

# QUANTUM GIAMBELLI FORMULAS FOR ISOTROPIC GRASSMANNIANS

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ABSTRACT. Let  $X$  be a symplectic or odd orthogonal Grassmannian which parametrizes isotropic subspaces in a vector space equipped with a nondegenerate (skew) symmetric form. We prove quantum Giambelli formulas which express an arbitrary Schubert class in the small quantum cohomology ring of  $X$  as a polynomial in certain special Schubert classes, extending the authors' cohomological Giambelli formulas.

## 0. INTRODUCTION

Let  $E$  be an even (respectively, odd) dimensional complex vector space equipped with a nondegenerate skew-symmetric (respectively, symmetric) bilinear form. Let  $X$  denote the Grassmannian which parametrizes the isotropic subspaces of  $E$  of some fixed dimension. The cohomology ring  $H^*(X, \mathbb{Z})$  is generated by certain special Schubert classes, which for us are (up to a factor of two) the Chern classes of the universal quotient vector bundle over  $X$ . These special classes also generate the small quantum cohomology ring  $\text{QH}(X)$ , a  $q$ -deformation of  $H^*(X, \mathbb{Z})$  whose structure constants are the three point, genus zero Gromov-Witten invariants of  $X$ . In [BKT3], we proved a Giambelli formula in  $H^*(X, \mathbb{Z})$ , that is, a formula expressing a general Schubert class as an explicit polynomial in the special classes. Our goal in the present work is to extend this result to a formula that holds in  $\text{QH}(X)$ .

The quantum Giambelli formula for the usual type A Grassmannian was obtained by Bertram [Be], and is in fact identical to the classical Giambelli formula. In the case of maximal isotropic Grassmannians, the corresponding questions were answered in [KT1, KT2]. The main conclusions here are similar to those of loc. cit., provided that one uses the raising operator Giambelli formulas of [BKT3] as the classical starting point. For an odd orthogonal Grassmannian, we prove that the quantum Giambelli formula is the same as the classical one. The result is more interesting when  $X$  is the Grassmannian  $\text{IG}(n-k, 2n)$  parametrizing  $(n-k)$ -dimensional isotropic subspaces of a symplectic vector space  $E$  of dimension  $2n$ . Our theorem in this case states that the quantum Giambelli formula for  $\text{IG}(n-k, 2n)$  coincides with the classical Giambelli formula for  $\text{IG}(n+1-k, 2n+2)$ , provided that the special Schubert class  $\sigma_{n+k+1}$  is replaced with  $q/2$ .

Although the two theorems in this article are analogous to those of [KT1, KT2], their proofs are quite different. We prove the quantum Giambelli formula by using the quantum Pieri rule of [BKT2], in a manner similar to [Bu] and [BKT1, Remark

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3]. However, unlike the previously known examples, for non-maximal isotropic Grassmannians no explicit recursion formula for the cohomological Giambelli polynomials is available, other than that given by the Pieri rule itself. We circumvent this difficulty by showing that a suitable recursion exists (Proposition 3). We also make essential use of a ring homomorphism from the stable cohomology ring of  $X$  to  $\mathrm{QH}(X)$  that is the identity on Schubert classes coming from  $H^*(X, \mathbb{Z})$ . The existence of this map (Propositions 4 and 5) may be of independent interest.

In a sequel to this paper, we will discuss the classical and quantum Giambelli formulas for even orthogonal Grassmannians.

## 1. PRELIMINARY RESULTS

**1.1. Classical Giambelli for IG.** Choose  $k \geq 0$  and consider the Grassmannian  $\mathrm{IG} = \mathrm{IG}(n - k, 2n)$  of isotropic  $(n - k)$ -dimensional subspaces of  $\mathbb{C}^{2n}$ , equipped with a symplectic form. A partition  $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell)$  is  $k$ -strict if all of its parts greater than  $k$  are distinct integers. Following [BKT2], the Schubert classes on  $\mathrm{IG}$  are parametrized by the  $k$ -strict partitions whose diagrams fit in an  $(n - k) \times (n + k)$  rectangle, i.e.  $\lambda_1 \leq n + k$  and  $\ell(\lambda) \leq n - k$ ; we denote the set of all such partitions by  $\mathcal{P}(k, n)$ . Given any partition  $\lambda \in \mathcal{P}(k, n)$  and a complete flag of subspaces

$$F_\bullet : 0 = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_{2n} = \mathbb{C}^{2n}$$

such that  $F_{n+i} = F_{n-i}^\perp$  for  $0 \leq i \leq n$ , we have a Schubert variety

$$X_\lambda(F_\bullet) := \{\Sigma \in \mathrm{IG} \mid \dim(\Sigma \cap F_{p_j(\lambda)}) \geq j \quad \forall 1 \leq j \leq \ell(\lambda)\},$$

where  $\ell(\lambda)$  denotes the number of (non-zero) parts of  $\lambda$  and

$$p_j(\lambda) := n + k + j - \lambda_j - \#\{i < j : \lambda_i + \lambda_j > 2k + j - i\}.$$

This variety has codimension  $|\lambda| = \sum \lambda_i$  and defines, via Poincaré duality, a Schubert class  $\sigma_\lambda = [X_\lambda(F_\bullet)]$  in  $H^{2|\lambda|}(\mathrm{IG}, \mathbb{Z})$ . The Schubert classes  $\sigma_\lambda$  for  $\lambda \in \mathcal{P}(k, n)$  form a free  $\mathbb{Z}$ -basis for the cohomology ring of  $\mathrm{IG}$ . The *special Schubert classes* are defined by  $\sigma_r = [X_r(F_\bullet)] = c_r(\mathcal{Q})$  for  $1 \leq r \leq n + k$ , where  $\mathcal{Q}$  denotes the universal quotient bundle over  $\mathrm{IG}$ .

The classical Giambelli formula for  $\mathrm{IG}$  is expressed using Young's *raising operators* [Y, p. 199]. We first agree that  $\sigma_0 = 1$  and  $\sigma_r = 0$  for  $r < 0$ . For any integer sequence  $\alpha = (\alpha_1, \alpha_2, \dots)$  with finite support and  $i < j$ , we set  $R_{ij}(\alpha) = (\alpha_1, \dots, \alpha_i + 1, \dots, \alpha_j - 1, \dots)$ ; a raising operator  $R$  is any monomial in these  $R_{ij}$ 's. Define  $m_\alpha = \prod_i \sigma_{\alpha_i}$  and  $Rm_\alpha = m_{R\alpha}$  for any raising operator  $R$ . For any  $k$ -strict partition  $\lambda$ , we consider the operator

$$R^\lambda = \prod (1 - R_{ij}) \prod_{\lambda_i + \lambda_j > 2k + j - i} (1 + R_{ij})^{-1}$$

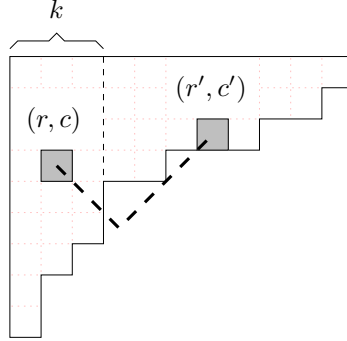
where the first product is over all pairs  $i < j$  and second product is over pairs  $i < j$  such that  $\lambda_i + \lambda_j > 2k + j - i$ . The main result of [BKT3] states that the *Giambelli formula*

$$(1) \quad \sigma_\lambda = R^\lambda m_\lambda$$

holds in the cohomology ring of  $\mathrm{IG}(n - k, 2n)$ .

**1.2. Classical Pieri for IG.** As is customary, we will represent a partition by its Young diagram of boxes; this is used to define the containment relation for partitions. Given two diagrams  $\mu$  and  $\nu$  with  $\mu \subset \nu$ , the skew diagram  $\nu/\mu$  (i.e., the set-theoretic difference  $\nu \setminus \mu$ ) is called a horizontal (resp. vertical) strip if it does not contain two boxes in the same column (resp. row).

We say that the box  $[r, c]$  in row  $r$  and column  $c$  of a  $k$ -strict partition  $\lambda$  is  $k$ -related to the box  $[r', c']$  if  $|c - k - 1| + r = |c' - k - 1| + r'$ . For instance, the grey boxes in the following partition are  $k$ -related.



For any two  $k$ -strict partitions  $\lambda$  and  $\mu$ , we write  $\lambda \rightarrow \mu$  if  $\mu$  may be obtained by removing a vertical strip from the first  $k$  columns of  $\lambda$  and adding a horizontal strip to the result, so that

- (1) if one of the first  $k$  columns of  $\mu$  has the same number of boxes as the same column of  $\lambda$ , then the bottom box of this column is  $k$ -related to at most one box of  $\mu \setminus \lambda$ ; and
- (2) if a column of  $\mu$  has fewer boxes than the same column of  $\lambda$ , then the removed boxes and the bottom box of  $\mu$  in this column must each be  $k$ -related to exactly one box of  $\mu \setminus \lambda$ , and these boxes of  $\mu \setminus \lambda$  must all lie in the same row.

Let  $\mathbb{A}$  denote the set of boxes of  $\mu \setminus \lambda$  in columns  $k + 1$  through  $k + n$  which are not mentioned in (1) or (2) above, and define  $N(\lambda, \mu)$  to be the number of connected components of  $\mathbb{A}$  which do not have a box in column  $k + 1$ . Here two boxes are connected if they share at least a vertex. In [BKT2, Thm. 1.1] we proved that the Pieri rule

$$(2) \quad \sigma_p \cdot \sigma_\lambda = \sum_{\substack{\lambda \rightarrow \mu \\ |\mu| = |\lambda| + p}} 2^{N(\lambda, \mu)} \sigma_\mu$$

holds in  $H^*(IG, \mathbb{Z})$ , for any  $p \in [1, n + k]$ .

**1.3. A recursion formula for IG.** In the following sections we will work in the stable cohomology ring  $\mathbb{H}(IG_k)$ , which is the inverse limit in the category of graded rings of the system

$$\cdots \leftarrow H^*(IG(n - k, 2n), \mathbb{Z}) \leftarrow H^*(IG(n + 1 - k, 2n + 2), \mathbb{Z}) \leftarrow \cdots$$

The ring  $\mathbb{H}(IG_k)$  has a free  $\mathbb{Z}$ -basis of Schubert classes  $\sigma_\lambda$ , one for each  $k$ -strict partition  $\lambda$ , and may be presented as a quotient of the polynomial ring  $\mathbb{Z}[\sigma_1, \sigma_2, \dots]$

modulo the relations

$$(3) \quad \sigma_r^2 + 2 \sum_{i=1}^r (-1)^i \sigma_{r+i} \sigma_{r-i} = 0 \quad \text{for } r > k.$$

There is a natural surjective ring homomorphism  $\mathbb{H}(\text{IG}_k) \rightarrow \mathbb{H}(\text{IG}(n-k, 2n), \mathbb{Z})$  that maps  $\sigma_\lambda$  to  $\sigma_\lambda$ , when  $\lambda \in \mathcal{P}(k, n)$ , and to zero, otherwise. The Giambelli formula (1) and Pieri rule (2) are both valid in  $\mathbb{H}(\text{IG}_k)$ . We begin with some elementary consequences of these theorems.

For any  $k$ -strict partition  $\lambda$  of length  $\ell$ , we define the sets of pairs

$$\mathcal{A}(\lambda) = \{(i, j) \mid \lambda_i + \lambda_j \leq 2k + j - i \text{ and } 1 \leq i < j \leq \ell\}$$

$$\mathcal{C}(\lambda) = \{(i, j) \mid \lambda_i + \lambda_j > 2k + j - i \text{ and } 1 \leq i < j \leq \ell\}$$

and two integer vectors  $a = (a_1, \dots, a_\ell)$  and  $c = (c_1, \dots, c_\ell)$  by setting

$$a_i = \#\{j \mid (i, j) \in \mathcal{A}(\lambda)\}, \quad c_i = \#\{j \mid (i, j) \in \mathcal{C}(\lambda)\}$$

for each  $i$ .

**Proposition 1.** *We have  $\lambda_i - c_i \geq \lambda_j - c_j$  for each  $i < j \leq \ell$ .*

*Proof.* Observe that the desired inequality is equivalent to

$$(4) \quad \lambda_i - \lambda_j \geq \#\{r \leq \ell \mid (i, r) \in \mathcal{C}(\lambda)\} - \#\{r \leq \ell \mid (j, r) \in \mathcal{C}(\lambda)\}.$$

Let  $j = i + r$  and let  $s$  (respectively  $t$ ) be maximal such that  $(i, s) \in \mathcal{C}(\lambda)$  (respectively,  $(j, t) \in \mathcal{C}(\lambda)$ ). Assume first that  $t$  exists, hence  $s$  exists and  $s \geq t$ . The inequality (4) then becomes  $\lambda_i - \lambda_{i+r} \geq s - t + r$ . If  $t = s$ , this is true because  $(j, j+1) \in \mathcal{C}(\lambda)$  and  $\lambda$  is  $k$ -strict, hence  $\lambda_i > \lambda_{i+1} > \dots > \lambda_{i+r}$ . Otherwise we have  $t < s \leq \ell$ ,  $\lambda_i + \lambda_s \geq 2k + 1 + s - i$ , and  $\lambda_{i+r} + \lambda_{t+1} \leq 2k + t + 1 - i - r$ . It follows that  $\lambda_i - \lambda_{i+r} \geq s - t + r + (\lambda_{t+1} - \lambda_s) \geq s - t + r$ .

Next we assume that  $t$  does not exist, so that either  $j = \ell$  or the pair  $(j, j+1)$  lies in  $\mathcal{A}(\lambda)$  and

$$(5) \quad \lambda_j + \lambda_{j+1} \leq 2k + 1.$$

If  $s$  does not exist, the inequality is obvious. Otherwise, we must show that  $\lambda_i - \lambda_j \geq s - i$ , knowing that  $(i, s) \in \mathcal{C}(\lambda)$ , that is,

$$(6) \quad \lambda_i + \lambda_s \geq 2k + 1 + s - i.$$

Suppose first that  $\lambda_s \geq \lambda_j$ . If  $\lambda_s > k$  then we have  $\lambda_i > \lambda_{i+1} > \dots > \lambda_s$  and hence  $\lambda_i - \lambda_j \geq \lambda_i - \lambda_s \geq s - i$ . Otherwise  $\lambda_s \leq k$  and (6) gives

$$\lambda_i - \lambda_j \geq \lambda_i - \lambda_s \geq \lambda_i - k \geq s - i + 1 + (k - \lambda_s) \geq s - i.$$

Finally, suppose that  $\lambda_s < \lambda_j$ , so in particular  $j + 1 \leq s$ . Then (5) and (6) give

$$\begin{aligned} \lambda_i - \lambda_j &\geq \lambda_i + (\lambda_{j+1} - 2k - 1) \geq (2k + 1 + s - i - \lambda_s) + \lambda_{j+1} - 2k - 1 \\ &= (\lambda_{j+1} - \lambda_s) + (s - i) \geq s - i. \end{aligned} \quad \square$$

Proposition 1 implies that for any  $\lambda$ , the composition  $\lambda - c$  is a partition, while  $\lambda + a$  is a strict partition.

**Proposition 2.** *For any  $k$ -strict partition  $\lambda$ , the Giambelli polynomial  $R^\lambda m_\lambda$  for  $\sigma_\lambda$  involves only generators  $\sigma_p$  with  $p \leq \lambda_1 + a_1 + \lambda_2 + a_2$ .*

*Proof.* We have

$$R^\lambda m_\lambda = \prod_{1 \leq i < j \leq \ell} \frac{1 - R_{ij}}{1 + R_{ij}} \prod_{(i,j) \in \mathcal{A}(\lambda)} (1 + R_{ij}) m_\lambda = \sum_{\nu \in N} \prod_{1 \leq i < j \leq \ell} \frac{1 - R_{ij}}{1 + R_{ij}} m_\nu$$

where  $N$  is the multiset of integer vectors defined by

$$N = \left\{ \prod_{(i,j) \in S} R_{ij} \lambda \mid S \subset \mathcal{A}(\lambda) \right\}.$$

If  $m > 0$  is the least integer such that  $2m \geq \ell$ , then we have

$$(7) \quad \prod_{1 \leq i < j \leq 2m} \frac{1 - R_{ij}}{1 + R_{ij}} = \text{Pfaffian} \left( \frac{1 - R_{ij}}{1 + R_{ij}} \right)_{1 \leq i < j \leq 2m}.$$

Equation (7) follows from Schur's classical identity [S, Sect. IX]

$$\prod_{1 \leq i < j \leq 2m} \frac{x_i - x_j}{x_i + x_j} = \text{Pfaffian} \left( \frac{x_i - x_j}{x_i + x_j} \right)_{1 \leq i, j \leq 2m}.$$

Note that each single entry in the Pfaffian (7) expands according to the formula

$$\frac{1 - R_{12}}{1 + R_{12}} m_{c,d} = \sigma_c \sigma_d - 2 \sigma_{c+1} \sigma_{d-1} + 2 \sigma_{c+2} \sigma_{d-2} - \cdots + (-1)^d 2 \sigma_{c+d}.$$

By Proposition 1, we know that  $\lambda + a = (\lambda_1 + a_1, \lambda_2 + a_2, \dots, \lambda_\ell + a_\ell)$  is a strict partition, hence  $\lambda_i + a_i + \lambda_j + a_j \leq \lambda_1 + a_1 + \lambda_2 + a_2$  for any distinct  $i$  and  $j$ . Since we furthermore have  $\nu_i \leq \lambda_i + a_i$ , for any  $\nu \in N$ , the result follows.  $\square$

**Corollary 1.** *For any  $\lambda \in \mathcal{P}(k, n)$  the stable Giambelli polynomial for  $\sigma_\lambda$  involves only special classes  $\sigma_p$  with  $p \leq 2n + 2k - 1$ .*

Given any partition  $\lambda$ , we set  $\lambda^* = (\lambda_2, \lambda_3, \dots, \lambda_\ell)$ .

**Lemma 1.** *Let  $\lambda$  and  $\nu$  be  $k$ -strict partitions such that  $\nu_1 > \max(\lambda_1, \ell(\lambda) + 2k)$  and let  $p, m \geq 0$ . Then the coefficient of  $\sigma_\nu$  in the Pieri product  $\sigma_p \cdot \sigma_\lambda$  is equal to the coefficient of  $\sigma_{(\nu_1+m, \nu^*)}$  in the product  $\sigma_{p+m} \cdot \sigma_\lambda$ .*

*Proof.* Since the box  $[\ell(\lambda), 1]$  is  $k$ -related to  $[1, \ell(\lambda) + 2k]$  and  $\nu_1 > \ell(\lambda) + 2k$ , it follows that  $\lambda \rightarrow \nu$  if and only if  $\lambda \rightarrow (\nu_1 + m, \nu^*)$ . In this case all of the boxes  $[1, c]$  for  $\max(\lambda_1, \ell(\lambda) + 2k) < c \leq \nu_1$  are contained in the rightmost component of the subset  $\mathbb{A}$  of  $\nu \setminus \lambda$  defined in §1.2. Since replacing  $\nu$  with  $(\nu_1 + m, \nu^*)$  simply adds  $m$  boxes to this component, we deduce that  $N(\lambda, \nu) = N(\lambda, (\nu_1 + m, \nu^*))$ .  $\square$

**Proposition 3.** *Let  $\lambda$  be a  $k$ -strict partition. Then there exist unique coefficients  $a_{p,\mu} \in \mathbb{Z}$  for  $p \geq \lambda_1$  and  $(p, \mu)$  a  $k$ -strict partition, such that the recursive identity*

$$(8) \quad \sigma_\lambda = \sum_{p \geq \lambda_1} \sum_{\mu: (p,\mu) \text{ } k\text{-strict}} a_{p,\mu} \sigma_p \sigma_\mu$$

*holds in  $\mathbb{H}(\text{IG}_k)$ . Furthermore,  $a_{p,\mu} = 0$  whenever  $\mu \not\subset \lambda^*$ , or when  $\lambda \in \mathcal{P}(k, n)$  and  $p \geq 2n + 2k$ .*

*Proof.* The Pieri rule (2) implies that

$$\sigma_\lambda = \sigma_{\lambda_1} \sigma_{\lambda^*} - \sum_{\substack{\lambda^* \rightarrow \nu \neq \lambda \\ |\nu| = |\lambda|}} 2^{N(\lambda^*, \nu)} \sigma_\nu.$$

Since all partitions  $\nu$  in the sum satisfy  $\nu_1 > \lambda_1$  and  $\nu^* \subset \lambda^*$ , the existence of the coefficients  $a_{p,\mu}$  follows by descending induction on  $\lambda_1$ , and they satisfy  $a_{(p,\mu)} = 0$  for  $\mu \not\subset \lambda^*$ . The uniqueness is true because the set of all products  $\sigma_p \cdot \sigma_\mu$  for which  $(p, \mu)$  is a  $k$ -strict partition is linearly independent in  $\mathbb{H}(\text{IG}_k)$ . In fact, if the Schubert classes of  $\mathbb{H}(\text{IG}_k)$  are ordered by the dominance order of partitions, then the lowest term of the product  $\sigma_p \cdot \sigma_\mu$  is the class  $\sigma_{(p,\mu)}$ .

On the other hand, Proposition 2 implies that there are coefficients  $b_{p,\mu}$ , indexed by integers  $p \in [\lambda_1, \lambda_1 + a_1 + \lambda_2 + a_2]$  and  $k$ -strict partitions  $\mu$ , such that

$$\sigma_\lambda = \sum_{p=\lambda_1}^{\lambda_1+a_1+\lambda_2+a_2} \sum_{|\mu|=\lambda_1-p} b_{p,\mu} \sigma_p \sigma_\mu.$$

In fact, if  $m_\nu$  is any monomial appearing in the stable Giambelli formula  $\sigma_\lambda = R^\lambda m_\lambda$ , then  $\lambda_1 \leq \max_i(\nu_i) \leq \lambda_1 + a_1 + \lambda_2 + a_2$ . If  $\lambda_1 > |\lambda^*|$ , then the uniqueness of the coefficients  $a_{p,\mu}$  implies that  $b_{p,\mu} = a_{p,\mu}$ . In particular, we have  $a_{p,\mu} = 0$  for  $p > \lambda_1 + a_1 + \lambda_2 + a_2$  in this case.

Now let  $\lambda \in \mathcal{P}(k, n)$ . Choose  $m > |\lambda^*|$  and set  $\lambda' = (\lambda_1 + m, \lambda^*)$ . By the above discussion, there are coefficients  $c_{p,\mu} \in \mathbb{Z}$  such that

$$(9) \quad \sigma_{\lambda'} = \sum_{p=\lambda_1+m}^{2n+2k-1+m} \sum_{\mu \subset \lambda^*} c_{p,\mu} \sigma_p \sigma_\mu.$$

We claim that the difference

$$(10) \quad \sigma_\lambda - \sum_{p=\lambda_1}^{2n+2k-1} \sum_{\mu \subset \lambda^*} c_{p+m,\mu} \sigma_p \sigma_\mu$$

is a linear combination of classes  $\sigma_\nu$  for partitions  $\nu \in \mathcal{P}(k, n)$  with  $\nu_1 > \lambda_1$ . To see this, notice that we must have  $c_{\lambda_1+m,\lambda^*} = 1$ , and hence the coefficient of  $\sigma_\lambda$  in the sum is equal to one. It follows that the difference (10) is equal to a linear combination of classes  $\sigma_\nu$  for which  $\nu_1 > \lambda_1$ . Furthermore, if  $\nu_1 > n + k$ , then Lemma 1 implies that the coefficient of  $\sigma_\nu$  in the sum in (10) is equal to the coefficient of  $\sigma_{(\nu_1+m,\nu^*)}$  on the right hand side of (9), which is zero. This proves the claim. Finally, the proposition follows from the claim by descending induction on  $\lambda_1$ .  $\square$

**Remark.** One can be more precise about the recursion formula (8) in the case when the  $k$ -strict partition  $\lambda$  satisfies  $\lambda_1 > \ell(\lambda) + 2k$ . If the Pieri rule reads

$$\sigma_{\lambda_1} \cdot \sigma_{\lambda^*} = \sum_{p \geq \lambda_1} \sum_{\mu \subset \lambda^*} 2^{n(p,\mu)} \sigma_p \sigma_\mu$$

then we have

$$\sigma_\lambda = \sum_{p \geq \lambda_1} \sum_{\mu \subset \lambda^*} (-1)^{p-\lambda_1} 2^{n(p,\mu)} \sigma_p \sigma_\mu.$$

This result is proved in [T].

## 2. QUANTUM GIAMBELLI FOR $\text{IG}(n-k, 2n)$

The quantum cohomology ring  $\text{QH}(\text{IG})$  is a  $\mathbb{Z}[q]$ -algebra which is isomorphic to  $\text{H}^*(\text{IG}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$  as a module over  $\mathbb{Z}[q]$ . The degree of the formal variable  $q$  here

is  $n + k + 1$ . We begin by recalling the quantum Pieri rule of [BKT2]. This states that for any  $k$ -strict partition  $\lambda \in \mathcal{P}(k, n)$  and integer  $p \in [1, n + k]$ , we have

$$(11) \quad \sigma_p \cdot \sigma_\lambda = \sum_{\lambda \rightarrow \mu} 2^{N(\lambda, \mu)} \sigma_\mu + \sum_{\lambda \rightarrow \nu} 2^{N(\lambda, \nu)-1} \sigma_{\nu^*} q$$

in the quantum cohomology ring of  $\text{IG}(n - k, 2n)$ . The first sum in (11) is over partitions  $\mu \in \mathcal{P}(k, n)$  such that  $|\mu| = |\lambda| + p$ , and the second sum is over partitions  $\nu \in \mathcal{P}(k, n + 1)$  with  $|\nu| = |\lambda| + p$  and  $\nu_1 = n + k + 1$ .

**Proposition 4.** *There exists a unique ring homomorphism*

$$\pi : \mathbb{H}(\text{IG}_k) \rightarrow \text{QH}(\text{IG}(n - k, 2n)) \otimes \mathbb{Q}$$

such that the following relations are satisfied:

$$\pi(\sigma_i) = \begin{cases} \sigma_i & \text{if } 1 \leq i \leq n + k, \\ q/2 & \text{if } i = n + k + 1, \\ 0 & \text{if } n + k + 1 < i \leq 2n + 2k, \\ 0 & \text{if } i \text{ is odd and } i > 2n + 2k. \end{cases}$$

Furthermore, we have  $\pi(\sigma_\lambda) = \sigma_\lambda$  for each  $\lambda \in \mathcal{P}(k, n)$ .

*Proof.* Recall that  $\mathbb{H}(\text{IG}_k)$  is the polynomial ring generated by all classes  $\sigma_i$  for  $i \geq 1$ , modulo the relations (3). These relations for  $r > n + k$  uniquely specify the values  $\pi(\sigma_i)$  for even integers  $i > 2n + 2k$ . The quantum Pieri rule implies that the remaining relations (3) for  $k < r \leq n + k$  are preserved by  $\pi$ .

We next prove that  $\pi(\sigma_\lambda) = \sigma_\lambda$  for each  $\lambda \in \mathcal{P}(k, n)$ . This is clear when  $\lambda$  has only one part. When  $\lambda$  has more than one part, we apply the ring homomorphism  $\pi$  to both sides of (8) and use induction on  $\ell(\lambda)$  to show that

$$(12) \quad \pi(\sigma_\lambda) = \sum_{p=\lambda_1}^{n+k} \sum_{\mu \subset \lambda^*} a_{p, \mu} \sigma_p \sigma_\mu + \frac{q}{2} \sum_{\mu \subset \lambda^*} a_{n+k+1, \mu} \sigma_\mu$$

holds in  $\text{QH}(\text{IG}(n - k, 2n)) \otimes \mathbb{Q}$ . We also deduce from Proposition 3 that

$$(13) \quad \sigma_\lambda = \sum_{p=\lambda_1}^{n+k} \sum_{\mu \subset \lambda^*} a_{p, \mu} \sigma_p \sigma_\mu + \sum_{\mu \subset \lambda^*} a_{n+k+1, \mu} \sigma_{(n+k+1, \mu)}$$

holds in the cohomology ring of  $\text{IG}(n + 1 - k, 2n + 2)$ . The quantum Pieri rule and (13) imply that the right hand side of (12) evaluates to  $\sigma_\lambda$ , as desired.  $\square$

**Theorem 1** (Quantum Giambelli for IG). *For every  $\lambda \in \mathcal{P}(k, n)$ , the quantum Giambelli formula for  $\sigma_\lambda$  in  $\text{QH}(\text{IG}(n - k, 2n))$  is obtained from the classical Giambelli formula  $\sigma_\lambda = R^\lambda m_\lambda$  in  $H^*(\text{IG}(n + 1 - k, 2n + 2), \mathbb{Z})$  by replacing the special Schubert class  $\sigma_{n+k+1}$  with  $q/2$ .*

*Proof.* This follows from Proposition 4 and Corollary 1.  $\square$

### 3. QUANTUM GIAMBELLI FOR $\text{OG}(n - k, 2n + 1)$

**3.1. Classical Giambelli for OG.** For each  $k \geq 0$ , let  $\text{OG} = \text{OG}(n - k, 2n + 1)$  denote the odd orthogonal Grassmannian which parametrizes the  $(n - k)$ -dimensional isotropic subspaces in  $\mathbb{C}^{2n+1}$ , equipped with a non-degenerate symmetric bilinear

form. The Schubert varieties in OG are indexed by the same set of  $k$ -strict partitions  $\mathcal{P}(k, n)$  as for  $\text{IG}(n - k, 2n)$ . Given any  $\lambda \in \mathcal{P}(k, n)$  and a complete flag of subspaces

$$F_\bullet : 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n+1} = \mathbb{C}^{2n+1}$$

such that  $F_{n+i} = F_{n+1-i}^\perp$  for  $1 \leq i \leq n+1$ , we define the codimension  $|\lambda|$  Schubert variety

$$X_\lambda(F_\bullet) = \{\Sigma \in \text{OG} \mid \dim(\Sigma \cap F_{\bar{p}_j(\lambda)}^\perp) \geq j \quad \forall 1 \leq j \leq \ell(\lambda)\},$$

where

$$\bar{p}_j(\lambda) = n + k + 1 + j - \lambda_j - \#\{i \leq j : \lambda_i + \lambda_j > 2k + j - i\}.$$

Let  $\tau_\lambda \in H^{2|\lambda|}(\text{OG}, \mathbb{Z})$  denote the cohomology class dual to the cycle given by  $X_\lambda(F_\bullet)$ .

Let  $\ell_k(\lambda)$  be the number of parts  $\lambda_i$  which are strictly greater than  $k$ , and let  $\mathcal{Q}_{\text{IG}}$  and  $\mathcal{Q}_{\text{OG}}$  denote the universal quotient vector bundles over  $\text{IG}(n - k, 2n)$  and  $\text{OG}(n - k, 2n + 1)$ , respectively. It is known (see e.g. [BS, §3.1]) that the map which sends  $\sigma_p = c_p(\mathcal{Q}_{\text{IG}})$  to  $c_p(\mathcal{Q}_{\text{OG}})$  for all  $p$  extends to a ring isomorphism  $\varphi : H^*(\text{IG}, \mathbb{Q}) \rightarrow H^*(\text{OG}, \mathbb{Q})$  such that  $\varphi(\sigma_\lambda) = 2^{\ell_k(\lambda)} \tau_\lambda$  for all  $\lambda \in \mathcal{P}(k, n)$ .

We let  $c_p = c_p(\mathcal{Q}_{\text{OG}})$ . The *special Schubert classes* on OG are related to the Chern classes  $c_p$  by the equations

$$c_p = \begin{cases} \tau_p & \text{if } p \leq k, \\ 2\tau_p & \text{if } p > k. \end{cases}$$

For any integer sequence  $\alpha$ , set  $m_\alpha = \prod_i c_{\alpha_i}$ . Then for every  $\lambda \in \mathcal{P}(k, n)$ , the classical Giambelli formula

$$\tau_\lambda = 2^{-\ell_k(\lambda)} R^\lambda m_\lambda$$

holds in  $H^*(\text{OG}, \mathbb{Z})$ .

**3.2. From classical to quantum Giambelli.** Suppose  $k \geq 1$ . The quantum cohomology ring  $\text{QH}(\text{OG}(n - k, 2n + 1))$  is defined similarly to that of IG, but the degree of  $q$  here is  $n + k$ . More notation is required to state the quantum Pieri rule for OG. For each  $\lambda$  and  $\mu$  with  $\lambda \rightarrow \mu$ , we define  $N'(\lambda, \mu)$  to be equal to the number (respectively, one less than the number) of connected components of  $\mathbb{A}$ , if  $p \leq k$  (respectively, if  $p > k$ ). Let  $\mathcal{P}'(k, n + 1)$  be the set of  $\nu \in \mathcal{P}(k, n + 1)$  for which  $\ell(\nu) = n + 1 - k$ ,  $2k \leq \nu_1 \leq n + k$ , and the number of boxes in the second column of  $\nu$  is at most  $\nu_1 - 2k + 1$ . For any  $\nu \in \mathcal{P}'(k, n + 1)$ , we let  $\tilde{\nu} \in \mathcal{P}(k, n)$  be the partition obtained by removing the first row of  $\nu$  as well as  $n + k - \nu_1$  boxes from the first column. That is,

$$\tilde{\nu} = (\nu_2, \nu_3, \dots, \nu_r), \quad \text{where } r = \nu_1 - 2k + 1.$$

According to [BKT2, Thm. 2.4], for any  $k$ -strict partition  $\lambda \in \mathcal{P}(k, n)$  and integer  $p \in [1, n + k]$ , the following quantum Pieri rule holds in  $\text{QH}(\text{OG}(n - k, 2n + 1))$ .

$$(14) \quad \tau_p \cdot \tau_\lambda = \sum_{\lambda \rightarrow \mu} 2^{N'(\lambda, \mu)} \tau_\mu + \sum_{\lambda \rightarrow \nu} 2^{N'(\lambda, \nu)} \tau_{\tilde{\nu}} q + \sum_{\lambda^* \rightarrow \rho} 2^{N'(\lambda^*, \rho)} \tau_\rho q^2.$$

Here the first sum is classical, the second sum is over  $\nu \in \mathcal{P}'(k, n + 1)$  with  $\lambda \rightarrow \nu$  and  $|\nu| = |\lambda| + p$ , and the third sum is empty unless  $\lambda_1 = n + k$ , and over  $\rho \in \mathcal{P}(k, n)$  such that  $\rho_1 = n + k$ ,  $\lambda^* \rightarrow \rho$ , and  $|\rho| = |\lambda| - n - k + p$ .



Let  $\delta_p = 1$ , if  $p \leq k$ , and  $\delta_p = 2$ , otherwise. The stable cohomology ring  $\mathbb{H}(\text{OG}_k)$  has a free  $\mathbb{Z}$ -basis of Schubert classes  $\tau_\lambda$  for  $k$ -strict partitions  $\lambda$ , and is presented as a quotient of the polynomial ring  $\mathbb{Z}[\tau_1, \tau_2, \dots]$  modulo the relations

$$(15) \quad \tau_r^2 + 2 \sum_{i=1}^r (-1)^i \delta_{r-i} \tau_{r+i} \tau_{r-i} = 0 \quad \text{for } r > k.$$

**Proposition 5.** *There exists a unique ring homomorphism*

$$\tilde{\pi} : \mathbb{H}(\text{OG}_k) \rightarrow \text{QH}(\text{OG}(n-k, 2n+1))$$

such that the following relations are satisfied:

$$\tilde{\pi}(\tau_i) = \begin{cases} \tau_i & \text{if } 1 \leq i \leq n+k, \\ 0 & \text{if } n+k < i < 2n+2k, \\ 0 & \text{if } i \text{ is odd and } i > 2n+2k. \end{cases}$$

Furthermore, we have  $\tilde{\pi}(\tau_\lambda) = \tau_\lambda$  for each  $\lambda \in \mathcal{P}(k, n)$ .

*Proof.* The relations (15) for  $r \geq n+k$  uniquely specify the values  $\tilde{\pi}(\tau_i)$  for even integers  $i \geq 2n+2k$ . We must show that the remaining relations for  $k < r < n+k$  are mapped to zero by  $\tilde{\pi}$ . Observe that when  $k < n-1$  the individual terms in these relations carry no  $q$  correction. Indeed, we are applying the quantum Pieri rule (14) to partitions of length one, hence the  $q$  term vanishes (since  $1 < n-k$ ) and the  $q^2$  term vanishes (since  $\deg(q^2) = 2n+2k$ ). It remains only to consider the case  $k = n-1$ , which uses the quantum Pieri rule for the quadric  $\text{OG}(1, 2n+1)$ . The computation is then done as in [BKT2, Thm. 2.5] (which treats the case  $r = n$ ), and involves computing the coefficient  $c$  of  $q \tau_{2(r-n)+1}$  in the corresponding expression. As in loc. cit., the result is  $c = 1 - 2 + 2 - \dots \pm 2 \mp 1$  when  $r \leq (3n-2)/2$ , and otherwise  $c = 2 - 4 + 4 - \dots \pm 4 \mp 2$ ; hence  $c = 0$  in both cases.

To prove that  $\tilde{\pi}(\tau_\lambda) = \tau_\lambda$  for every  $\lambda \in \mathcal{P}(k, n)$ , we use an orthogonal analogue of Proposition 3, which follows from the isomorphism  $\mathbb{H}(\text{OG}_k) \otimes \mathbb{Q} \cong \mathbb{H}(\text{IG}_k) \otimes \mathbb{Q}$ . Arguing by induction on  $\ell(\lambda)$  as in Proposition 4, we obtain that

$$(16) \quad \tilde{\pi}(\tau_\lambda) = \sum_{p=\lambda_1}^{n+k} \sum_{\mu \subset \lambda^*} a'_{p,\mu} \tau_p \tau_\mu$$

holds in  $\text{QH}(\text{OG}(n-k, 2n+1)) \otimes \mathbb{Q}$ , where  $a'_{p,\mu} \in \mathbb{Q}$ . The quantum Pieri rule (14) implies that any product  $\tau_p \tau_\mu$  in (16) carries no  $q$  correction terms. It follows that the right hand side of (16) evaluates to  $\tau_\lambda$ .  $\square$

**Theorem 2** (Quantum Giambelli for OG). *For every  $\lambda \in \mathcal{P}(k, n)$ , we have*

$$\tau_\lambda = 2^{-\ell_k(\lambda)} R^\lambda m_\lambda$$

in the quantum cohomology ring  $\text{QH}(\text{OG}(n-k, 2n+1))$ . In other words, the quantum Giambelli formula for OG is the same as the classical Giambelli formula.

*Proof.* This follows from Proposition 5 and the orthogonal version of Corollary 1.  $\square$

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