_3} \cdots c_m^{\lambda_m}$, where the monomials $c_1^{p_1} c_2^{p_2} \dots c_m^{p_m}$ of S are ordered according to the lexicographic order on the exponent vectors $(p_m, p_{m-1}, \dots, p_1)$. The lemma follows from this. □

Lemma 10. *The set $\{\sigma_\lambda \mid \lambda \subset m \times k\}$ is a $\mathbb{Q}$-basis of the ring $S / \langle \sigma_{k+1}, \dots, \sigma_n \rangle$.*

Proof. By Lemma 9 it is enough to show that $I = \langle \sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_n \rangle$ has a basis consisting of all elements σ_λ for which $\ell(\lambda) \leq m$ and $\lambda_1 > k$. Fact 7 implies that I is contained in the span of these basis elements. By using the identity $\sigma_p = \sum_{i=1}^m (-1)^{i-1} c_i \sigma_{p-i}$, it follows by induction on p that $\sigma_p \in I$ for all $p > k$. Therefore all elements in the top row of the determinant defining σ_λ are contained in I when $\lambda_1 > k$. □

Corollary 11. *The element σ_n is a non-zero divisor in $S/\langle\sigma_{k+1},\dots,\sigma_{n-1}\rangle$.*

Proof. Set $J = \langle\sigma_{k+1},\dots,\sigma_{n-1}\rangle \subset S$ and $K = \{f \in S \mid \sigma_n f \in J\}$. We must show that $K = J$. Otherwise choose a prime ideal $P \subset S$ such that $(K/J)_P \neq 0$. Then P contains $I = \langle\sigma_{k+1},\sigma_{k+2},\dots,\sigma_n\rangle$, and S_P/I_P has Krull dimension zero by Lemma 10. It follows that the generators of I form a regular sequence in S_P [47, Thm. 17.4], which contradicts that σ_n is a zero-divisor in S_P/J_P. □

3.3. **Cohomology of Grassmannians.** We summarize the main facts about the cohomology of Grassmannians. More details and proofs can be found in e.g. [23, §9.4] or [46, §3].

Each partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m)$ contained in the rectangle $m \times k$ defines a Schubert variety in $X = \mathrm{Gr}(m,n)$ given by

$$X^\lambda = \{V \in X \mid \dim(V \cap F^{k+i-\lambda_i}) \geq i \ \forall 1 \leq i \leq m\}\,,$$

where $F^{k+i-\lambda_i}$ denotes the span of the last $k+i-\lambda_i$ coordinate basis vectors in $\mathbb{C}^n$. We have $\operatorname{codim}(X^\lambda, X) = |\lambda|$. With the notation of the introduction we have $X = G/P$, where $G = \mathrm{GL}(n,\mathbb{C})$ and P is the stabilizer of the span of the first m basis vectors in $\mathbb{C}^n$. If we use the maximal torus T of diagonal matrices and the Borel subgroup B of upper triangular matrices, and identify the Weyl group W with the subgroup of permutation matrices in G, then the set W^P consists of all permutations $w = (w_1, w_2, \dots, w_n)$ with descent only at position m, i.e. $w_i < w_{i+1}$ for $i \neq m$. We then have $X^w = X^\lambda$ where $\lambda = (w_m - m, \dots, w_2 - 2, w_1 - 1)$.

The Schubert classes $[X^\lambda]$ form a basis of $H^*(X,\mathbb{Q})$ as a vector space over $\mathbb{Q}$. The special Schubert classes $[X^{(1^p)}]$ are the Chern classes of the dual of the tautological subbundle $\mathcal{S} \subset X \times \mathbb{C}^n$ of rank m. The classical Pieri formula states that multiplication with these classes in $H^*(X,\mathbb{Q})$ is determined by

$$[X^{(1^p)}] \cdot [X^\lambda] = \sum_\mu [X^\mu]\,, \tag{1}$$

where the sum is over all partitions $\mu \subset m \times k$ obtained by adding a vertical strip of p boxes to λ.

Lemma 12. *The linear map $S/\langle\sigma_{k+1},\dots,\sigma_n\rangle \to H^*(X,\mathbb{Q})$ defined by $\sigma_\lambda \mapsto [X^\lambda]$ is an isomorphism of graded rings.*

Proof. Since $c_p = \sigma_{(1^p)}$ is mapped to $[X^{(1^p)}]$, it follows from Fact 7 and the Pieri formula (1) that any product $c_p\sigma_\lambda$ is mapped to $[X^{(1^p)}] \cdot [X^\lambda]$. Since S is generated by $c_1, \dots, c_m$, this implies that the map is a ring homomorphism. □

3.4. **Quantum deformations of Grassmannians.** The quantum cohomology ring $\mathrm{QH}(X)$ contains a single deformation parameter q of degree n. We next give a combinatorial construction of the non-negative quantum deformations of $H^*(X,\mathbb{Q})$.

Given any rational number $\alpha \in \mathbb{Q}$, define the ring

$$\mathrm{QH}_\alpha = S[q]/\langle\sigma_{k+1},\dots,\sigma_{n-1},\sigma_n + (-1)^m \alpha q\rangle\,.$$

This ring is a graded $\mathbb{Q}[q]$-algebra, where the grading is defined by $\deg(q) = n$ and $\deg(c_p) = p$ for $1 \leq p \leq m$. Lemma 12 implies that the assignments $\sigma_\lambda \mapsto [X^\lambda]$ and $q \mapsto 0$ define a ring homomorphism $\mathrm{QH}_\alpha \to H^*(X,\mathbb{Q})$, and we show in Corollary 14 that QH_α is a quantum deformation of $H^*(X,\mathbb{Q})$ with basis $\{\sigma_\lambda \mid \lambda \subset m \times k\}$. We first show that QH_α satisfies the following rescaled version of Bertram's quantum Pieri formula [4].

Proposition 13. (a) *For $1 \leq p \leq m$ and $\lambda \subset m \times k$ we have in QH_α that*

$$c_p\, \sigma_\lambda = \sum_\mu \sigma_\mu + \sum_\nu \alpha q\, \sigma_\nu\,.$$

The first sum is over all partitions $\mu \subset m \times k$ that can be obtained by adding a vertical strip of p boxes to λ. The second sum is empty unless $\lambda_1 = k$, in which case it is over all partitions ν obtained by removing $n-p$ boxes from the outer rim of λ, with at least one box removed from each column.

(b) *For $1 \leq p \leq k$ and $\lambda \subset m \times k$ we have in QH_α that*

$$\sigma_p\, \sigma_\lambda = \sum_\mu \sigma_\mu + \sum_\nu \alpha q\, \sigma_\nu\,.$$

The first sum is over all partitions $\mu \subset m \times k$ that can be obtained by adding a horizontal strip of p boxes to λ. The second sum is empty unless $\lambda_m \neq 0$, in which case it is over all partitions ν obtained by removing $n-p$ boxes from the outer rim of λ, with at least one box removed from each row.

Proof. It follows from Fact 7 and Lemma 9 that

$$c_p\, \sigma_\lambda = \sum_\mu \sigma_\mu$$

where the sum is over all partitions μ with at most m rows that can be obtained by adding a vertical strip of size p to λ. If such a partition satisfies $\mu_1 \leq k$, then the term σ_μ is included in the first sum of part (a). Otherwise we have $\lambda_1 = k$ and $\mu_1 = k+1$, in which case the first row of the $m \times m$ determinant defining σ_μ is equal to $(0, \dots, 0, (-1)^{m-1}\alpha q)$ modulo the ideal defining QH_α. If $\mu_m = 0$, then the last row of the same determinant is $(0, \dots, 0, 1)$, in which case $\sigma_\mu = 0$. On the other hand, if $\mu_m > 0$, then $\sigma_\mu = \alpha q\, \sigma_\nu \in \mathrm{QH}_\alpha$ where $\nu = (\mu_2 - 1, \dots, \mu_m - 1, 0)$. The second sum of part (a) contains exactly the terms obtained in this way.

Part (b) is equivalent to part (a) with m and k interchanged. To make this explicit, let $c' = (c'_1, \dots, c'_k, 0, 0, \dots)$ be a sequence of k variables, and set $\sigma'_p = \Delta_{(1^p)}(c')$ for $p \geq 0$ and $\sigma'_\mu = \Delta_\mu(\sigma')$ for $\mu \subset k \times m$. Let $\phi : \mathbb{Q}[c'_1, \dots, c'_k] \to \mathbb{Q}[c_1, \dots, c_m]$ be the ring homomorphism defined by $\phi(c'_j) = \sigma_j$ for $1 \leq j \leq k$. For $1 \leq p < n$ we have $\phi(\sigma'_p) = \Delta_{(1^p)}(\sigma_1, \dots, \sigma_k, 0, \dots) \equiv \Delta_{(1^p)}(\sigma) = c_p$ modulo the ideal $J = \langle \sigma_{k+1}, \dots, \sigma_{n-1} \rangle$. This implies that $\phi(\sigma'_\mu) \equiv \Delta_\mu(c) = \Delta_{\mu'}(\sigma) = \sigma_{\mu'}$ (mod J) for $\mu \subset k \times m$. We also have $\phi(\sigma'_n) = \Delta_{(1^n)}(\sigma_1, \dots, \sigma_k, 0, \dots) \equiv \Delta_{(1^n)}(\sigma) + (-1)^n \sigma_n = (-1)^n \sigma_n$ (mod J); this term appears because σ_n is the upper-right entry of the determinant defining $\Delta_{(1^n)}(\sigma)$. Define

$$\mathrm{QH}'_\alpha = \mathbb{Q}[q, c'_1, \dots, c'_k] / \langle \sigma'_{m+1}, \dots, \sigma'_{n-1}, \sigma'_n + (-1)^k \alpha q \rangle\,.$$

The observed properties of ϕ show that this map factors as a $\mathbb{Q}[q]$-algebra homomorphism $\phi : \mathrm{QH}'_\alpha \to \mathrm{QH}_\alpha$ given by $\phi(\sigma'_\mu) = \sigma_{\mu'}$ for $\mu \subset k \times m$. Part (b) of the proposition therefore follows from part (a) applied to QH'_α. $\square$

Corollary 14. QH_α *is a free $\mathbb{Q}[q]$-module with basis $\{\sigma_\lambda \mid \lambda \subset m \times k\}$.*

Proof. It follows from Proposition 13(a) that QH_α is generated as a $\mathbb{Q}[q]$-module by the classes σ_λ for $\lambda \subset m \times k$, and Corollary 11 implies that q is a non-zero divisor in QH_α. Assume that $\sum_\lambda a_\lambda \sigma_\lambda = 0$ is a non-trivial relation in QH_α, where the sum is over $\lambda \subset m \times k$ and $a_\lambda \in \mathbb{Q}[q]$. Since q is a non-zero divisor, we may assume that $a_\lambda(0) \neq 0$ for some λ. But then the relation $\sum_\lambda a_\lambda(0)\sigma_\lambda = 0$ in $\mathrm{QH}_\alpha/\langle q\rangle$ contradicts Lemma 10. $\square$

3.5. The Seidel representation.

Definition 15. Given a partition $\lambda \subset m \times k$, define the first *Seidel shift* of λ by

$$\lambda \uparrow 1 = \begin{cases} (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_m + 1) & \text{if } \lambda_1 < k, \\ (\lambda_2, \lambda_3, \ldots, \lambda_m, 0) & \text{if } \lambda_1 = k. \end{cases}$$

The p-th Seidel shift $\lambda \uparrow p$ is defined inductively by $\lambda \uparrow 0 = \lambda$ and $\lambda \uparrow (p+1) = (\lambda \uparrow p) \uparrow 1$ for $p \geq 0$.

The first Seidel shift captures the following case of Bertram's quantum Pieri formula (Proposition 13):

$$c_m\,\sigma_\lambda = (\alpha q)^{\frac{m+|\lambda|-|\lambda\uparrow 1|}{n}}\sigma_{\lambda\uparrow 1}\,. \tag{2}$$

Notice that Seidel shifts of the zero partition are given by

$$0 \uparrow p = \begin{cases} (p^m) & \text{for } 0 \leq p \leq k, \\ (k^{n-p}) & \text{for } k \leq p \leq n. \end{cases} \tag{3}$$

It follows that $c_m^n = (\alpha q)^m$ and $(\alpha q)^m \sigma_\lambda = c_m^n \sigma_\lambda = (\alpha q)^{\frac{mn+|\lambda|-|\lambda\uparrow n|}{n}}\sigma_{\lambda\uparrow n}$, so we obtain $\lambda \uparrow n = \lambda$ for any partition $\lambda \subset m \times k$. For any integer $p \in \mathbb{Z}$ we can therefore define $\lambda \uparrow p = \lambda \uparrow p'$ where $p' \geq 0$ is chosen so that $p' \equiv p \pmod{n}$. We write $\lambda \downarrow p = \lambda \uparrow (-p)$ for convenience. With this notation we have

$$\sigma_k\,\sigma_\lambda = (\alpha q)^{\frac{k+|\lambda|-|\lambda\downarrow 1|}{n}}\sigma_{\lambda\downarrow 1} \tag{4}$$

in QH_α by Proposition 13(b).

The following identity was proved by Propp and Roby in [51, Thm. 19].

Lemma 16 ([51])**.** *For any partition $\lambda \subset m \times k$ we have $\sum_{p=0}^{n-1} |\lambda \uparrow p| = \frac{1}{2}kmn$.*

Proof. The lemma follows from (3) when $\lambda = 0$, so it suffices to show that $\sum_{p=0}^{n-1} |\mu \uparrow p| = \sum_{p=0}^{n-1} |\lambda \uparrow p|$ holds whenever the partition μ is obtained by adding a single box to λ, say in row i and column j. By replacing the partitions with $\lambda \uparrow (k-j+i-1)$ and $\mu \uparrow (k-j+i-1)$, we may assume that the added box is in the upper-right corner of the rectangle $m \times k$, so that $\lambda_1 = k-1$ and $\mu_1 = k$. The identity then follows because $|\mu \downarrow p| = |\lambda \downarrow p| + 1$ for $0 \leq p \leq n-2$ and $|\mu \uparrow 1| = |\lambda \uparrow 1| - n + 1$. $\square$

Corollary 17. *Let λ and μ be partitions contained in $m \times k$ such that $|\lambda| > |\mu|$. Then there exists $p \in \mathbb{Z}$ for which $|\lambda \uparrow p| < |\mu \uparrow p|$.*

3.6. Main results.

Our classification of non-negative quantum deformations of Grassmannians is based on the following theorem.

Theorem 18. *Let $\{\tau_\lambda \mid \lambda \subset m \times k\}$ be any $\mathbb{Q}[q]$-basis of QH_α such that* (a) *τ_λ is homogeneous of degree $|\lambda|$,* (b) *$\tau_\lambda - \sigma_\lambda \in \langle q\rangle$, and* (c) *the structure constants of QH_α with respect to the $\mathbb{Q}$-basis $\{q^d\tau_\lambda \mid \lambda \subset m \times k, d \geq 0\}$ are non-negative. Then we have $\tau_\lambda = \sigma_\lambda$ for all λ.*

Proof. For each $\lambda \subset m \times k$ it follows from the hypotheses (a) and (b) that we can write

$$\tau_\lambda = \sigma_\lambda + \sum_{|\mu|<|\lambda|} a_{\lambda,\mu}\, q^{\frac{|\lambda|-|\mu|}{n}}\, \sigma_\mu \,, \tag{5}$$

where $a_{\lambda,\mu} \in \mathbb{Q}$ is non-zero only if $|\lambda| - |\mu|$ is a positive multiple of n. We must show that $a_{\lambda,\mu} = 0$ for all λ and μ. Since $c_m \tau_k = \alpha q$, we must have $\alpha \geq 0$ by hypothesis (c).

Assume first that $\alpha = 0$. If the theorem is false, then let $\lambda \subset m \times k$ be minimal in the degree-lexicographic order such that $\tau_\lambda \neq \sigma_\lambda$. Set $\ell = \ell(\lambda)$ and let $\widehat{\lambda} = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_\ell - 1)$ be the partition obtained by removing the first column of λ. Proposition 13(a) shows that

$$c_\ell \tau_{\widehat{\lambda}} = c_\ell \sigma_{\widehat{\lambda}} = \sum_\nu \sigma_\nu \,,$$

where the sum is over all partitions $\nu \subset m \times k$ that can be obtained by adding a vertical strip of ℓ boxes to $\widehat{\lambda}$. Since λ is the maximal such partition in the degree-lexicographic order, we obtain

$$c_\ell \tau_{\widehat{\lambda}} = \sigma_\lambda + \sum_{\nu \neq \lambda} \sigma_\nu = \tau_\lambda - \sum_{|\mu|<|\lambda|} a_{\lambda,\mu}\, q^{\frac{|\lambda|-|\mu|}{n}}\, \tau_\mu + \sum_{\nu \neq \lambda} \tau_\nu \,.$$

This shows that $-a_{\lambda,\mu}$ is a structure constant of the basis $\mathcal{T} = \{q^d \tau_\lambda\}$ for each μ, so hypothesis (c) implies that $a_{\lambda,\mu} \leq 0$. Now choose ν with $a_{\lambda,\nu} < 0$ such that the dual partition $\nu^\vee = (\nu^\vee_1, \dots, \nu^\vee_m) = (k - \nu_m, \dots, k - \nu_1)$ is maximal in the degree-lexicographic order. If μ is any partition for which $a_{\lambda,\mu} \neq 0$, then either $\nu^\vee$ is larger than $\mu^\vee$ in the lexicographic order, in which case it follows from Proposition 13(b) and induction on p that

$$\sigma_{\nu^\vee_1} \cdots \sigma_{\nu^\vee_p}\, \sigma_\mu = \begin{cases} \sigma_{(k^p, \mu_1, \dots, \mu_{m-p})} & \text{if } \mu_i = \nu_i \text{ for } m - p < i \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise we have $|\nu^\vee| > |\mu^\vee|$, in which case $|\nu^\vee| + |\mu| > mk$ and $\sigma_{\nu^\vee_1} \cdots \sigma_{\nu^\vee_m}\, \sigma_\mu = 0$.

Applying this to equation (5) we therefore obtain

$$\tau_{\nu^\vee_1} \cdots \tau_{\nu^\vee_m} \tau_\lambda = a_{\lambda,\nu}\, q^{\frac{|\lambda|-|\nu|}{n}} \sigma_{(k^m)} \,.$$

Finally, since the coefficient of $\tau_{(k^m)}$ in the $\mathcal{T}$-expansion of $\sigma_{(k^m)}$ is equal to 1, we deduce that $a_{\lambda,\nu}$ is an iterated structure constant, so hypothesis (c) implies that $a_{\lambda,\nu} \geq 0$, a contradiction.

Assume now that $\alpha > 0$. Then $c_m \tau_k = \alpha q$ is a non-zero multiple of q. If $\lambda \subset m \times k$ is any partition with $\lambda_1 < k$, then since $[X^{(1^m)}] \cdot [X^\lambda] = [X^{\lambda \uparrow 1}]$ holds in $H^*(X, \mathbb{Q})$, it follows from Lemma 3 that $c_m \tau_\lambda = \tau_{\lambda \uparrow 1}$. On the other hand, if $\lambda_1 = k$, then $[X^k] \cdot [X^{\lambda \uparrow 1}] = [X^\lambda]$ holds in $H^*(X, \mathbb{Q})$, so Lemma 3 implies that $c_m \tau_\lambda = c_m \tau_k \tau_{\lambda \uparrow 1} = \alpha q\, \tau_{\lambda \uparrow 1}$. This shows that multiplication by c_m in the basis $\mathcal{T}$ is determined by

$$c_m \tau_\lambda = (\alpha q)^{\frac{m+|\lambda|-|\lambda \uparrow 1|}{n}} \tau_{\lambda \uparrow 1} \,. \tag{6}$$

If $\tau_\lambda \neq \sigma_\lambda$, then choose μ such that $a_{\lambda,\mu} \neq 0$. By Corollary 17 we can choose p such that $|\lambda \uparrow p| < |\mu \uparrow p|$. Equation (6) then implies that $c_m^p \tau_\lambda = (\alpha q)^d \tau_{\lambda \uparrow p}$ where

$d = \frac{mp+|\lambda|-|\lambda\uparrow p|}{n}$, so $c_m^p \tau_\lambda$ is divisible by q^d. However, equation (2) shows that the product of c_m^p with the right hand side of (5) contains the term

$$a_{\lambda,\mu}\, q^{\frac{|\lambda|-|\mu|}{n}}\, (\alpha q)^{\frac{mp+|\mu|-|\mu\uparrow p|}{n}} \sigma_{\mu\uparrow p}\,,$$

and since $|\lambda \uparrow p| < |\mu \uparrow p|$, this term is not divisible by q^d. This contradicts that $c_m^p \tau_\lambda$ is divisible by q^d, which completes the proof. $\square$

Corollary 19. *Let $X = \mathrm{Gr}(m,n)$ and let $\mathrm{QH}_\alpha = (\mathrm{QH}_\alpha, \{\sigma_\lambda\})$ be the quantum deformation of $H^*(X,\mathbb{Q})$ defined in Section 3.4.*

(a) *The deformation QH_α is non-negative if and only if $\alpha \geq 0$.*

(b) *Any non-negative quantum deformation of $H^*(X,\mathbb{Q})$ is isomorphic to QH_α for a unique rational number $\alpha \geq 0$.*

(c) *$\mathrm{QH}(X) \cong \mathrm{QH}_1$ is the only non-negative change-of-basis quantum deformation of $H^*(X,\mathbb{Q})$.*

Proof. Given any quantum deformation $(\mathrm{QH}, \{\tau_\lambda\})$ of $H^*(X,\mathbb{Q})$, it follows from Definition 2 that $\tau_{(1^m)}\tau_k = \alpha q$ for some $\alpha \in \mathbb{Q}$. Define a ring homomorphism $\phi : S[q] \to \mathrm{QH}$ by $\phi(c_p) = \tau_{(1^p)}$ and $\phi(q) = q$. Definition 2 then implies that $\phi(\sigma_p) = \tau_p$ for $1 \leq p \leq k$, $\phi(\sigma_p) = 0$ for $k < p < n$, and $\phi(\sigma_n) = \phi\left(\sum_{p=1}^m (-1)^{p-1} c_p \sigma_{n-p}\right) = (-1)^{m-1}\tau_{(1^m)}\tau_k = (-1)^{m-1}\alpha q$. We deduce that ϕ factors as an isomorphism of $\mathbb{Q}[q]$-algebras $\psi : \mathrm{QH}_\alpha \to \mathrm{QH}$, and the set $\{\psi^{-1}(\tau_\lambda)\}$ is a deformation of the basis $\{\sigma_\lambda\}$ of QH_α. If QH is a non-negative quantum deformation, then Theorem 18 implies that $\psi^{-1}(\tau_\lambda) = \sigma_\lambda$ for each λ. This proves part (b).

Part (c) follows from part (b) by using that $\mathrm{QH}(X)$ is a non-negative quantum deformation of $H^*(X,\mathbb{Q})$ satisfying the relation $[X^{(1^m)}] \star [X^k] = q$. This relation is equivalent to Witten's presentation of $\mathrm{QH}(X)$ [55, §3.2] (see also [54, Prop. 3.1]). The uniqueness statement of part (c) follows from Theorem 18 applied to QH_1.

For $\alpha > 0$ the assignments $q \mapsto \alpha q$ and $c_p \mapsto c_p$ define an isomorphism of graded rings $\mathrm{QH}_1 \cong \mathrm{QH}_\alpha$. Since this map sends σ_λ to σ_λ, it follows that the structure constants of QH_α are the rescaled Gromov-Witten invariants $\alpha^d \langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d$. This shows that QH_α is non-negative for $\alpha > 0$. The trivial deformation $\mathrm{QH}_0 = H^*(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[q]$ is also non-negative, and the relation $c_m \sigma_k = \alpha q$ shows that QH_α is not non-negative for $\alpha < 0$. This proves part (a). $\square$

Corollary 20. *The quantum cohomology ring $\mathrm{QH}(X)$ satisfies Bertram's quantum Giambelli formula*

$$[X^\lambda] = \det\left([X^{\lambda_i+j-i}]\right)_{m\times m}$$

and quantum Pieri formula

$$[X^p] \star [X^\lambda] = \sum_\mu [X^\mu] + \sum_\nu q\,[X^\nu]\,,$$

where the sums are over the same partitions as in Proposition 13(b).

Proof. This follows from Corollary 19, Proposition 13, and the definition of σ_λ. $\square$

Example 21. Let $X = \mathrm{GL}(6)/B$ be the variety of complete flags in $\mathbb{C}^6$ and set $t = (6,1,2,3,4,5)$. We have $\pi_1(\mathrm{Aut}(X)) \cong \mathbb{Z}/6\mathbb{Z}$, and the Seidel representation of this group on $\mathrm{QH}(X)/\langle q-1 \rangle$ is defined by quantum multiplication with $[X^t]$, see [53, 19]. Products with this class modulo $\langle q-1 \rangle$ are given by $[X^t] \star [X^w] = [X^{tw}]$

[21]. The following table shows the Seidel orbits of the Schubert classes given by $e = (1,2,3,4,5,6)$ and $v = (3,2,1,6,5,4)$, using the notation $w \uparrow r = t^r w$.

r	$e \uparrow r$	$\ell(e \uparrow r)$	$v \uparrow r$	$\ell(v \uparrow r)$
0	123456	0	321654	6
1	612345	5	216543	7
2	561234	8	165432	10
3	456123	9	654321	15
4	345612	8	543216	10
5	234561	5	432165	7

This shows that the analogues of Lemma 16 and Corollary 17 for varieties of complete flags are false.

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