# LITTLEWOOD-RICHARDSON RULES FOR GRASSMANNIANS

ANDERS SKOVSTED BUCH, ANDREW KRESCH, AND HARRY TAMVAKIS

## 1. Introduction

The classical Littlewood-Richardson rule [LR] describes the structure constants obtained when the cup product of two Schubert classes in the cohomology ring of a complex Grassmannian is written as a linear combination of Schubert classes. It also gives a rule for decomposing the tensor product of two irreducible polynomial representations of the general linear group into irreducibles, or equivalently, for expanding the product of two Schur $S$-functions in the basis of Schur $S$-functions. In this paper we give a short and self-contained argument which shows that this rule is a direct consequence of Pieri's formula [P] for the product of a Schubert class with a special Schubert class.

There is an analogous Littlewood-Richardson rule for the Grassmannians which parametrize maximal isotropic subspaces of $\mathbb{C}^n$, equipped with a symplectic or orthogonal form. The precise formulation of this rule is due to Stembridge [St], working in the context of Schur's $Q$-functions [S]; the connection to geometry was shown by Hiller and Boe [HB] and Pragacz [Pr]. The argument here for the type $A$ rule works equally well in these more difficult cases and gives a simple derivation of Stembridge's rule from the Pieri formula of [HB].

Currently there are many proofs available for the classical Littlewood-Richardson rule, some of them quite short. The proof of Remmel and Shimozono [RS] is also based on the Pieri rule; see the recent survey of van Leeuwen [vL] for alternatives. In contrast, we know of only two prior approaches to Stembridge's rule (described in [St, HH] and [Sh], respectively), both of which are rather involved.

The argument presented here proceeds by defining an abelian group $\mathbb{H}$ with a basis of Schubert symbols, and a bilinear product on $\mathbb{H}$ with structure constants coming from the Littlewood-Richardson rule in each case. Since this rule is compatible with the Pieri products, it suffices to show that $\mathbb{H}$ is an *associative* algebra. The proof of associativity is based on Schützenberger slides in type $A$, and uses the more general slides for marked shifted tableaux due to Worley [W] and Sagan [Sa] in the other Lie types. In each case, we need only basic properties of these operations which are easily verified from the definitions. Our paper is self-contained, once the Pieri rules are granted.

The work on this article was completed during a fruitful visit to the Mathematisches Forschungsinstitut Oberwolfach, as part of the Research in Pairs program. It is a pleasure to thank the Institut for its hospitality and stimulating atmosphere.

---

*Date*: June 22, 2003.

2000 *Mathematics Subject Classification.* 05E15; 14M15.

The authors were supported in part by NSF Grant DMS-0070479 (Buch), an NSF Postdoctoral Research Fellowship (Kresch), and NSF Grant DMS-0296023 (Tamvakis).

We also thank the referee for a careful reading of our paper and for some well-placed final touches to the exposition.

## 2. The Littlewood-Richardson rule for type $A$ Grassmannians

Let $X = G(k,n)$ be the Grassmannian of $k$-dimensional linear subspaces of $\mathbb{C}^n$ and set $m = n-k$. For each partition $\lambda$ whose Young diagram is contained in the $k\times m$ rectangle $(m^k)$, there is a *Schubert class* $\sigma_\lambda$ in the cohomology ring $H^*(X,\mathbb{Z})$. If a partition $\widetilde{\lambda} \subset (m^k)$ can be obtained from $\lambda$ by adding a horizontal strip with $p$ boxes, then we write $\lambda \xrightarrow{p} \widetilde{\lambda}$. The Pieri rule [P] states that for each $p \leqslant m$, $\sigma_p \cdot \sigma_\lambda$ is equal to the sum of all $\sigma_{\widetilde{\lambda}}$ for which $\lambda \xrightarrow{p} \widetilde{\lambda}$.

In this section, we will prove that the Littlewood-Richardson rule holds in the ring $H^*X = H^*(X,\mathbb{Z})$. We note however that the argument requires only two facts about this ring: (i) the classes $\sigma_\lambda$ for $\lambda \subset (m^k)$ form a basis of $H^*X$, and (ii) the Pieri rule holds in $H^*X$. An easy induction shows that the special Schubert classes $\sigma_p$ for $1 \leqslant p \leqslant m$ generate the entire ring $H^*X$. This also follows from the Giambelli formula, which is a direct consequence of Pieri's rule. Let $\lambda^\vee = (m-\lambda_k,\ldots,m-\lambda_1)$ denote the dual partition of $\lambda$.

A *tableau* $T$ of skew shape $\lambda/\mu$ is a filling of the boxes of $\lambda/\mu$ with positive integers such that the entries are weakly increasing along each row and strictly increasing down each column. The *content* of $T$ is the sequence whose $i$th element is the number of boxes of $T$ containing $i$. The *word* $\mathbf{w} = \mathbf{w}(T)$ of $T$ is the sequence obtained by reading the entries of $T$ going from right to left in successive rows, starting with the top row. We say that $\mathbf{w} = w_1 \ldots w_r$ is a *lattice word* and that $T$ is a *Littlewood-Richardson tableau* (or *LR tableau*) if the number of occurrences of $i$ among $w_1 \ldots w_j$ is not less than the number of occurrences of $i+1$, for all $i$ and $j$ with $1 \leqslant j \leqslant r$.

Given three partitions $\lambda,\mu,\nu \subset (m^k)$, define $c(\lambda,\mu;\nu)$ to be the number of LR tableaux of shape $\lambda^\vee/\mu$ with content $\nu^\vee$. (If $\mu$ is not contained in $\lambda^\vee$ then we set $c(\lambda,\mu;\nu) = 0$.)

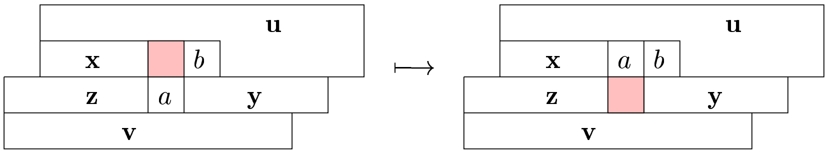
**Proposition 1.** *For any three partitions $\lambda,\mu,\nu \subset (m^k)$ and integer $p \leqslant m$, we have*

$$\sum_{\lambda \xrightarrow{p} \widetilde{\lambda}} c(\widetilde{\lambda},\mu;\nu) = \sum_{\mu \xrightarrow{p} \widetilde{\mu}} c(\lambda,\widetilde{\mu};\nu)\,. \tag{1}$$

*Proof.* We assume here familiarity with Schützenberger's jeu de taquin (explained e.g. in [F, §1.2]). Given a skew tableau $T$ and an empty box which is an inner corner of $T$, we may perform Schützenberger slides to obtain a new skew tableaux $T'$; the empty box slides to an outer corner of $T$.

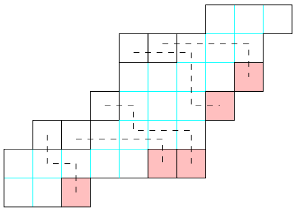
**Fact 1.** *$T$ is an LR tableau if and only if $T'$ is an LR tableau.*

This follows immediately from the definitions; alternatively, it is a consequence of the well-known fact that plactic relations on words preserve the lattice property. For the direct implication, it suffices to consider a single vertical slide as displayed below. In the figure, the symbols $\mathbf{u}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{v}$ denote the words of their respective subsets in the tableau. In particular, they are read from right to left.



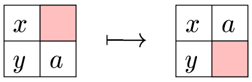

We must check that the word $\mathbf{u}ba\mathbf{xyzv}$ of the resulting tableau is a lattice word. This is true because $\mathbf{u}b\mathbf{xy}a\mathbf{zv}$ is a lattice word, and the tableau inequalities imply that there are at least as many $a$'s in the word $\mathbf{z}$ as there are $(a-1)$'s in $\mathbf{x}$. A similar argument shows that reverse slides also preserve the lattice property.

We shall call an empty box contained inside the skew shape $\lambda^\vee/\mu$ a *hole*. Given an LR tableau on a shape $\lambda^\vee/\widetilde{\mu}$ such that $\mu \xrightarrow{p} \widetilde{\mu}$, we can use Schützenberger slides starting from the holes contained in $\widetilde{\mu}/\mu$, in right to left order, to obtain another LR tableau of some shape $\widetilde{\lambda}^\vee/\mu$. Define the *sliding path* of each such hole to be the set of boxes it occupies during the sliding process.



**Fact 2.** *Two distinct sliding paths cannot cross each other.*

More precisely, if a hole is at a given position during its slide, then the boxes in any subsequent sliding path must all lie strictly left or weakly below that position. For otherwise, at some point a hole will slide right to occupy the position vacated by a vertical slide in the previous sliding path. Depicting the vertical slide as




we must have $y \leqslant a$, and hence a subsequent hole, having arrived at position $x$, will slide down to position $y$. Since different sliding paths cannot cross each other, it follows that $\lambda \xrightarrow{p} \widetilde{\lambda}$. Furthermore the entire process can be inverted using reverse slides. This gives a bijective proof of identity (1). $\square$

The following theorem is one out of many equivalent statements of the classical Littlewood-Richardson rule.

**Theorem 1.** *The constant $c(\lambda, \mu; \nu)$ is the coefficient of $\sigma_\nu$ in the product $\sigma_\lambda \cdot \sigma_\mu$.*

*Proof.* Let $\mathbb{H}$ be the free abelian group generated by symbols $s_\lambda$ for all partitions $\lambda \subset (m^k)$. We define a bilinear operator "$\circ$" on $\mathbb{H}$ by

$$s_\lambda \circ s_\mu = \sum_\nu c(\lambda, \mu; \nu)\, s_\nu \,.$$

The operator $\circ$ is, a priori, neither commutative nor associative.

It is easy to see that there is a unique LR tableau of shape $\lambda^\vee$ and a unique LR tableau of shape $(m^k)/\mu$, and that these tableaux have contents $\lambda^\vee$ and $\mu^\vee$, respectively. It follows that $s_\emptyset$ acts as a left and right identity in $\mathbb{H}$. By taking $\lambda = \emptyset$ in Proposition 1 we deduce that $s_p \circ s_\mu = \sum s_{\widetilde{\mu}}$ where the sum is over $\mu \xrightarrow{p} \widetilde{\mu}$. Similarly one obtains $s_\lambda \circ s_p = \sum s_{\widetilde{\lambda}}$ by setting $\mu = \emptyset$; in other words, the operator $\circ$ satisfies the Pieri rule.

Equation (1) is therefore equivalent to the associativity relation $(s_\lambda \circ s_p) \circ s_\mu = s_\lambda \circ (s_p \circ s_\mu)$. It follows that the elements $s_p$ for $1 \leqslant p \leqslant m$ generate an associative subalgebra of $\mathbb{H}$. Using the same Pieri induction as before, one sees that this subalgebra is the entire algebra $\mathbb{H}$. We conclude that the linear map $H^*X \to \mathbb{H}$ given by $\sigma_\lambda \mapsto s_\lambda$ is an isomorphism of (associative) rings. □

**Remark. 1)** In its usual formulation, the Littlewood-Richardson rule states that the coefficient $c(\lambda, \mu; \nu)$ is equal to the number of LR tableaux of shape $\nu/\lambda$ with content $\mu$. To see this, note that the identity $c(\lambda, \mu; (m^k)) = \delta_{\lambda, \mu^\vee}$ holds by definition (this corresponds to Poincaré duality in $H^*X$). It follows that

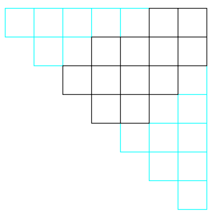
$$c(\lambda, \mu; \nu)\sigma_{(m^k)} = \sigma_{\nu^\vee}(\sigma_\lambda \sigma_\mu) = (\sigma_{\nu^\vee}\sigma_\lambda)\sigma_\mu = c(\nu^\vee, \lambda; \mu^\vee)\sigma_{(m^k)}$$

and hence $c(\lambda, \mu; \nu) = c(\nu^\vee, \lambda; \mu^\vee)$, as required. Alternatively, a bijective proof of this equality may be obtained using [F, Prop. 5.1.2].

**2)** The above argument may be applied to derive other forms of the Littlewood-Richardson rule. For example, it gives a short proof of the puzzle rule of Knutson, Tao and Woodward [KTW]. In the language of puzzles, Schützenberger slides correspond to a subset of the propagations described in [KT] (those which involve only non-equivariant puzzle pieces).

## 3. The Littlewood-Richardson-Stembridge rule for maximal isotropic Grassmannians

The odd orthogonal Grassmannian $Y = OG(n, 2n+1)$ parametrizes $n$-dimensional isotropic linear subspaces of $\mathbb{C}^{2n+1}$ with respect to a nondegenerate orthogonal form. The cohomology ring $H^*(Y, \mathbb{Z})$ has a basis of Schubert classes $\tau_\lambda$, indexed by strict partitions $\lambda$ (i.e. with distinct parts) such that $\lambda \subset \rho_n$, where $\rho_n = (n, n-1, \ldots, 1)$. For each strict $\lambda \subset \rho_n$, define $\lambda^\vee \subset \rho_n$ as the strict partition whose parts complement the parts of $\lambda$ in the set $\{1, \ldots, n\}$. The *shifted diagram* $\mathcal{S}(\lambda)$ is obtained from the Young diagram of $\lambda$ by indenting the $i$th row by $i-1$ columns, for each $i \geqslant 1$. For skew diagrams we set $\mathcal{S}(\lambda/\mu) = \mathcal{S}(\lambda) \smallsetminus \mathcal{S}(\mu)$. For example, if $n = 7$, $\lambda = (5,3,1)$, and $\mu = (5,2)$ then $\mathcal{S}(\lambda^\vee/\mu)$ is the diagram:



Recall that a *border strip* is an edge-connected skew diagram that contains no $2 \times 2$ block of squares. As before, we write $\lambda \xrightarrow{p} \widetilde{\lambda}$ if the partition $\widetilde{\lambda} \subset \rho_n$ can be obtained from $\lambda$ by adding a horizontal strip of length $p$. In this case, the shifted skew diagram $\mathcal{S}(\widetilde{\lambda}/\lambda)$ is a union of border strips. The Pieri rule for $OG(n, 2n+1)$, due to Hiller and Boe [HB], states that

$$\tau_p \cdot \tau_\lambda = \sum 2^{N(\widetilde{\lambda}/\lambda)}\, \tau_{\widetilde{\lambda}}, \tag{2}$$

where the sum is over strict $\widetilde{\lambda} \subset \rho_n$ with $\lambda \xrightarrow{p} \widetilde{\lambda}$, and $N(\widetilde{\lambda}/\lambda)$ is one less than the number of border strip components of $\mathcal{S}(\widetilde{\lambda}/\lambda)$. The Pieri rule implies that the special Schubert classes $\tau_p$ for $1 \leqslant p \leqslant n$ generate $H^*(Y, \mathbb{Z})$.

Let $A$ be the ordered alphabet $1' < 1 < 2' < 2 < \cdots$; the symbols $1', 2', \ldots$ are said to be *marked*. A *shifted tableau* $T$ on the shifted skew shape $\mathcal{S}(\lambda/\mu)$ is a filling of the boxes of $\mathcal{S}(\lambda/\mu)$ with symbols from $A$ such that (i) the entries are weakly increasing along each row and down each column, and (ii) each row contains at most one $i'$ and each column contains at most one $i$, for every integer $i \geqslant 1$. The content of $T$ is the partition whose $i$th part is the number of boxes with entry $i$ or $i'$ in $T$, while the word $\mathbf{w} = \mathbf{w}(T)$ of $T$ is defined as in Section 2.

For any integer $i$ we set $\widehat{i'} = i$ and $\widehat{i} = (i{+}1)'$. If $\mathbf{w} = w_1 w_2 \ldots w_p$ is a word of marked and unmarked integers $w_j$, then we write $\widehat{\mathbf{w}} = \widehat{w}_p \ldots \widehat{w}_2 \widehat{w}_1$. We say that $\mathbf{w}$ is an *LRS word* if (i) $\mathbf{w}\widehat{\mathbf{w}}$ is a *lattice word*, i.e. every $i$ or $i'$ in $\mathbf{w}\widehat{\mathbf{w}}$ is preceded by more occurrences of $i-1$ than of $i$, for all $i$, and (ii) the last occurrence of $i'$ in $\mathbf{w}$ (if any) is followed by at least one $i$, for all $i \geqslant 1$. A tableau $T$ is a *Littlewood-Richardson-Stembridge tableau* (or *LRS tableau*) if $\mathbf{w}(T)$ is an LRS word.

Given three strict partitions $\lambda, \mu, \nu \subset \rho_n$, define $f(\lambda, \mu; \nu)$ to be the number of LRS tableaux of shape $\mathcal{S}(\lambda^\vee/\mu)$ with content $\nu^\vee$. (If $\mu$ is not contained in $\lambda^\vee$ then we set $f(\lambda, \mu; \nu) = 0$.) For example, if $n = 7$ we have $f((5{,}3{,}1), (5{,}2); (6{,}5{,}4{,}1)) = 4$ as counted by the following list of LRS tableaux:

$$\begin{array}{ccccc} & & & 1' & 1 \\ & 1' & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 & \\ & 3 & 3 & & \end{array} \qquad \begin{array}{ccccc} & & & 1' & 1 \\ & 1' & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 & \\ & 2 & 3 & & \end{array} \qquad \begin{array}{ccccc} & & & 1' & 1 \\ & 1' & 1 & 1 & 2 \\ 1 & 1 & 2 & 3' & \\ & 2 & 3 & & \end{array} \qquad \begin{array}{ccccc} & & & 1' & 1 \\ & 1' & 1 & 1 & 2 \\ 1 & 1 & 2' & 3' & \\ & 2 & 3 & & \end{array}$$

**Theorem 2.** *The constant $f(\lambda, \mu; \nu)$ is the coefficient of $\tau_\nu$ in the product $\tau_\lambda \cdot \tau_\mu$.*

Using the same argument as in the proof of Theorem 1, Theorem 2 follows from the Pieri rule (2) and the next proposition, which comes from the associativity relation in $H^*(Y, \mathbb{Z})$.

**Proposition 2.** *For any three strict partitions $\lambda, \mu, \nu \subset \rho_n$ and integer $p \leqslant n$, we have*

$$\sum_{\lambda \xrightarrow{p} \widetilde{\lambda}} 2^{N(\widetilde{\lambda}/\lambda)} f(\widetilde{\lambda}, \mu; \nu) = \sum_{\mu \xrightarrow{p} \widetilde{\mu}} 2^{N(\widetilde{\mu}/\mu)} f(\lambda, \widetilde{\mu}; \nu)\,. \tag{3}$$

The proof of Proposition 2 occupies the remainder of this section. Define the *main diagonal* $\Delta$ to be the set of squares along the southwest border of $\mathcal{S}(\rho_n)$. We will apply the shifted analogue of Schützenberger's sliding operation, constructed by Worley [W] and Sagan [Sa], to LRS tableaux. This involves the usual sliding moves which refer to the alphabet $A$, with the exception of the horizontal slide in case (a) below, when a different rule applies. In addition, there is a *special slide* in case (b), which is used only when the empty box is on the diagonal $\Delta$.

$$(a) \quad \begin{array}{|c|c|} \hline \phantom{i} & i' \\ \hline i' & x \\ \hline \end{array} \longmapsto \begin{array}{|c|c|} \hline i' & \phantom{i} \\ \hline i' & x \\ \hline \end{array} \qquad\qquad (b) \quad \begin{array}{|c|c|} \hline \phantom{i} & i' \\ \hline \multicolumn{1}{c|}{} & i \\ \cline{2-2} \end{array} \longmapsto \begin{array}{|c|c|} \hline i & i \\ \hline \multicolumn{1}{c|}{} & \phantom{i} \\ \cline{2-2} \end{array}$$

These operations are invertible using the obvious reverse slides.

Suppose that we are given an LRS tableau $T$ and an empty box at an inner corner of $T$, and let $T'$ be the result of performing a shifted sliding operation to $T$. The next lemma is parallel to Fact 1, and follows from the fact that the shifted analogues of the plactic relations preserve the Littlewood-Richardson-Stembridge property (see [W, Sa, St] for details.) We give a direct proof here.

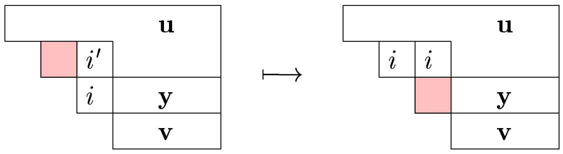
**Lemma 1.** *$T$ is an LRS tableau if and only if $T'$ is an LRS tableau.*

*Proof.* For any $a$ in the alphabet $A$, let $N_a(\mathbf{w})$ denote the number of occurrences of $a$ in $\mathbf{w}$. It follows immediately from the definitions that for any LRS word $\mathbf{w}$,

$$N_i(\mathbf{w}) > N_{i+1}(\mathbf{w}), \quad \text{for each unmarked } i \in A. \tag{4}$$

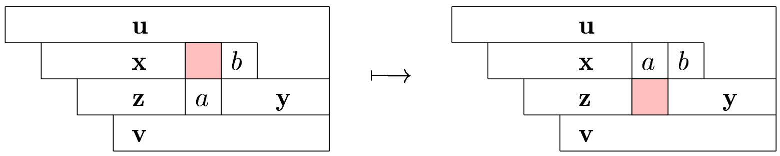
Since horizontal slides do not change the word of a tableau, we need only consider special and vertical slides. Observe that in either case, condition (ii) in the definition of an LRS tableau is easily verified; hence we concentrate on condition (i).

We start with a special slide as displayed below.



We must show that if $\mathbf{w}_1 = \mathbf{u}i'\mathbf{y}i\mathbf{v}$ is an LRS word, then so is $\mathbf{w}_2 = \mathbf{u}ii\mathbf{y}\mathbf{v}$. Using (4) we see that $N_i(\mathbf{u}) + 1 \leqslant N_i(\mathbf{w}_1) < N_{i-1}(\mathbf{w}_1) = N_{i-1}(\mathbf{u})$. Since $i', i \notin \mathbf{y}\mathbf{v}\widehat{\mathbf{v}}\widehat{\mathbf{y}}$ this implies that every $i'$ and $i$ in the word $\mathbf{w}_2\widehat{\mathbf{w}}_2$ is preceded by more occurrences of $i-1$ than of $i$. Furthermore, since $N_i(\mathbf{w}_1\widehat{\mathbf{v}}(i+1)'\widehat{\mathbf{y}}) \geqslant N_{i+1}(\mathbf{w}_1\widehat{\mathbf{v}}(i+1)'\widehat{\mathbf{y}})$ it also follows that every $(i+1)'$ and $i+1$ in $\mathbf{w}_2\widehat{\mathbf{w}}_2$ is preceded by more occurrences of $i$ than of $i+1$. All other symbols are not affected by the slide.

Next, consider a vertical slide. In the figure, $a$ and $b$ are symbols such that $a \leqslant b$ (if $b$ is marked then $a < b$).



We must show that if $\mathbf{w}_1 = \mathbf{u}b\mathbf{x}\mathbf{y}a\mathbf{z}\mathbf{v}$ is an LRS word then so is $\mathbf{w}_2 = \mathbf{u}ba\mathbf{x}\mathbf{y}\mathbf{z}\mathbf{v}$. Assume first that $a = i$ is unmarked. To see that every $i'$ and $i$ in the new word $\mathbf{w}_2\widehat{\mathbf{w}}_2$ is preceded by more occurrences of $i-1$ than of $i$, we must show that $N_i(\mathbf{u}b) + N_{i'}(\mathbf{x}) < N_{i-1}(\mathbf{u}b)$. If $N_i(\mathbf{z}) + N_{i'}(\mathbf{z}) \geqslant N_{i-1}(\mathbf{x}) + N_{i'}(\mathbf{x})$ then this follows from the LRS condition $N_i(\mathbf{u}b\mathbf{x}\mathbf{y}i\mathbf{z}) + N_{i'}(\mathbf{z}) \leqslant N_{i-1}(\mathbf{u}b\mathbf{x})$. Otherwise $N_i(\mathbf{z}) + N_{i'}(\mathbf{z}) < N_{i-1}(\mathbf{x}) + N_{i'}(\mathbf{x})$ which can only happen when $\mathbf{z}$ is a string of copies of $i$ terminating at the diagonal $\Delta$, in which case we have $N_i(\mathbf{z}) = N_{i-1}(\mathbf{x}) + N_{i'}(\mathbf{x}) - 1$ and $i-1 \notin \mathbf{z}\mathbf{v}$. The word $\mathbf{z}$ here cannot contain $i'$ because of condition (ii) in the definition of an LRS tableau. Using (4) we get

$$N_i(\mathbf{u}b) + 1 + N_i(\mathbf{z}) \leqslant N_i(\mathbf{w}_1) < N_{i-1}(\mathbf{w}_1) = N_{i-1}(\mathbf{u}b) + N_{i-1}(\mathbf{x})$$

which also implies the required inequality.

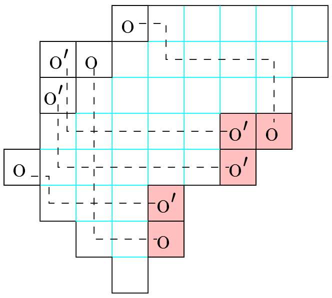
Since $\widehat{a} = (i+1)'$, we also must check that the string $\mathbf{w}_2\widehat{\mathbf{v}}\widehat{\mathbf{z}}\widehat{\mathbf{y}}\widehat{\mathbf{x}}$ contains more occurrences of $i$ than of $i+1$. The only way this can fail is if $\widehat{\mathbf{y}}\widehat{\mathbf{x}}$ contains an $i+1$, i.e. if $(i{+}1)' \in \mathbf{xy}$. Now all symbols in $\mathbf{x}$ are less than $i$, so $(i{+}1)' \notin \mathbf{x}$. If $(i{+}1)' \in \mathbf{y}$ then $\widehat{b} = (i{+}1)'$ or $\widehat{b} = i{+}1$, so the lattice property of the original word $\mathbf{w}_1\widehat{\mathbf{w}}_1$ implies the desired one.

Now suppose that $a = i'$ is marked. To see that the displaced $i'$ is not a problem, we must verify that $N_i(\mathbf{u}b) < N_{i-1}(\mathbf{u})$. Since $i \notin \mathbf{xy}$ and $i{-}1 \notin \mathbf{xy}$, this follows from the LRS property of the original word. We also need to check that all symbols $(i{+}1)'$ and $i{+}1$ in $\mathbf{w}_2\widehat{\mathbf{w}}_2$ are preceded by enough occurrences of $i$. This can only fail if $\widehat{\mathbf{y}}\widehat{\mathbf{x}}$ contains $(i{+}1)'$ or $i{+}1$, i.e., if $\mathbf{xy}$ contains $i$ or $(i{+}1)'$. These symbols cannot be in $\mathbf{x}$ since all symbols in $\mathbf{x}$ are less than $i'$. The only symbol among the two that can be in $\mathbf{y}$ is $(i{+}1)'$, and this can only occur once in $\mathbf{y}$. Furthermore, we must have $\widehat{b} = (i{+}1)'$ or $\widehat{b} = i{+}1$. Since $i \notin \widehat{\mathbf{y}}\widehat{\mathbf{x}}$ and $i{+}1 \in \widehat{\mathbf{y}}$, we deduce that $\mathbf{w}_1\widehat{\mathbf{v}}\widehat{\mathbf{z}}$ contains more occurrences of $i$ than of $i{+}1$, as required.

By inverting these arguments, one can show that reverse slides also send LRS tableaux to LRS tableaux. The details are left to the reader. $\square$

As in the proof of Proposition 1, we shall call an empty box contained inside the skew shape $\mathcal{S}(\lambda^\vee/\mu)$ a *hole*, but we will need to distinguish between two kinds of holes. For this purpose, we extend the ordered alphabet $A$ to $\widetilde{A} = A \cup \{\mathrm{o}', \mathrm{o}\}$, where $\mathrm{o}' < \mathrm{o}$ and the new symbols represent a marked and an unmarked hole. Define a *NW-holed tableau* (respectively, a *SE-holed tableau*) to be a filling of a shifted shape $\mathcal{S}(\lambda^\vee/\mu)$ with symbols from $\widetilde{A}$ so that the entries in $A$ satisfy the usual conditions and the holes form a shifted horizontal strip $L$ along its northwest (respectively, southeast) border, such that $\mathbf{w}(L)$ is an LRS word. This means that the holes in a NW-holed tableau occupy a skew shape $\mathcal{S}(\widetilde{\mu}/\mu)$ for which $\mu \xrightarrow{p} \widetilde{\mu}$ so that any hole above another hole is marked, any hole to the right of another hole is unmarked, and the most southwest hole is unmarked; the conditions for a SE-holed tableau are similar.

The identity (3) is equivalent to the statement that there are equally many NW-holed and SE-holed LRS tableaux with content $\nu$ on the shape $\mathcal{S}(\lambda^\vee/\mu)$. We will use shifted slides to construct an explicit bijection between these two kinds of tableaux. Given a NW-holed LRS tableau, we first slide the unmarked holes to the south-east border, in right to left order, after which we slide the marked holes, proceeding from bottom to top. If the final position of an unmarked hole is in a row above the final position of the previous hole, then we change it to a marked hole. Marked holes always stay marked.




For the reverse bijection, we begin by sliding the marked holes in top to bottom order, followed by the unmarked holes in left to right order. If the path of a marked hole intersects the diagonal $\Delta$ then we erase its marking; the unmarked holes remain unmarked. To verify that these two transformations are inverse to each other, we must check that after all the holes have been slid by one of them, the other will slide them back in the opposite order.

Let $P$ be a set of boxes in the shifted diagram $\mathcal{S}(\lambda^\vee/\mu)$, and let $B$ be any box in this diagram. We say that $B$ lies *west of* $P$ if $P$ contains a box which is strictly east and weakly north of $B$. And we say that $B$ lies *north of* $P$ if $P$ contains a box which is strictly south and weakly west of $B$.

**Lemma 2.** *Consider the path of a hole $\mathrm{o}_2$ which slides directly after a hole $\mathrm{o}_1$.*

(a) *At any given step, if $\mathrm{o}_2$ lies west of the sliding path of $\mathrm{o}_1$, and $\mathrm{o}_2$ is not on $\Delta$, then at the next step $\mathrm{o}_2$ will remain west of the path of $\mathrm{o}_1$.*
(b) *At any given step, if $\mathrm{o}_2$ lies north of the sliding path of $\mathrm{o}_1$, then the same is true at the next step.*

*Proof.* Suppose the position of the hole $\mathrm{o}_2$ is as indicated in the figure.

$$\begin{array}{|c|c|}\hline \mathrm{o}_2 & b \\ \hline a & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

The only way (a) can fail is if $\mathrm{o}_1$ was in the position of $b$ and moved down from there. But then $a \leqslant b$ (and if $b$ is marked then $a < b$), so $\mathrm{o}_2$ will also move down. Notice that there must be a symbol from $A$ in the square occupied by $a$, because $\mathrm{o}_2$ is not on the diagonal $\Delta$.

The only way (b) can fail is if the first hole $\mathrm{o}_1$ was in the position of $a$ and moved east from there. But this means that $a \geqslant b$ (and if $b$ is unmarked then $a > b$), hence $\mathrm{o}_2$ will move east as well. This time there must be a symbol from $A$ in the square occupied by $b$. ∎

Consider the sequence of slides from northwest to southeast, beginning with the unmarked holes. If the path of an unmarked hole crosses the previous path, then by Lemma 2 (a) this must be at a corner, and Lemma 2 (b) then implies that the hole will remain north of the previous path from that point onwards. Since this creates a path which meets the diagonal $\Delta$, the next unmarked hole will be forced to cross it, and so on. The result is that all of the remaining unmarked holes will become marked and land in reverse order. After all the unmarked holes have been slid, Lemma 2 (b) will force every subsequent marked hole to stay north of the previous hole's path, thus all the marked holes retain their order. It follows that the reverse slides are performed in the opposite order, as required. Similar arguments can be used to show that reverse slides will deposit the holes along the northwest border in the opposite order. This completes the proof of Proposition 2.

**Example.** The following gives an example of the bijection:

$$\begin{array}{ccccc}  &  & \mathrm{o} & 1 & 1 \\  & 1' & 1 & 2 & 2 \\ \mathrm{o}' & 1' & 2' & 3' & 3 \\ \mathrm{o} & 1' & 2' & 3' &  \\  & 1 & 2 & 4' &  \\  &  & 3 & 4 &  \end{array} \mapsto \begin{array}{ccccc}  &  & 1 & 1 & 1 \\  & 1' & 2' & 2 & 2 \\ \mathrm{o}' & 1' & 2' & 3' & 3 \\ \mathrm{o} & 1' & 2 & 3' &  \\  & 1 & 3 & 4' &  \\  &  & 4 & \mathrm{o} &  \end{array} \mapsto \begin{array}{ccccc}  &  & 1 & 1 & 1 \\  & 1' & 2' & 2 & 2 \\ \mathrm{o}' & 1' & 2' & 3' & 3 \\ 1 & 1 & 2 & 3' &  \\  & 3 & 4' & \mathrm{o}' &  \\  &  & 4 & \mathrm{o} &  \end{array} \mapsto \begin{array}{ccccc}  &  & 1 & 1 & 1 \\  & 1' & 2' & 2 & 2 \\ 1' & 1 & 2' & 3' & 3 \\ 1 & 2 & 3' & \mathrm{o}' &  \\  & 3 & 4' & \mathrm{o}' &  \\  &  & 4 & \mathrm{o} &  \end{array}$$

**Remark.** Arguing as in Section 2, we can show that $f(\lambda, \mu; \nu)$ is equal to the number of LRS tableaux of shape $\mathcal{S}(\nu/\lambda)$ with content $\mu$, which is Stembridge's original statement of the rule. Note also that the even orthogonal Grassmannian $OG(n+1, 2n+2)$ is isomorphic to the odd orthogonal Grassmannian $OG(n, 2n+1)$, and the Schubert structure constants for these two spaces coincide. The Schubert classes on the Lagrangian Grassmannian $LG(n, 2n)$ are also indexed by strict partitions $\lambda$ contained in $\rho_n$, and the corresponding structure constants $e(\lambda, \mu; \nu)$ satisfy the identity $e(\lambda, \mu; \nu) = 2^{\ell(\lambda)+\ell(\mu)-\ell(\nu)} f(\lambda, \mu; \nu)$. This follows by comparing the Pieri formulas for these spaces; see [Pr] for more details. Therefore, the proof of the Littlewood-Richardson-Stembridge rule given here also covers the maximal isotropic Grassmannians in Lie types $C$ and $D$.

## References


[F] W. Fulton : *Young Tableaux*, L.M.S. Student Texts **35**, Cambridge Univ. Press, Cambridge, 1997.

[HB] H. Hiller and B. Boe : *Pieri formula for $SO_{2n+1}/U_n$ and $Sp_n/U_n$*, Adv. Math. **62** (1986), 49–67.

[HH] P. N. Hoffman and J. F. Humphreys : *Projective Representations of the Symmetric Groups; Q-Functions and Shifted Tableaux*, Oxford Univ. Press, New York, 1992.

[KT] A Knutson and T. Tao : *Puzzles and (equivariant) cohomology of Grassmannians*, to appear in Duke Math. J.

[KTW] A. Knutson, T. Tao, and C. Woodward : *The honeycomb model of $GL_n$ tensor products II: Puzzles determine facets of the Littlewood-Richardson cone*, to appear in J. Amer. Math. Soc.

[LR] D. E. Littlewood and A. R. Richardson : *Group characters and algebra*, Phil. Trans. R. Soc., A, **233** (1934), 99–141.

[P] M. Pieri : *Sul problema degli spazi secanti. Nota 1$^a$*, Rend. Ist. Lombardo (2) **26** (1893), 534–546.

[Pr] P. Pragacz : *Algebro-geometric applications of Schur $S$- and $Q$-polynomials*, Topics in invariant theory (Paris, 1989/1990), Lecture Notes in Math. **1478**, 130–191, Springer, Berlin, 1991.

[RS] J. B. Remmel and M. Shimozono: *A simple proof of the Littlewood-Richardson rule and applications*, Discrete Math. **193** no. 1-3 (1998), 257–266.

[Sa] B. Sagan : *Shifted tableaux, Schur Q-functions, and a conjecture of R. Stanley*, J. Combin. Theory Ser. A **45** (1987), 62–103.

[S] I. Schur : *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. reine angew. Math. **139** (1911), 155–250.

[Sh] M. Shimozono : *Multiplying Schur Q-functions*, J. Combin. Theory Ser. A **87** (1999), 198–232.

[St] J. R. Stembridge : *Shifted tableaux and the projective representations of symmetric groups*, Adv. Math. **74** (1989), 87–134.

[vL] M. A. A. van Leeuwen : *The Littlewood-Richardson rule, and related combinatorics*, in 'Interactions of Combinatorics and Representation Theory', MSJ Memoirs **11**, 95–145, Math. Soc. Japan, Tokyo, 2001.

[W] D. R. Worley : *A theory of shifted Young tableaux*, Ph. D. Thesis, M.I.T., 1984.



Matematisk Institut, Aarhus Universitet, Ny Munkegade, 8000 Århus C, Denmark
*E-mail address*: `abuch@imf.au.dk`

Department of Mathematics, University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104-6395, USA
*E-mail address*: `kresch@math.upenn.edu`

Department of Mathematics, Brandeis University - MS 050, P. O. Box 9110, Waltham, MA 02454-9110, USA
*E-mail address*: `harryt@brandeis.edu`