K-THEORY OF MINUSCULE VARIETIES

ANDERS SKOVSTED BUCH AND MATTHEW J. SAMUEL

Abstract. Based on Thomas and Yong's K-theoretic jeu de taquin algorithm, we prove a uniform Littlewood-Richardson rule for the K-theoretic Schubert structure constants of all minuscule homogeneous spaces. Our formula is new in all types. For the main examples of Grassmannians of type A and maximal orthogonal Grassmannians it has the advantage that the tableaux to be counted can be recognized without reference to the jeu de taquin algorithm.

1. Introduction

The goal of this paper is to prove a uniform Littlewood-Richardson rule for the K-theoretic Schubert structure constants of any minuscule homogeneous space, also called a *minuscule variety*. The family of minuscule varieties includes Grassmann varieties of type A, maximal orthogonal Grassmannians, even dimensional quadric hypersurfaces, as well as two exceptional varieties called the Cayley plane and the Freudenthal variety. The slightly larger family of *cominuscule varieties* also includes Lagrangian Grassmannians and odd dimensional quadrics. Several papers illustrate that many aspects of the geometry and combinatorics of (co)minuscule varieties are natural generalizations of Grassmannians of type A [27, 31, 24, 33, 8].

A result of Proctor [27] implies that the Schubert varieties X_λ in a cominuscule variety $X = G/P$ can be indexed by order ideals λ in a partially ordered set Λ_X. These order ideals can be identified with shapes that generalize Young diagrams. The Schubert structure sheaves $\mathcal{O}_\lambda := [\mathcal{O}_{X_\lambda}]$ form a $\mathbb{Z}$-basis of the Grothendieck ring $K(X)$ of algebraic vector bundles on X. Brion has proved that the structure constants of $K(X)$ with respect to this basis have signs that alternate with codimension [2]. Equivalently, there are unique non-negative integers $c^\nu_{\lambda,\mu}$ for which the identity

$$\mathcal{O}_\lambda \cdot \mathcal{O}_\mu = \sum_\nu (-1)^{|\nu|-|\lambda|-|\mu|}\, c^\nu_{\lambda,\mu}\, \mathcal{O}_\nu$$

holds in $K(X)$, where $|\lambda|$ denotes the codimension of X_λ. The structure constants $c^\nu_{\lambda,\mu}$ are generalizations of the classical Littlewood-Richardson coefficients and describe the geometry of intersections of Schubert varieties in X.

When X is a Grassmann variety of type A, it was proved in [3] that the structure constant $c^\nu_{\lambda,\mu}$ is equal to the number of certain combinatorial objects called *set-valued tableaux*. So far no generalization of this rule has been found for other homogeneous spaces. More recently, Thomas and Yong have defined a K-theoretic version of Schützenberger's *jeu de taquin* algorithm and conjectured that, if X is any minuscule variety, then the structure constant $c^\nu_{\lambda,\mu}$ is equal to the number of

Date: April 15, 2014.
2010 *Mathematics Subject Classification.* Primary 14N15; Secondary 05E15, 19E08, 14M15.
The first author was supported in part by NSF grants DMS-0906148 and DMS-1205351.

increasing tableaux of skew shape ν/λ that rectify to a *superstandard tableau* S_μ by using K-theoretic jeu de taquin slides. Thomas and Yong proved their conjecture for Grassmannians of type A in [34], while the conjecture for maximal orthogonal Grassmannians follows from combinatorial results of Clifford, Thomas, and Yong [9] together with a K-theoretic Pieri formula proved by Buch and Ravikumar [7]. It turns out that the conjecture is also true for the Cayley Plane and for even dimensional quadrics, but it fails for the Freudenthal variety, see Remark 3.9 and Example 3.21 below. In this paper we show that, if the superstandard tableau S_μ is replaced with a *minimal increasing tableau* M_μ, then Thomas and Yong's conjecture gives a correct formula for the K-theoretic Schubert structure constants of all minuscule varieties. We note that the possibility of replacing the superstandard tableau S_μ with something else was mentioned in [34]. Notice also that Littlewood-Richardson rules are known for the cohomology of all cominuscule varieties [23, 29, 32, 30, 25, 33].

In contrast to Schützenberger's classical jeu de taquin algorithm, the K-theoretic version operates with several empty boxes at the same time. Another important difference is that the K-theoretic algorithm is not independent of choices, in the sense that different choices of initial empty boxes may lead to several different rectifications of the same skew tableau. Thomas and Yong proved that any superstandard tableau S_μ associated to a Grassmann variety of type A has the property that, if any skew tableau T has S_μ as a rectification, then S_μ is the only rectification of T [34]. We will call such a tableau a *unique rectification target*. The main combinatorial result of [9] states that superstandard tableaux associated to maximal orthogonal Grassmannians are also unique rectification targets.

In this paper we attempt a systematic study of unique rectification targets. In particular, we prove that any minimal increasing tableau associated to a minuscule variety is a unique rectification target. We also prove several alternative versions of the Littlewood-Richardson rule. For example, if T_0 is *any* increasing tableau of shape μ, then we prove that $c^\nu_{\lambda,\mu}$ is the number of increasing tableaux T of shape ν/λ for which the *greedy rectification* of T is equal to T_0. This rectification is obtained by consistently choosing *all* inner corners as initial empty boxes during the rectification process.

When X is a Grassmannian of type A, we also prove versions of the Littlewood-Richardson rule that are formulated without reference to the jeu de taquin algorithm. To each permutation w there is a *stable Grothendieck polynomial* G_w [20, 12] which can be identified with an element of the K-theory ring $K(X)$ [3]. For example, if w is the Grassmannian permutation associated to a partition μ, then G_w is equal to $\mathcal{O}_\mu$. It was proved in [11, 5] that the coefficient of $\mathcal{O}_\nu$ in the expansion of G_w is equal to $(-1)^{|\nu|-\ell(w)}$ times the number of increasing tableaux T of shape ν for which the *Hecke permutation* $w(T)$ is equal to w^{-1}. This permutation $w(T)$ is defined as the Hecke product of the simple transpositions given by the row word of T. We prove more generally that the coefficient of $\mathcal{O}_\nu$ in the expansion of $\mathcal{O}_\lambda \cdot G_w$ is equal to $(-1)^{|\nu/\lambda|-\ell(w)}$ times the number of increasing tableaux T of shape ν/λ such that $w(T) = w^{-1}$. By choosing w to be the Grassmannian permutation for μ, this formula specializes to express the structure constants $c^\nu_{\lambda,\mu}$. Similar identities are obtained for maximal orthogonal Grassmannians.

The main technical tool introduced in this paper is an equivalence relation on words of integers that we call *K-Knuth equivalence*. This relation is defined by using a mixture of the basic relations defining the Hecke monoid and the plactic algebra. We prove that if T and T' are increasing tableaux defined on skew Young diagrams, then T' can be obtained from T by a sequence of forward and reverse K-theoretic jeu de taquin slides if and only if the the row words of T and T' are K-Knuth equivalent. This result implies that the Hecke permutation $w(T)$ is an invariant under jeu de taquin slides. It also implies that the length of the longest strictly increasing subsequence of the row word of a tableau is an invariant, as proved earlier by Thomas and Yong [34].

Our paper is organized as follows. Section 2 explains Proctor's bijection between order ideals and Schubert classes and gives self-contained proofs of its main properties. Section 3 explains Thomas and Yong's K-theoretic jeu de taquin algorithm and gives several examples. We also define a *combinatorial K-theory ring* associated to any partially ordered set that satisfies certain properties. In Section 4 we then show that, if X is any minuscule variety, then the Grothendieck ring $K(X)$ is isomorphic to the combinatorial K-theory ring associated to Λ_X. We then use the geometry of X to establish additional combinatorial facts, including a criterion for unique rectification targets and new versions of the Littlewood-Richardson rule. Section 5 defines the K-Knuth equivalence relation, as well as *resolutions* of increasing tableaux with empty boxes, and *reading words* of these resolutions. The main result in this section states that all resolutions of a tableau with empty boxes have K-Knuth equivalent reading words. Section 6 is devoted to Grassmannians of type A. We first use the K-Knuth equivalence of resolutions to establish that the K-Knuth class of the row word of an increasing tableau is an invariant under jeu de taquin slides. We apply this to prove that minimal increasing tableaux are unique rectification targets, as well as a criterion that generalizes Thomas and Yong's result that superstandard tableaux are unique rectification targets. We also prove the above mentioned formula for products involving a stable Grothendieck polynomial. Finally, we prove that the rectification of a minimal increasing tableau of skew shape is always a minimal increasing tableau. This fact implies that *maximal increasing tableaux* are also unique rectification targets, which in turn is used in the proof of the Littlewood-Richardson rule based on greedy rectifications. Section 7 finally gives analogues of the results of Section 6 for maximal orthogonal Grassmannians. While many of these results can be translated from type A, we also define a weak version of the K-Knuth relation that governs the jeu de taquin algorithm for maximal orthogonal Grassmannians.

We thank Robert Proctor and John Stembridge for answers to questions about the history of minuscule posets. We also thank Alex Yong for inspiring discussions. Finally, we benefited from helpful comments to our exposition from both Proctor and an anonymous referee.

2. Schubert varieties in cominuscule spaces

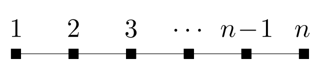
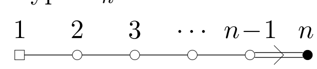
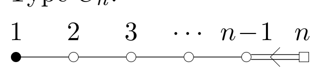
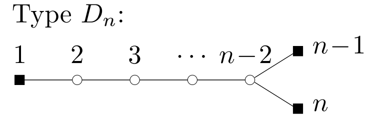
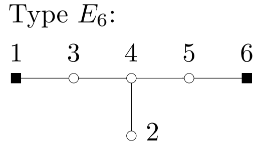
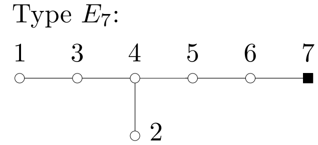
Let $X = G/P$ be a homogeneous space defined by a semisimple complex linear algebraic group G and a parabolic subgroup P. Fix also a maximal torus T and a Borel subgroup B such that $T \subset B \subset P \subset G$. Let R be the corresponding root system, with positive roots R^+ and simple roots $\Delta \subset R^+$. Let $W = N_G(T)/T$ be the Weyl group of G and $W_P = N_P(T)/T \subset W$ the Weyl group of P. Each element $w \in W$ defines a Schubert variety $X(w) = \overline{Bw.P} \subset X$, which depends only on the coset of w in W/W_P. Let $W^P \subset W$ be the set of minimal length representatives for the cosets in W/W_P. An element $w \in W$ belongs to W^P if and only if $w.\beta \in R^+$ for each $\beta \in \Delta$ with $s_\beta \in W_P$. The set W^P is in one-to-one correspondence with the B-stable Schubert varieties in X, and for $w \in W^P$ we have $\dim X(w) = \ell(w)$.

Assume that P is a maximal parabolic subgroup of G. Then P corresponds to a unique simple root $\gamma \in \Delta$ such that W_P is generated by all simple reflections except s_γ. Given any root $\alpha \in R$, let $\gamma(\alpha)$ denote the coefficient of γ when α is written as a linear combination of simple roots. The simple root γ and the variety X are called *cominuscule* if $|\gamma(\alpha)| \leq 1$ for all $\alpha \in R$, and γ and X are called *minuscule* if $\gamma^\vee$ is a cominuscule simple root in the dual root system $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$. Here $\alpha^\vee = \frac{2\,\alpha}{(\alpha,\alpha)}$ denotes the coroot of α.

The collection of all minuscule and cominuscule varieties is listed in Table 1. Notice that all minuscule varieties are also cominuscule, possibly for a different Lie type. For example, the minuscule odd orthogonal Grassmannian $\mathrm{OG}(n, 2n+1)$ of type B_n is isomorphic to the even orthogonal Grassmannian $\mathrm{OG}(n+1, 2n+2)$ of type D_{n+1} which is both minuscule and cominuscule. In this section we assume that γ is a cominuscule simple root.

TABLE 1. Minuscule and cominuscule varieties

● Minuscule simple root.
□ Cominuscule simple root.
■ Minuscule and cominuscule simple root.

Type A_n:  $A_n/P_m = \mathrm{Gr}(m, n+1)$ Grassmannian of type A.	Type B_n:  $B_n/P_1 = Q^{2n-1}$ Odd quadric. $B_n/P_n = \mathrm{OG}(n, 2n+1)$ Max. orthogonal Grassmannian.
Type C_n:  $C_n/P_1 = \mathbb{P}^{2n-1}$ Projective space. $C_n/P_n = \mathrm{LG}(n, 2n)$ Lagrangian Grassmannian.	Type D_n:  $D_n/P_1 = Q^{2n-2}$ Even quadric. $D_n/P_{n-1} \cong D_n/P_n = \mathrm{OG}(n, 2n)$ Max. orthogonal Grassmannian.
Type E_6:  $E_6/P_1 \cong E_6/P_6$ Cayley plane.	Type E_7:  E_7/P_7 Freudenthal variety.

A result of Proctor [27] shows that the Bruhat order on W^P is a distributive lattice. If we let Λ_X denote the partially ordered set of join irreducible elements of this lattice, then Birkhoff's representation theorem gives a bijection between W^P

and the set of order ideals in Λ_X. Proctor identified W^P with the set of weights of a minuscule representation, and in this way Λ_X is identified with a subset of the positive roots R^+.[1] Stembridge later gave a different construction of Λ_X as the heap of the longest element of W^P [31]. In this expository section we start by defining the set Λ_X using the description given in [27, Thm. 11], then prove directly from the cominuscule condition on γ that the order ideals of Λ_X are in one-to-one correspondence with the Schubert classes of X. An alternative treatment can be found in [33, §2]. Let $\leq$ denote the partial order on $\mathrm{Span}_{\mathbb{Z}}(\Delta)$ defined by $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a sum of simple roots.

Definition 2.1. Let $\Lambda_X = \{\alpha \in R \mid \gamma(\alpha) = 1\}$ be the set of positive roots for which the coefficient of γ is non-zero, and equip this set with the restriction of the partial order $\leq$ on $\mathrm{Span}_{\mathbb{Z}}(\Delta)$. A subset $\lambda \subset \Lambda_X$ is called a *straight shape* if it is a lower order ideal, i.e. if $\alpha \in \lambda$ then λ also contains all roots $\beta \in \Lambda_X$ for which $\beta \leq \alpha$. A *skew shape* is any set theoretic difference $\lambda/\mu := \lambda \smallsetminus \mu$ of straight shapes λ and μ; this notation will be used only when $\mu \subset \lambda$.

The partially ordered set Λ_X can be identified with a subset of $\mathbb{N}^2$, where the order on $\mathbb{N}^2$ is defined by $[r_1, c_1] \leq [r_2, c_2]$ if and only if $r_1 \leq r_2$ and $c_1 \leq c_2$. We will represent $\mathbb{N}^2$ as a grid of boxes $[r, c]$, where the first coordinate is a row number (increasing from top to bottom) and the second coordinate is a column number (increasing from left to right). The shape of the resulting set of boxes Λ_X is displayed in Table 2. We will henceforth call the roots in Λ_X *boxes*.

We remark that there are other ways to identify Λ_X with a subset of $\mathbb{N}^2$, and the choices made in Table 2 are slightly different from those made in [33]. The shapes chosen here satisfy that the longest element w_X in W_P acts on Λ_X as a 180 degree rotation for shapes in the left column of Table 2, while w_X acts by reflection in a south-west to north-east diagonal for shapes in the right column of the table. The action of w_X is related to Poincare duality, see Corollary 2.8 below. We have also chosen the shapes so that the number of boxes in each row decreases from top to bottom. As a result, if $\lambda \subset \Lambda_X$ is any straight shape, then λ can be represented by the *partition* $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$, where λ_r is the number of boxes in row r of λ.

One can check that for each box $\alpha \in \Lambda_X$ with $\alpha > \gamma$, there exists a simple root $\beta \in \Delta \smallsetminus \{\gamma\}$ such that $\alpha - \beta \in \Lambda_X$. This implies that $\mathrm{height}(\alpha)$, defined as the sum of the coefficients obtained when α is written as a linear combination of simple roots, is equal to the maximal cardinality of a totally ordered subset of Λ_X with α as its maximal element.

Lemma 2.2. *Let $\alpha, \beta \in \Lambda_X$. If $\alpha \not\leq \beta$ and $\beta \not\leq \alpha$, then $(\alpha, \beta) = 0$.*

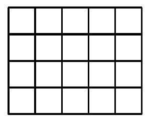
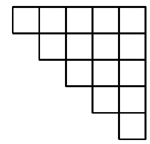
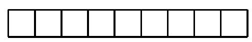
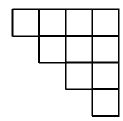
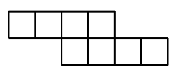
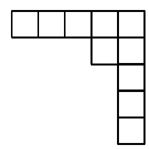
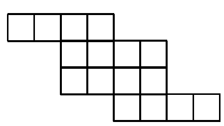
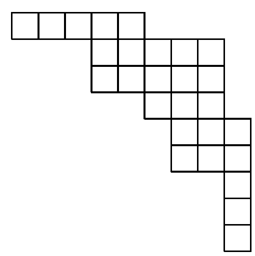
Proof. The cominuscule condition implies that $\alpha + \beta \notin R$, and the incomparability gives $\alpha - \beta \notin R$. The lemma therefore follows from [15, Lemma 9.4]. □

Definition 2.3. Given a straight shape $\lambda \subset \Lambda_X$, define an element $w_\lambda \in W$ as follows. If $\lambda = \emptyset$, then set $w_\lambda = 1$. Otherwise set $w_\lambda = w_{\lambda \smallsetminus \{\alpha\}}\, s_\alpha \in W$, where $\alpha \in \lambda$ is any maximal box.

To see that w_λ is well defined, assume that α and β are two distinct maximal boxes of the straight shape λ. Then Lemma 2.2 implies that $(\alpha, \beta) = 0$, so s_α commutes with s_β. By induction on $|\lambda|$ we may assume that $w_{\lambda \smallsetminus \{\alpha\}} = w_{\lambda \smallsetminus \{\alpha,\beta\}}\, s_\beta$ and

[1] Strictly speaking Proctor's conventions are closer to the situation in Remark 2.5.

TABLE 2. The shape Λ_X of a cominuscule variety X

Grassmannian of type A $\Lambda_{\mathrm{Gr}(m,m+k)}$ =  (m rows and k columns)	Lagrangian Grassmannian $\Lambda_{\mathrm{LG}(n,2n)}$ =  (n rows)
Odd quadric $\Lambda_{Q^{2n-1}}$ =  ($2n-1$ boxes)	Max. orthogonal Grassmannian $\Lambda_{\mathrm{OG}(n,2n)}$ =  ($n-1$ rows)
Even quadric, n even $\Lambda_{Q^{2n}}$ =  (n boxes in each row)	Even quadric, n odd $\Lambda_{Q^{2n}}$ =  ($2n$ boxes)
Cayley plane Λ_{E_6/P_6} = 	Freudenthal variety Λ_{E_7/P_7} = 

$w_{\lambda \smallsetminus \{\beta\}} = w_{\lambda \smallsetminus \{\alpha,\beta\}}\, s_\alpha$ are well defined, so we obtain $w_{\lambda \smallsetminus \{\alpha\}}\, s_\alpha = w_{\lambda \smallsetminus \{\alpha,\beta\}}\, s_\beta\, s_\alpha = w_{\lambda \smallsetminus \{\alpha,\beta\}}\, s_\alpha\, s_\beta = w_{\lambda \smallsetminus \{\beta\}}\, s_\beta$, as required.

For any element $w \in W$, let $I(w) = \{\alpha \in R^+ \mid w.\alpha < 0\}$ denote the inversion set of w. This set uniquely determines w [15, Thm. 10.3], and we have $\ell(w) = |I(w)|$. The following proposition shows that the map from W^P to straight shapes in Λ_X defined by $w \mapsto I(w)$ is a bijection, see also [33, Prop. 2.1].

Theorem 2.4. *If $\lambda \subset \Lambda_X$ is a straight shape, then $w_\lambda \in W^P$ and $I(w_\lambda) = \lambda$. On the other hand, if $u \in W^P$, then $I(u)$ is a straight shape in Λ_X and $w_{I(u)} = u$.*

Proof. The first claim is clear if $\lambda = \emptyset$, so assume that $\lambda \neq \emptyset$ and let $\alpha \in \lambda$ be a maximal box. Then $w_\lambda = w_{\lambda \smallsetminus \{\alpha\}}\, s_\alpha$, and by induction on $|\lambda|$ we have $I(w_{\lambda \smallsetminus \{\alpha\}}) = \lambda \smallsetminus \{\alpha\}$. Let $\beta \in R^+$ be any positive root. We claim that $w_\lambda.\beta < 0$ if and only if $\beta \in \lambda$. We have $s_\alpha.\beta = \beta - (\beta, \alpha^\vee)\alpha$, so the cominuscule condition implies that $\gamma(\beta) - (\beta, \alpha^\vee) = \gamma(s_\alpha.\beta) \in \{0, \pm 1\}$. The claim is clear if $(\beta, \alpha^\vee) = 0$, so assume that

$(\beta, \alpha^\vee) \neq 0$. If $\gamma(\beta) = 0$, then we must have $(\beta, \alpha^\vee) = \pm 1$, and the claim is true because we have either $s_\alpha.\beta = \beta + \alpha > \alpha$ or $0 < -s_\alpha.\beta = \alpha - \beta < \alpha$. Otherwise we have $\gamma(\beta) = 1$, and the cominuscule condition implies that $(\beta, \alpha^\vee) \in \{1, 2\}$. In this case [15, Lemma 9.4] implies that $\beta \leq \alpha$ or $\beta > \alpha$. If $\beta > \alpha$, then the claim is true because we have either $s_\alpha.\beta = \beta - \alpha > 0$ and $\gamma(s_\alpha.\beta) = 0$, or $0 < -s_\alpha.\beta = 2\alpha - \beta < \alpha$. Finally, if $\beta \leq \alpha$ then the claim follows because we have either $-s_\alpha.\beta = \alpha - \beta > 0$ and $\gamma(s_\alpha.\beta) = 0$, or $-s_\alpha.\beta = 2\alpha - \beta \geq \alpha$. It follows from the claim that $I(w_\lambda) = \lambda$, which in turn implies that $w_\lambda \in W^P$.

Let $u \in W^P$. Since $u.(\Delta \smallsetminus \{\gamma\}) \subset R^+$ we get $u.(R^+ \smallsetminus \Lambda_X) \subset R^+$. It follows that $I(u) \subset \Lambda_X$. To see that $I(u)$ is a straight shape in Λ_X, let $\alpha \in I(u)$, $\beta \in \Lambda_X$, and assume that $\beta \leq \alpha$. Then $\alpha - \beta$ is a sum of simple roots from the set $\Delta \smallsetminus \{\gamma\}$, so $u.(\alpha - \beta) \geq 0$, hence $u.\beta \leq u.\alpha < 0$ and $\beta \in I(u)$. Finally, the identity $w_{I(u)} = u$ follows because $I(w_{I(u)}) = I(u)$. $\square$

Remark 2.5. If the simple root γ is minuscule but not cominuscule, then the Schubert varieties in X are indexed by straight shapes in the partially ordered set $\Lambda_X := \{\alpha^\vee \in R^\vee \mid \gamma^\vee(\alpha^\vee) = 1\}$, consisting of positive coroots for which the coefficient of the cominuscule coroot $\gamma^\vee$ is non-zero. Alternatively, one can construct X using a group G of a different Lie type, so that γ is both minuscule and cominuscule.

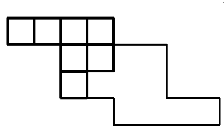
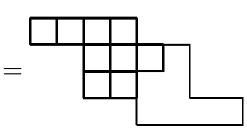
The Bruhat order on W^P is defined by $u \leq v$ if and only if $X(u) \subset X(v)$.

Corollary 2.6. *Let λ and μ be straight shapes in Λ_X. Then $w_\mu \leq w_\lambda$ if and only if $\mu \subset \lambda$.*

Proof. We may assume that $|\lambda| = |\mu| + 1$, in which case $w_\mu \leq w_\lambda$ if and only if $w_\mu^{-1} w_\lambda$ is a reflection [16, §5.9]. If $\mu \subset \lambda$, then this is true because $w_\lambda = w_\mu\, s_\alpha$ where $\{\alpha\} = \lambda \smallsetminus \mu$. Assume that $w_\mu \leq w_\lambda$ and choose $\alpha \in R^+$ such that $w_\lambda = w_\mu\, s_\alpha$. It follows from [16, Prop. 5.7] that $\alpha \in \lambda \smallsetminus \mu$. We must show that $w_\lambda.\beta < 0$ for each $\beta \in \mu$. This is clear if $(\beta, \alpha^\vee) = 0$, so assume that $(\beta, \alpha^\vee) \neq 0$. Since $\gamma(s_\alpha.\beta) = 1 - (\beta, \alpha^\vee) \in \{0, \pm 1\}$, we have $(\beta, \alpha^\vee) > 0$, and we deduce from [15, Lemma 9.4] that $\beta < \alpha$. This implies that $\beta \in \lambda$, as required. $\square$

Let w_X be the longest element of W_P. Then $w_X.\Lambda_X = \Lambda_X$. In fact, since $w_X.(\Delta \smallsetminus \{\gamma\}) \subset -R^+$, it follows that w_X is an order-reversing involution of Λ_X. As mentioned above, w_X acts as a rotation on the shapes in the left column of Table 2 and as a reflection on the shapes in the right column. If $\lambda \subset \Lambda_X$ is a straight shape, then $w_X.\lambda$ is an upper order ideal of Λ_X. The *Poincare dual shape* of λ is the straight shape $\lambda^\vee = \Lambda_X \smallsetminus w_X.\lambda$.

Example 2.7. Let $X = E_6/P_6$ be the Cayley plane. Then the straight shape $\lambda = (4, 2, 1)$ has Poincare dual shape $\lambda^\vee = (4, 3, 2)$.

$\lambda =$  $\lambda^\vee =$ 

Corollary 2.8. *Let $\lambda \subset \Lambda_X$ be a straight shape. Then $w_{\lambda^\vee} = w_0 w_\lambda w_X$ is the Poincare dual Weyl group element of w_λ, where w_0 is the longest element in W.*

Proof. If $\beta \in R^+$ is any positive root, then we have $w_0 w_\lambda w_X.\beta < 0$ if and only if $w_X.\beta \in R^+ \smallsetminus \lambda$, and since $w_X.(\Delta \smallsetminus \{\gamma\}) \subset -R^+$, this holds if and only if $w_X.\beta \in \Lambda_X \smallsetminus \lambda = w_X.\lambda^\vee$. It follows that $I(w_0 w_\lambda w_X) = \lambda^\vee$, as required. $\square$

To each straight shape $\lambda \subset \Lambda_X$ we assign the Schubert variety $X_\lambda := X(w_{\lambda^\vee})$ of codimension $|\lambda|$ in X. If μ is an additional straight shape, then Corollary 2.6 implies that $X_\lambda \subset X_\mu$ if and only if $\mu \subset \lambda$. For the classical Grassmannians $\mathrm{Gr}(m,n)$, $\mathrm{LG}(n,2n)$, and $\mathrm{OG}(n,2n)$ that parametrize subspaces of a vector space, one can also define the Schubert variety corresponding to a partition λ by using incidence conditions relative to a fixed flag of subspaces (see e.g. [7] for these standard constructions). Since the Bruhat order is also determined by containment of partitions in these constructions, one may deduce from the following lemma that the standard constructions agree with the assignment $\lambda \mapsto X_\lambda$ used here. In particular, the K-theoretic Pieri formulas proved in [21, 7] are valid with the notation used here.

Lemma 2.9. *Any automorphism of the set W^P that preserves the Bruhat order arises from an automorphism of the Dynkin diagram of Δ that fixes γ.*

Proof. An order preserving automorphism of W^P is equivalent to an inclusion preserving automorphism of the set of straight shapes in Λ_X. Since such an automorphism must restrict to an automorphism of the straight shapes that contain a unique maximal box, it must be given by an automorphism of the partially ordered set Λ_X. If X is the Cayley plane or the Freudenthal variety, then we leave it as an exercise to check that the only automorphism of Λ_X is the identity.

We will say that a box $\alpha \in \Lambda_X$ is *extreme* if α belongs to a unique maximal totally ordered subset of Λ_X. If X is not the Cayley plane or the Freudenthal variety, then an inspection of Table 2 shows that Λ_X contains at least one extreme box. For example, if $X = \mathrm{Gr}(m,n)$ is a Grassmannian of type A, then the upper-right box and the lower-left box of Λ_X are extreme boxes. Furthermore, Λ_X contains two distinct extreme boxes of the same height if and only if $X = \mathrm{Gr}(n,2n)$ or $X = Q^{2n}$ for some $n \geq 2$. These are also the cases where the Dynkin diagram has a nontrivial automorphism that fixes γ, and this automorphism defines an automorphism ι of Λ_X that interchanges the two extreme boxes.

Let ψ be any automorphism of Λ_X. Then ψ maps each extreme box α to an extreme box α' of the same height. If $\alpha' \neq \alpha$, then we can replace ψ with $\psi\iota$ to obtain that ψ maps every extreme box to itself. It is enough to show that ψ is the identity. Let $\alpha \in \Lambda_X$ be an extreme box and let $P \subset \Lambda_X$ be the unique maximal totally ordered subset containing α. Then ψ is the identity on P, so ψ restricts to an automorphism of $\Lambda_X \smallsetminus P$. This partially ordered set is isomorphic to $\Lambda_{X'}$ for a cominuscule variety X' of smaller dimension. Since ψ fixes at least one extreme box of $\Lambda_X \smallsetminus P$, it follows by induction that ψ is the identity on $\Lambda_X \smallsetminus P$, as required. $\square$

The methods of this section can be used to prove that every element $w \in W^P$ is *fully commutative*, i.e. any reduced expression for w can be obtained from any other by interchanging commuting simple reflections. This was proved by Fan for simply laced root systems [10] and by Stembridge for non-simply laced root systems [31]. The remainder of this section will not be used in the rest of our paper.

Let $\mu \subset \lambda$ be straight shapes in Λ_X. Then Definition 2.3 implies that $w_\mu^{-1} w_\lambda$ is the product of all reflections s_α for $\alpha \in \lambda/\mu$, in any order compatible with the partial order $\leq$ on Λ_X. However, the length of this product is hard to predict.

On the other hand, it follows from Theorem 2.4 that $I(w_\lambda w_\mu^{-1}) = w_\mu.(\lambda/\mu)$, so $\ell(w_\lambda w_\mu^{-1}) = |\lambda/\mu|$. Furthermore, the product $w_\lambda w_\mu^{-1}$ depends only on the skew shape λ/μ. Indeed, if $\alpha \in \Lambda_X \smallsetminus \lambda$ is any box such that $\mu \cup \{\alpha\}$ is a straight shape, then $w_{\lambda\cup\{\alpha\}}\, {w_{\mu\cup\{\alpha\}}}^{-1} = w_\lambda s_\alpha (w_\mu s_\alpha)^{-1} = w_\lambda w_\mu^{-1}$. The elements $w_{\lambda/\mu} :=$

$w_\lambda w_\mu^{-1}$ provide cominuscule analogues of the 321-avoiding permutations in type A [1]. Notice that $w_{\{\alpha\}}$ is a simple reflection for each $\alpha \in \Lambda_X$. In fact, if we label each box $\alpha \in \Lambda_X$ with the corresponding simple reflection $w_{\{\alpha\}}$ as in [27, Thm. 11], then we recover the heap of the longest element in W^P used by Stembridge in [31]. Notice also that if $\mu \subset \lambda \subset \nu \subset \Lambda_X$ are straight shapes, then $w_{\nu/\mu} = w_{\nu/\lambda} w_{\lambda/\mu}$.

Corollary 2.10. *Let $\lambda \subset \Lambda_X$ be a straight shape. Then the reduced expressions for w_λ are exactly the expressions of the form $w_\lambda = w_{\alpha_{|\lambda|}} \cdots w_{\alpha_2} w_{\alpha_1}$, where $\alpha_1, \alpha_2, \ldots, \alpha_{|\lambda|}$ is any ordering of the boxes of λ compatible with the partial order $\leq$ on Λ_X. Furthermore, w_λ is fully commutative.*

Proof. Let $\beta \in \Delta$ be any simple root such that $\ell(s_\beta w_\lambda) = |\lambda| - 1$. Then $s_\beta w_\lambda \in W^P$, so we have $s_\beta w_\lambda = w_\mu$ for some straight shape $\mu \subset \lambda$, such that $\lambda/\mu = \{\alpha\}$ is a single box. Since α is a maximal box of λ and $w_{\{\alpha\}} = w_\lambda w_\mu^{-1} = s_\beta$, it follows by induction on $|\lambda|$ that every reduced expression for w_λ has the indicated form. It is therefore enough to show that, if $\alpha, \alpha' \in \Lambda_X$ are incomparable boxes, then the simple reflections $w_{\{\alpha\}}$ and $w_{\{\alpha'\}}$ commute. To see this, let $\nu \subset \Lambda_X$ be any straight shape such that $\nu \cup \{\alpha\}$ and $\nu \cup \{\alpha'\}$ are strictly larger straight shapes. Then we have $w_{\{\alpha\}} w_{\{\alpha'\}} = w_{\nu \cup \{\alpha,\alpha'\}/\nu \cup \{\alpha'\}}\, w_{\nu \cup \{\alpha'\}/\nu} = w_{\nu \cup \{\alpha,\alpha'\}/\nu} = w_{\nu \cup \{\alpha,\alpha'\}/\nu \cup \{\alpha\}}\, w_{\nu \cup \{\alpha\}/\nu} = w_{\{\alpha'\}} w_{\{\alpha\}}$, as required. $\square$

3. Increasing tableaux and jeu de taquin

3.1. **Increasing tableaux.** In this section we let Λ denote a partially ordered set. The elements of Λ will be called *boxes*. We write $\alpha \lessdot \beta$ if the box β covers α, i.e. we have $\alpha < \beta$ and no box $\beta' \in \Lambda$ satisfies $\alpha < \beta' < \beta$. We will assume that Λ contains finitely many minimal boxes and each box α has finitely many covers. A finite lower order ideal $\lambda \subset \Lambda$ is called a *straight shape* in Λ, and a difference $\lambda/\mu := \lambda \smallsetminus \mu$ of straight shapes is called a *skew shape*.

Definition 3.1. Let $\nu/\lambda \subset \Lambda$ be a skew shape and S a set. A *tableau* of shape ν/λ with values in S is a map $T : \nu/\lambda \to S$. An *increasing tableau* of shape ν/λ is a map $T : \nu/\lambda \to \mathbb{Z}$ such that $T(\alpha) < T(\beta)$ for all boxes $\alpha, \beta \in \nu/\lambda$ with $\alpha < \beta$.

We will identify the tableau $T : \nu/\lambda \to S$ with the filling of the boxes of ν/λ with the values specified by T. If $\Lambda = \Lambda_X$ is the partially ordered set associated to a minuscule variety X as in Table 2, then a tableau $T : \nu/\lambda \to \mathbb{Z}$ is increasing exactly when the rows of T are strictly increasing from left to right and the columns of T are strictly increasing from top to bottom. The shape of T is denoted $\mathrm{sh}(T) = \nu/\lambda$. If $S' \subset S$ is a subset, then we let $T|_{S'}$ denote the restriction of T to the subset $T^{-1}(S') \subset \mathrm{sh}(T)$. If $T^{-1}(S')$ is itself a skew shape in Λ, then $T|_{S'}$ is the tableau obtained from T by removing all boxes with values in $S \smallsetminus S'$. Given straight shapes $\lambda \subset \mu \subset \nu$ and tableaux $T_1 : \mu/\lambda \to S$ and $T_2 : \nu/\mu \to S$, we let $T_1 \cup T_2 : \nu/\lambda \to S$ denote their union, defined by $(T_1 \cup T_2)(\alpha) = T_1(\alpha)$ for $\alpha \in \mu/\lambda$ and $(T_1 \cup T_2)(\alpha) = T_2(\alpha)$ for $\alpha \in \nu/\mu$.

3.2. ***K*-theoretic jeu de taquin.** We next describe the K-theoretic jeu de taquin algorithm of Thomas and Yong [34]. We will say that two boxes $\alpha, \beta \in \Lambda$ are *neighbors* if $\alpha \lessdot \beta$ or $\beta \lessdot \alpha$. Given a tableau T with values in S and two elements

$s, s' \in S$, define a new tableau $\mathrm{swap}_{s,s'}(T)$ of the same shape by

$$\mathrm{swap}_{s,s'}(T) : \alpha \mapsto \begin{cases} s' & \text{if } T(\alpha) = s \text{ and } T(\beta) = s' \text{ for some neighbor } \beta \text{ of } \alpha; \\ s & \text{if } T(\alpha) = s' \text{ and } T(\beta) = s \text{ for some neighbor } \beta \text{ of } \alpha; \\ T(\alpha) & \text{otherwise.} \end{cases}$$

Let T be an increasing tableau of shape ν/λ with values in the interval $[a,b] \subset \mathbb{Z}$ and let $C \subset \lambda$ be a subset of the maximal boxes in λ. Then the *forward slide* of T starting from C is defined by

$$\mathrm{jdt}_C(T) = (\mathrm{swap}_{b,\bullet}\, \mathrm{swap}_{b-1,\bullet} \cdots \mathrm{swap}_{a+1,\bullet}\, \mathrm{swap}_{a,\bullet}([C \to \bullet] \cup T))|_{\mathbb{Z}}\,,$$

where $[C \to \bullet]$ denotes the constant tableau of shape C that puts a dot "$\bullet$" in each box. It is easy to see that $\mathrm{jdt}_C(T)$ is again an increasing tableau. Similarly, if $\widehat{C} \subset \Lambda \smallsetminus \nu$ is a subset of the minimal boxes of $\Lambda \smallsetminus \nu$, then the *reverse slide* of T starting from $\widehat{C}$ is the increasing tableau

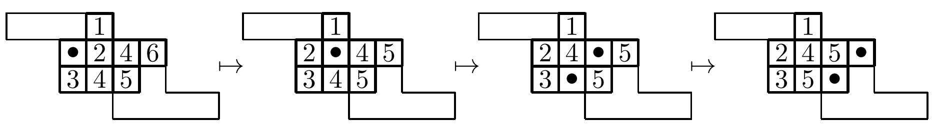
$$\widehat{\mathrm{jdt}}_{\widehat{C}}(T) = (\mathrm{swap}_{a,\bullet}\, \mathrm{swap}_{a+1,\bullet} \cdots \mathrm{swap}_{b-1,\bullet}\, \mathrm{swap}_{b,\bullet}(T \cup [\widehat{C} \to \bullet]))|_{\mathbb{Z}}\,.$$

Forward and reverse slides are inverse operations in the sense that

$$\widehat{\mathrm{jdt}}_{\widehat{C}'}(\mathrm{jdt}_C(T)) \;=\; T \;=\; \mathrm{jdt}_{C'}(\widehat{\mathrm{jdt}}_{\widehat{C}}(T))\,,$$

where $\widehat{C}' = \mathrm{sh}(T) \smallsetminus \mathrm{sh}(\mathrm{jdt}_C(T))$ and $C' = \mathrm{sh}(T) \smallsetminus \mathrm{sh}(\widehat{\mathrm{jdt}}_{\widehat{C}}(T))$.

Example 3.2. Let $\Lambda = \Lambda_{E_6/P_6}$ be the partially ordered set associated to the Cayley plane. We list the sequence of intermediate tableaux obtained when a forward slide is applied to a tableau T of shape $(4,4,3)/(3,1)$, starting from the box $[2,3]$ in row 2 and column 3. The resulting tableau $\mathrm{jdt}_{\{[2,3]\}}(T)$ has shape $(4,3,2)/(3)$.



We will say that two increasing tableaux S and T are *jeu de taquin equivalent* if S can be obtained by applying a sequence of forward and reverse jeu de taquin slides to T. The *jeu de taquin class* of T is the set $[T]$ of all increasing tableaux that are jeu de taquin equivalent to T. If the partially ordered set Λ is not understood from the context, then we will also denote this equivalence class by $[T]_\Lambda$. We remark that if Λ is not too large, then it is easy to generate a list of all tableaux in the jeu de taquin class $[T]$ using a computer. In particular, this can be done when Λ is the partially ordered set associated to the Cayley plane or the Freudenthal variety.

The following fact follows immediately from the definitions.

Lemma 3.3. *Let T and T' be jeu de taquin equivalent increasing tableaux and let $[a,b]$ be any integer interval. Then $T|_{[a,b]}$ and $T'|_{[a,b]}$ are jeu de taquin equivalent.*

3.3. **Rectifications.** A *rectification* of an increasing tableau T is any tableau of straight shape that can be obtained by applying a sequence of forward slides to T. A central feature of Schützenberger's cohomological jeu de taquin algorithm for semistandard Young tableaux is that each semistandard tableau has a unique rectification, i.e. the outcome of the jeu de taquin algorithm does not depend on the chosen inner corners (see [26] and the references therein). Thomas and Yong observed in [34, Ex. 1.3] that this does not hold for the K-theoretic jeu de taquin algorithm. The following example is slightly smaller.

Example 3.4. Let $\Lambda = \Lambda_{\mathrm{Gr}(3,6)}$. Then the increasing tableau $\begin{array}{ccc} & & \\ & & 2 \\ 1 & 3 & 4 \end{array}$ has the rectifications $\begin{array}{ccc} 1 & 2 & 4 \\ 3 & & \\ & & \end{array}$ and $\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 4 & \\ & & \end{array}$.

Definition 3.5. An increasing tableau U of straight shape is a *unique rectification target* if, for every increasing tableau T that has U as a rectification, U is the only rectification of T.

Unique rectification targets are essential for using the K-theoretic jeu de taquin algorithm to formulate Littlewood-Richardson rules. If $\lambda \subset \Lambda_X$ is a straight shape associated to a minuscule variety X, then Thomas and Yong define the *row-wise superstandard tableau* of shape λ to be the unique increasing tableau S_λ that fills the first row of boxes in λ with the integers $1, 2, \ldots, \lambda_1$, fills the second row with $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$, etc. Similarly, one can define a *column-wise superstandard tableau* $\widehat{S}_\lambda$ in which the integers increase consecutively in each column.

Example 3.6. Let $X = \mathrm{OG}(n, 2n)$ be a maximal orthogonal Grassmannian. Then the shape $\lambda = (5, 3, 2)$ has the following row-wise and column-wise superstandard tableaux:

$$S_\lambda = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ & 6 & 7 & 8 & \\ & & 9 & 10 & \end{array} \quad \text{and} \quad \widehat{S}_\lambda = \begin{array}{ccccc} 1 & 2 & 4 & 7 & 10 \\ & 3 & 5 & 8 & \\ & & 6 & 9 & \end{array}.$$

It was proved in [34, 9] that the row-wise superstandard tableau S_λ is a unique rectification target if X is a Grassmannian of type A or a maximal orthogonal Grassmannian; alternative proofs are given in sections 6 and 7 below. However, the following example shows that this fails for the Freudenthal variety.

Example 3.7. Let $X = E_7/P_7$ be the Freudenthal variety and set $\lambda = (5, 3, 3)$. Then the row-wise superstandard tableau S_λ and the column-wise superstandard tableau $\widehat{S}_\lambda$ both fail to be unique rectification targets. In fact, consider the tableaux

$$T = \begin{array}{ccccccccc} \square & \square & \square & \square & \square & & & & \\ & & & \square & 1 & 2 & 3 & 5 & \\ & & & 1 & 2 & 4 & 6 & 8 & \\ & & & & & 7 & 9 & \square & \\ & & & & & & 10 & \square & \square \\ & & & & & & 11 & \square & \square \\ & & & & & & & & \square \\ & & & & & & & & \square \\ & & & & & & & & \square \end{array} \quad \text{and} \quad \widehat{T} = \begin{array}{ccccccccc} \square & \square & \square & \square & \square & & & & \\ & & & \square & 1 & 2 & 4 & 5 & \\ & & & 1 & 3 & 4 & 6 & 8 & \\ & & & & & 7 & 9 & \square & \\ & & & & & & 10 & \square & \square \\ & & & & & & 11 & \square & \square \\ & & & & & & & & \square \\ & & & & & & & & \square \\ & & & & & & & & \square \end{array}.$$

Then S_λ is the only rectification of $\mathrm{jdt}_{\{[1,5]\}}(T)$, but S_λ is not a rectification of $\mathrm{jdt}_{\{[2,4]\}}(T)$. Similarly, $\widehat{S}_\lambda$ is the only rectification of $\mathrm{jdt}_{\{[1,5]\}}(\widehat{T})$, but not a rectification of $\mathrm{jdt}_{\{[2,4]\}}(\widehat{T})$.

The next example shows that column-wise superstandard tableaux may fail to be unique rectification targets for Λ_X when X is a maximal orthogonal Grassmannian.

Example 3.8. Let $X = \mathrm{OG}(6, 12)$ and set $\lambda = (4, 2)$. Then the column-wise superstandard tableau $\widehat{S}_\lambda$ is not a unique rectification target. In fact, if T is the tableau displayed below, then $\widehat{S}_\lambda$ is a rectification of $\mathrm{jdt}_{\{[1,4]\}}(T)$, but $\widehat{S}_\lambda$ is not a rectification of $\mathrm{jdt}_{\{[2,2]\}}(T)$.

$$T = \begin{array}{ccccc} \boxed{} & \boxed{} & \boxed{} & \boxed{} & \boxed{2} \\ & \boxed{} & \boxed{1} & \boxed{2} & \boxed{4} \\ & & \boxed{3} & \boxed{5} & \boxed{} \\ & & & \boxed{6} & \boxed{} \\ & & & & \boxed{} \end{array} \qquad \widehat{S}_\lambda = \begin{array}{ccccc} \boxed{1} & \boxed{2} & \boxed{4} & \boxed{6} & \boxed{} \\ & \boxed{3} & \boxed{5} & \boxed{} & \boxed{} \\ & & \boxed{} & \boxed{} & \boxed{} \\ & & & \boxed{} & \boxed{} \\ & & & & \boxed{} \end{array}$$

Remark 3.9. One can show that if $X = \mathrm{Gr}(m, m+k)$ with $\max(m,k) \leq 2$ then every increasing tableau of straight shape in Λ_X is a unique rectification target. The same is true if $X = \mathrm{OG}(n, 2n)$ with $n \leq 5$, if $X = Q^{2n}$ is an even quadric, or if $X = E_6/P_6$. On the other hand, examples 3.4, 3.7, and 3.8 show that if X is any other minuscule variety, then there exist tableaux of straight shapes in Λ_X that are not unique rectification targets.

In this paper we show that the following definition gives a uniform construction of unique rectification targets for all minuscule varieties.

Definition 3.10. Given a straight shape $\lambda \subset \Lambda$, define the *minimal increasing tableau* M_λ of shape λ by setting $M_\lambda(\alpha)$ equal to the maximal cardinality of a totally ordered subset of Λ with α as its maximal element, for each box $\alpha \in \lambda$. We will say that Λ is a *unique rectification poset* if M_λ is a unique rectification target for all straight shapes $\lambda \subset \Lambda$.

In other words, M_λ puts the integer 1 in the minimal boxes of λ and fills the rest of the boxes with the smallest possible values allowed in an increasing tableau. If X is a minuscule variety and $\Lambda = \Lambda_X$, then $M_\lambda(\alpha) = \mathrm{height}(\alpha)$ for each $\alpha \in \lambda$.

Example 3.11. The minimal increasing tableau of the shape λ from Example 3.6 is given by

$$M_\lambda = \begin{array}{ccccc} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} \\ & \boxed{3} & \boxed{4} & \boxed{5} & \\ & & \boxed{5} & \boxed{6} & \end{array} .$$

Theorem 3.12. *If X is any minuscule variety, then Λ_X is a unique rectification poset.*

Proof. We must show that if $\lambda \subset \Lambda_X$ is any straight shape, then M_λ is a unique rectification target. This follows from Theorem 6.6 if X is a Grassmannian of type A, and from Corollary 7.2 if X is a maximal orthogonal Grassmannian. If X is a quadric hypersurface, then it is an easy exercise to check that every increasing tableau of straight shape in Λ_X is a unique rectification target. Finally, when X is the Cayley plane or the Freudenthal variety, we have checked by computer that M_λ is the only tableau of straight shape in the jeu de taquin class $[M_\lambda]$. □

Lemma 3.13. *Let U be a unique rectification target. Then U is the only tableau of straight shape in its jeu de taquin class $[U]$.*

Proof. Let $T \in [U]$ be any tableau of straight shape, and choose a sequence $U = T_0, T_1, \ldots, T_m = T$ such that each tableau T_i can be obtained by applying a slide to T_{i-1}. Since T_{i-1} and T_i share at least one rectification, it follows by induction that U is the only rectification of T_i for each i. It follows that $T = U$. □

3.4. **K-infusion.** Jeu de taquin slides are special cases of the more general K-*infusion algorithm* from [34], which can be defined as follows. For each integer $a \in \mathbb{Z}$ we introduce a *barred* copy $\overline{a}$, and we set $\overline{\overline{a}} = a$. Let $\overline{\mathbb{Z}} = \{\overline{a} \mid a \in \mathbb{Z}\}$ be the set of all barred integers. If Y is a tableau with values in $\mathbb{Z} \cup \overline{\mathbb{Z}}$, then we let $\overline{Y}$ denote the tableau obtained by replacing $Y(\alpha)$ with $\overline{Y(\alpha)}$ for each $\alpha \in \mathrm{sh}(Y)$. Define an automorphism of the set of all tableaux with values in $\mathbb{Z} \cup \overline{\mathbb{Z}}$ by

$$\Psi = \prod_{a=-\infty}^{\infty} \prod_{b=\infty}^{-\infty} \mathrm{swap}_{\overline{a},b} \,.$$

This is well defined because only finitely many factors of Ψ will result in changes to any given tableau. Furthermore, since $\mathrm{swap}_{\overline{a},b}$ commutes with $\mathrm{swap}_{\overline{a'},b'}$ whenever $a \neq a'$ and $b \neq b'$, the factors of Ψ can be rearranged in any order such that $\mathrm{swap}_{\overline{a},b}$ is applied after $\mathrm{swap}_{\overline{a'},b'}$ whenever $a \leq a'$ and $b \geq b'$. This implies that $\Psi = \prod_{b=\infty}^{-\infty} \prod_{a=-\infty}^{\infty} \mathrm{swap}_{\overline{a},b}$, so we have $\overline{\Psi(Y)} = \Psi^{-1}(\overline{Y})$, hence the map $Y \mapsto \overline{\Psi(Y)}$ is an involution on the set of all tableaux Y with values in $\mathbb{Z} \cup \overline{\mathbb{Z}}$.

Example 3.14. The following tableaux are mapped to each other by the involution.

$$Y = \begin{array}{cccccccc} & & & \overline{2} & & & & \\ & & \overline{1} & \overline{3} & \overline{4} & 1 & & \\ & & \overline{3} & 1 & 2 & 3 & & \\ & & & & 3 & 4 & 5 & \end{array} \qquad ; \qquad \overline{\Psi(Y)} = \begin{array}{cccccccc} & & & \overline{1} & & & & \\ & & \overline{1} & \overline{2} & \overline{3} & \overline{4} & & \\ & & \overline{3} & \overline{4} & \overline{5} & 2 & & \\ & & & & 1 & 3 & 4 & \end{array}$$

Given increasing tableaux S and T such that $\mathrm{sh}(S) = \mu/\lambda$ and $\mathrm{sh}(T) = \nu/\mu$ for straight shapes $\lambda \subset \mu \subset \nu$, we define

$$\mathrm{jdt}_S(T) := \Psi(\overline{S} \cup T)|_{\mathbb{Z}} \quad \text{and} \quad \widehat{\mathrm{jdt}}_T(S) := \Psi^{-1}(S \cup \overline{T})|_{\mathbb{Z}} \,. \tag{1}$$

The following result is proved in [34, Thm. 3.1]. We sketch the short proof for convenience.

Proposition 3.15 (Thomas and Yong)**.** *Let $\lambda \subset \nu$ be straight shapes in Λ. Then the map $(S,T) \mapsto (\mathrm{jdt}_S(T), \widehat{\mathrm{jdt}}_T(S))$ is an involution on the set of all pairs (S,T) of increasing tableaux for which $\mathrm{sh}(S) = \mu/\lambda$ and $\mathrm{sh}(T) = \nu/\mu$ for some straight shape μ. Furthermore, the tableau $\mathrm{jdt}_S(T)$ can be obtained by applying a sequence of forward slides to T, and $\widehat{\mathrm{jdt}}_T(S)$ is obtained by applying a sequence of reverse slides to S.*

Proof. For each $m \in \mathbb{Z}$ we define the tableau

$$Y_m = \left(\prod_{a=m+1}^{\infty} \prod_{b=\infty}^{-\infty} \mathrm{swap}_{\overline{a},b} \right) (\overline{S} \cup T) \,.$$

Then we have $Y_{m-1}|_{\mathbb{Z}} = \mathrm{jdt}_{C_m}(Y_m|_{\mathbb{Z}})$, where $C_m = Y_m^{-1}(\overline{m}) = S^{-1}(m)$ is the set of boxes mapped to m by S. It follows by descending induction on m that $\overline{Y_m}|_{[-\infty,m]}$, $Y_m|_{\mathbb{Z}}$, and $\overline{Y_m}|_{[m+1,\infty]}$ are increasing tableaux, with shapes given by $\mathrm{sh}(\overline{Y_m}|_{[-\infty,m]}) = \mu'_m/\lambda$, $\mathrm{sh}(Y_m|_{\mathbb{Z}}) = \mu_m/\mu'_m$, and $\mathrm{sh}(\overline{Y_m}|_{[m+1,\infty]}) = \nu/\mu_m$, where μ'_m and μ_m are straight shapes satisfying $\lambda \subset \mu'_m \subset \mu_m \subset \nu$. In addition, $Y_m|_{\mathbb{Z}}$ can be obtained from T by a sequence of forward slides. If we choose m smaller than all the integers in S, then $Y_m = \Psi(\overline{S} \cup T)$, so that $\mathrm{jdt}_S(T) = Y_m|_{\mathbb{Z}}$ and $\widehat{\mathrm{jdt}}_T(S) = \overline{Y_m}|_{\mathbb{Z}}$. We deduce that $\mathrm{jdt}_S(T)$ and $\widehat{\mathrm{jdt}}_T(S)$ are increasing tableaux of shapes $\mathrm{sh}(\mathrm{jdt}_S(T)) = \mu_m/\lambda$ and $\mathrm{sh}(\widehat{\mathrm{jdt}}_T(S)) = \nu/\mu_m$, and $\mathrm{jdt}_S(T)$ can be obtained from T by a sequence of forward slides. The identity $\overline{\mathrm{jdt}_S(T)} \cup \widehat{\mathrm{jdt}}_T(S) = \overline{\Psi(\overline{S} \cup T)}$ and the fact that $Y \mapsto \overline{\Psi(Y)}$ is an involution imply that $(S,T) \mapsto (\mathrm{jdt}_S(T), \widehat{\mathrm{jdt}}_T(S))$ is an involution. Finally, set $(T', S') = (\mathrm{jdt}_S(T), \widehat{\mathrm{jdt}}_T(S))$ and notice that $S = \mathrm{jdt}_{T'}(S')$. This shows that S can be obtained by applying a sequence of forward slides to S', or equivalently, $S' = \widehat{\mathrm{jdt}}_T(S)$ is obtained by applying a sequence of reverse slides to S. $\square$

3.5. **Combinatorial K-theory rings.** Assume that Λ is a unique rectification poset. For each straight shape λ in Λ, let G_λ be a symbol, and let $\Gamma(\Lambda) = \bigoplus_\lambda \mathbb{Z}\, G_\lambda$ be the free abelian group generated by these symbols. We will say that a set τ of increasing tableaux is *locally finite* if all tableaux of τ have values in some finite interval $[-c, c]$. Assume that τ is locally finite and closed under (forward and reverse) jeu de taquin slides. For any skew shape ν/λ we let $\tau(\nu/\lambda) = \#\{T \in \tau \mid \mathrm{sh}(T) = \nu/\lambda\}$ be the number of tableaux of shape ν/λ in τ. Define a linear operator $F_\tau : \Gamma(\Lambda) \to \Gamma(\Lambda)$ by $F_\tau(G_\lambda) = \sum_\nu \tau(\nu/\lambda)\, G_\nu$. The assumption that Λ has finitely many minimal boxes and each box of Λ has finitely many covers implies that $F_\tau(G_\lambda)$ is a finite linear combination.

Lemma 3.16. *Let σ and τ be locally finite sets of skew increasing tableaux, each closed under jeu de taquin slides. Then the operators F_σ and F_τ commute.*

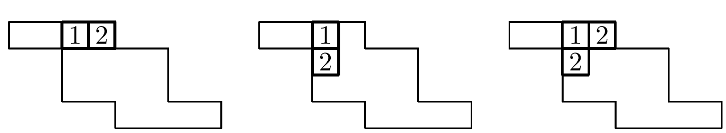
Proof. The coefficient of G_ν in $F_\sigma(F_\tau(G_\lambda))$ is equal to $\sum_\mu \sigma(\nu/\mu)\tau(\mu/\lambda)$, and the coefficient of G_ν in $F_\tau(F_\sigma(G_\lambda))$ is equal to $\sum_\mu \tau(\nu/\mu)\sigma(\mu/\lambda)$. It follows from Proposition 3.15 that these sums are equal to each other. $\square$

For each straight shape $\lambda \subset \Lambda$ we set $F_\lambda = F_{[M_\lambda]}$. It follows from Lemma 3.13 that the coefficient of G_ν in $F_\lambda(G_\mu)$ is the number of increasing tableaux of shape ν/μ that rectify to the minimal increasing tableau M_λ.

Proposition 3.17. *Assume that Λ is a unique rectification poset. Then the bilinear map $\Gamma(\Lambda) \times \Gamma(\Lambda) \to \Gamma(\Lambda)$ defined by $G_\lambda \cdot G_\mu = F_\lambda(G_\mu)$ gives $\Gamma(\Lambda)$ the structure of a commutative and associative ring with unit $1 = G_\emptyset$. Furthermore, for any locally finite set τ of increasing tableaux that is closed under jeu de taquin slides and any $f \in \Gamma(\Lambda)$, we have $F_\tau(f) = F_\tau(1) \cdot f$.*

Proof. It follows from Lemma 3.13 that $F_\lambda(1) = G_\lambda$, and Lemma 3.16 implies that $F_\tau(G_\lambda) = F_\tau(F_\lambda(1)) = F_\lambda(F_\tau(1)) = G_\lambda \cdot F_\tau(1)$. It follows that $F_\tau(f) = f \cdot F_\tau(1)$ for all $f \in \Gamma(\Lambda)$ by linearity. Commutativity follows from this because $G_\lambda \cdot G_\mu = F_\lambda(G_\mu) = G_\mu \cdot F_\lambda(1) = G_\mu \cdot G_\lambda$, and associativity follows because $G_\lambda \cdot (G_\mu \cdot G_\nu) = F_\lambda(F_\mu(F_\nu(1))) = F_\nu(F_\lambda(F_\mu(1))) = G_\nu \cdot (G_\lambda \cdot G_\mu) = (G_\lambda \cdot G_\mu) \cdot G_\nu$. $\square$

Example 3.18. Let $X = E_6/P_6$ be the Cayley plane. Then we have $G_2 \cdot G_2 = G_{(4)} + G_{(3,1)} + G_{(4,1)}$ in $\Gamma(\Lambda_X)$, due to the following tableaux that all rectify to $M_{(2)}$.



The *structure constants* of the ring $\Gamma(\Lambda)$ are the integers $c^\nu_{\lambda,\mu} = c^\nu_{\lambda,\mu}(\Lambda) := [M_\lambda](\nu/\mu)$. Equivalently, the identity $G_\lambda \cdot G_\mu = \sum_\nu c^\nu_{\lambda,\mu}\, G_\nu$ holds in $\Gamma(X)$.

The next two results have appeared earlier when Λ is the partially ordered set associated to a Grassmann variety of type A or a maximal orthogonal Grassmannian, the constants $c^{\nu}_{\lambda,\mu}$ are the K-theoretic Schubert structure constants for this space, and U is a superstandard tableau. Corollary 3.19 was proved in [34] for Grassmannians of type A and in [7, 9] for maximal orthogonal Grassmannians. Corollary 3.20 was obtained in [35].

Corollary 3.19. *Let λ, μ, and ν be straight shapes, and let U be any unique rectification target of shape μ. Then $c^{\nu}_{\lambda,\mu}$ is equal to the number of increasing tableaux of shape ν/λ that rectify to U.*

Proof. This is true because $G_\lambda \cdot G_\mu = G_\lambda \cdot F_{[U]}(1) = F_{[U]}(G_\lambda)$. □

Corollary 3.20. *Let λ, μ, and ν be straight shapes, let T_0 be any increasing tableau of shape μ, and let U be any unique rectification target of shape λ. Then $c^{\nu}_{\lambda,\mu}$ is the number of increasing tableaux T of shape ν/λ for which $\mathrm{jdt}_U(T) = T_0$.*

Proof. It follows from Proposition 3.15 that the map $T \mapsto \widehat{\mathrm{jdt}}_T(U)$ is a bijection from the set of tableaux T of shape ν/λ for which $\mathrm{jdt}_U(T) = T_0$ to the set of tableaux of shape ν/μ that rectify to U. □

Example 3.21. Let $X = E_7/P_7$ be the Freudenthal variety and consider the shapes $\lambda = (5,1)$, $\mu = (5,3,3)$, and $\nu = (5,5,5,2,1,1)$ in Λ_X. Then $c^{\nu}_{\lambda,\mu}(\Lambda_X) = 11$. There are 12 increasing tableaux of shape ν/λ that have S_μ as a rectification, and 10 of these tableaux have S_μ as the only rectification. Similarly, there are 12 increasing tableaux of shape ν/λ that have $\widehat{S}_\mu$ as a rectification, and 10 of these tableaux have $\widehat{S}_\mu$ as the only rectification.

Remark 3.22. Proposition 3.17 would be true with the weaker hypothesis that for each straight shape $\lambda \subset \Lambda$ there exists *some* unique rectification target U_λ of shape λ, and we set $F_\lambda = F_{[U_\lambda]}$. However, all partially ordered sets Λ with this property that we know about are unique rectification posets.

Remark 3.23. Let $\Lambda' \subset \Lambda$ be any lower order ideal and T an increasing tableau of straight shape $\mathrm{sh}(T) \subset \Lambda'$. If T is a unique rectification target for Λ then T is also a unique rectification target for Λ'. In particular, if Λ is a unique rectification poset, then so is Λ'.

Remark 3.24. Proctor has defined a notion of *d-complete posets* that generalize the partially ordered sets associated to minuscule varieties. He proves in [26] that every d-complete poset has the *jeu de taquin property*, i.e. the rectification of any standard skew tableau with Schützenberger's cohomological jeu de taquin algorithm is independent of choices. It would be interesting to know if all d-complete posets are also unique rectification posets. We have checked that this is the case for a few d-complete posets that are not associated to minuscule varieties. We note that Problem 9.1 in [34] similarly asks which tableaux of straight shape in a d-complete poset are unique rectification targets.

4. K-theory of minuscule varieties

4.1. **The Grothendieck ring.** In this section we let $X = G/P$ be a minuscule variety and Λ_X the associated partially ordered set (see Remark 2.5). Theorem 3.12 and Proposition 3.17 imply that the combinatorial K-theory ring $\Gamma(\Lambda_X)$ is a well defined associative ring. Let $K(X)$ be the Grothendieck ring of algebraic vector bundles on X. A brief description of this ring can be found in [7, §2]. In this section we first show that the ring $K(X)$ is isomorphic to $\Gamma(\Lambda_X)$. We then use the geometry of X to prove results about about increasing tableaux for Λ_X.

For each straight shape $\lambda \subset \Lambda_X$ we let $\mathcal{O}_\lambda := [\mathcal{O}_{X_\lambda}] \in K(X)$ denote the Grothendieck class of the Schubert variety X_λ. The next result shows that the K-theoretic Schubert structure constants of X are the integers $(-1)^{|\nu|-|\lambda|-|\mu|}\, c^\nu_{\lambda,\mu}(\Lambda_X)$, i.e. the identity $\mathcal{O}_\lambda \cdot \mathcal{O}_\mu = \sum_\nu (-1)^{|\nu|-|\lambda|-|\mu|}\, c^\nu_{\lambda,\mu}(\Lambda_X)\, \mathcal{O}_\nu$ holds in $K(X)$.

Theorem 4.1. *Let X be a minuscule variety. Then the $\mathbb{Z}$-linear isomorphism $\varphi : \Gamma(\Lambda_X) \to K(X)$ defined by $G_\lambda \mapsto (-1)^{|\lambda|}\, \mathcal{O}_{X_\lambda}$ is an isomorphism of rings.*

Proof. It is enough to show that there exist straight shapes $\mu_1, \dots, \mu_k$ and $\nu_1, \dots, \nu_\ell$ in Λ_X such that the following conditions hold.

(a) $\varphi(G_{\mu_j} \cdot G_\lambda) = \varphi(G_{\mu_j}) \cdot \varphi(G_\lambda)$ for every straight shape $\lambda \subset \Lambda_X$ and $1 \leq j \leq k$.

(b) $\varphi(G_{\nu_i} \cdot G_{\nu_j}) = \varphi(G_{\nu_i}) \cdot \varphi(G_{\nu_j})$ for all $1 \leq i \leq j \leq \ell$.

(c) The cohomology ring $H^*(X;\mathbb{Q})$ is spanned by the classes $1, [X_{\nu_1}], \dots, [X_{\nu_\ell}]$ as a module over the subring of $H^*(X;\mathbb{Q})$ generated by $[X_{\mu_1}], \dots, [X_{\mu_k}]$.

To see that these conditions are sufficient, notice first that since the cohomology ring $H^*(X;\mathbb{Q})$ is the associated graded ring of $K(X)_\mathbb{Q} := K(X) \otimes_\mathbb{Z} \mathbb{Q}$ modulo Grothendieck's gamma filtration (see examples 15.1.5 and 15.2.16 in [13]), condition (c) implies that $K(X)_\mathbb{Q}$ is spanned by $\mathcal{O}_{\nu_0}, \dots, \mathcal{O}_{\nu_\ell}$ as a module over the subring $R = \mathbb{Q}[\mathcal{O}_{\mu_1}, \dots, \mathcal{O}_{\mu_k}] \subset K(X)_\mathbb{Q}$, where we set $\nu_0 = \emptyset$ so that $\mathcal{O}_{\nu_0} = 1$. Set $S = \mathbb{Q}[G_{\mu_1}, \dots, G_{\mu_k}] \subset \Gamma(\Lambda_X)_\mathbb{Q}$. Then (a) implies that $\varphi : S \to R$ is an isomorphism of rings, and that $\varphi(s \cdot f) = \varphi(s) \cdot \varphi(f)$ for all $s \in S$ and $f \in \Gamma(\Lambda_X)_\mathbb{Q}$. It follows that $\Gamma(\Lambda_X)_\mathbb{Q}$ is spanned by $G_{\nu_0}, \dots, G_{\nu_\ell}$ as an S-module. Finally, given $f, g \in \Gamma(\Lambda_X)_\mathbb{Q}$ we may write $f = \sum a_i G_{\nu_i}$ and $g = \sum b_j G_{\nu_j}$ with $a_i, b_j \in S$. Using (b) we obtain

$$\varphi(fg) = \sum_{i,j} \varphi(a_i)\varphi(b_j)\varphi(G_{\nu_i}G_{\nu_j}) = \sum_{i,j} \varphi(a_i)\varphi(b_j)\varphi(G_{\nu_i})\varphi(G_{\nu_j}) = \varphi(f)\varphi(g)\,,$$

as required.

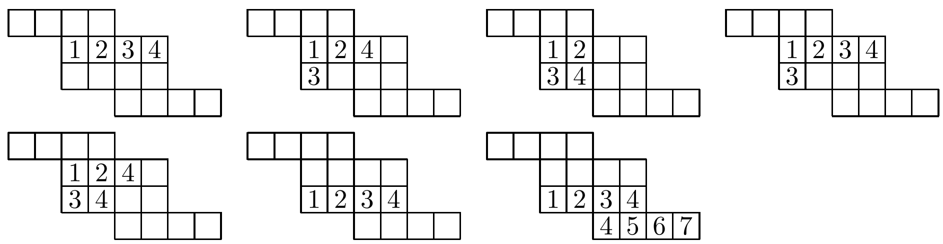
Assume first that $X = \mathrm{Gr}(m,n)$ is a Grassmannian of type A. In this case we can use the shapes $\mu_i = (i)$ for $1 \leq i \leq n-m$. Since the special Schubert classes $[X_{\mu_i}]$ generate $H^*(X;\mathbb{Q})$, no shapes ν_j are required for property (c). Property (a) follows by comparing Lenart's Pieri rule [21, Thm. 3.4] for products of the form $\mathcal{O}_{(i)} \cdot \mathcal{O}_\lambda$ in $K(X)$ to the corresponding Pieri rule for $\Gamma(\Lambda_X)$ given in Corollary 6.7.

Assume next that $X = \mathrm{OG}(n, 2n)$ is a maximal orthogonal Grassmannian. We use $\mu_i = (i)$ for $1 \leq i \leq n-1$; no shapes ν_j are required. Property (a) follows by comparing the K-theoretic Pieri rule for $K(X)$ proved in [7, Cor. 4.8] to Corollary 7.11.

Assume that $X = E_6/P_6$ is the Cayley Plane. Here we use $\mu_1 = (1)$, $\nu_1 = (4)$, and $\nu_2 = (4,4)$. Property (a) can be verified using Lenart and Postnikov's K-theoretic Chevalley formula [22]. The Betti numbers $d_i := \dim H^{2i}(X;\mathbb{Q})$ of X are given by $d_i = 1$ for $0 \leq i \leq 3$, $d_i = 2$ for $4 \leq i \leq 7$, $d_8 = 3$, and $d_{16-i} = d_i$ for $0 \leq i \leq 16$. Using the Chevalley formula, one can check that $[X_{(4)}] \notin [X_{(1)}] \cdot H^6(X;\mathbb{Q})$ and that $[X_{(4,4)}] \notin [X_{(1)}] \cdot H^{14}(X;\mathbb{Q})$. Property (c) therefore follows from the Hard Lefschetz theorem. We have computed the products $\mathcal{O}_{\nu_i} \cdot \mathcal{O}_{\nu_j}$ in the equivariant K-theory ring $K_T(X)$, where $T \subset G$ is a maximal torus. This was done with help from a computer, by utilizing that the restriction map $K_T(X) \to K_T(X^T)$ to the T-fixed points of X is an injective ring homomorphism [18]. Formulas for the restrictions of Schubert classes to T-fixed points can be found in [14, 36, 17]. The products $\mathcal{O}_{\nu_i} \cdot \mathcal{O}_{\nu_j}$ in the ordinary K-theory ring are given by

$$\begin{aligned} \mathcal{O}_{(4)} \cdot \mathcal{O}_{(4)} &= \mathcal{O}_{(4,4)} + \mathcal{O}_{(4,3,1)} + \mathcal{O}_{(4,2,2)} - \mathcal{O}_{(4,4,1)} - \mathcal{O}_{(4,3,2)} \ , \\ \mathcal{O}_{(4)} \cdot \mathcal{O}_{(4,4)} &= \mathcal{O}_{(4,4,4)} \ , \\ \mathcal{O}_{(4,4)} \cdot \mathcal{O}_{(4,4)} &= \mathcal{O}_{(4,4,4,4)} \ . \end{aligned}$$

Property (b) follows from this by observing that the following seven tableaux are the only increasing tableaux that contribute to these products.



If $X = E_7/P_7$ is the Freudenthal variety, then the proof is analogous to the argument given for the Cayley plane. We use $\mu_1 = (1)$, $\nu_1 = (5)$, and $\nu_2 = (5,4)$. The Betti numbers $d_i := \dim H^{2i}(X;\mathbb{Q})$ are given by $d_i = 1$ for $0 \leq i \leq 4$, $d_i = 2$ for $5 \leq i \leq 8$, $d_i = 3$ for $9 \leq i \leq 18$, and $d_{27-i} = d_i$ for $0 \leq i \leq 27$. The products $\mathcal{O}_{\nu_i} \cdot \mathcal{O}_{\nu_j}$ can be computed with equivariant methods and are given by

$$\begin{aligned} \mathcal{O}_{(5)} \cdot \mathcal{O}_{(5)} &= 2\,\mathcal{O}_{(5,4,1)} + 2\,\mathcal{O}_{(5,3,2)} - 3\,\mathcal{O}_{(5,4,2)} - \mathcal{O}_{(5,3,3)} + \mathcal{O}_{(5,4,3)} \ , \\ \mathcal{O}_{(5)} \cdot \mathcal{O}_{(5,4)} &= 2\,\mathcal{O}_{(5,5,4)} + 2\,\mathcal{O}_{(5,5,3,1)} + \mathcal{O}_{(5,4,4,1)} - 4\,\mathcal{O}_{(5,5,4,1)} \ , \\ \mathcal{O}_{(5,4)} \cdot \mathcal{O}_{(5,4)} &= 2\,\mathcal{O}_{(5,5,5,2,1)} + 2\,\mathcal{O}_{(5,5,4,2,1,1)} - 3\,\mathcal{O}_{(5,5,5,2,1,1)} \ . \end{aligned}$$

We leave it as an exercise to identify the corresponding 25 increasing tableaux.

Finally, assume that $X = Q^{2n} \subset \mathbb{P}^{2n+1}$ is an even quadric hypersurface. Fix the orthogonal form on $\mathbb{C}^{2n+2}$ defined by $(e_i, e_j) = \delta_{i+j,2n+3}$. Then $X = \{L \in \mathbb{P}^{2n+1} \mid (L,L) = 0\}$. For $0 \leq p \leq 2n$ we let $X_p \subset X$ be the subvariety defined by

$$X_p = \begin{cases} \mathbb{P}^{2n+1-p} \cap X & \text{if } 0 \leq p < n, \\ \mathbb{P}^{2n-p} & \text{if } n \leq p \leq 2n, \end{cases}$$

where $\mathbb{P}^m \subset \mathbb{P}^{2n-1}$ denotes the linear subspace spanned by $e_1, \dots, e_{m+1}$. Let $X'_n \subset \mathbb{P}^{2n+1}$ be the linear subspace spanned by $e_1, \dots, e_n, e_{n+2}$. Then $X_p \subset X$ is the unique Schubert variety of codimension p when $p \neq n$, while X_n and X'_n are the Schubert varieties of codimension n. We claim that multiplication with the divisor class $[\mathcal{O}_{X_1}]$ in $K(X)$ is determined by $[\mathcal{O}_{X_1}] \cdot [\mathcal{O}_{X'_n}] = [\mathcal{O}_{X_{n+1}}]$ and

$$[\mathcal{O}_{X_1}] \cdot [\mathcal{O}_{X_p}] = \begin{cases} [\mathcal{O}_{X_{p+1}}] & \text{if } p \notin \{n-1, 2n\}; \\ [\mathcal{O}_{X_n}] + [\mathcal{O}_{X'_n}] - [\mathcal{O}_{X_{n+1}}] & \text{if } p = n-1; \\ 0 & \text{if } p = 2n. \end{cases} \tag{2}$$

To see this, consider the linear map $i_* : K(X) \to K(\mathbb{P}^{2n+1})$ defined by the inclusion $i : X \subset \mathbb{P}^{2n+1}$. If we let $h = [\mathcal{O}_H] \in K(\mathbb{P}^{2n+1})$ be the class of a hyperplane, then we have $i_*[\mathcal{O}_{X_p}] = h^p(2h - h^2)$ for $0 \leq p < n$ and $i_*[\mathcal{O}_{X_p}] = h^{p+1}$ for $n \leq p \leq 2n$. Since the projection formula implies that $i_*([\mathcal{O}_{X_1}] \cdot [\mathcal{O}_{X_p}]) = h \cdot i_*[\mathcal{O}_{X_p}]$, it follows that we obtain a valid identity in $K(\mathbb{P}^{2n+1})$ after applying i_* to both sides of (2).

The claim now follows because $\mathrm{Ker}(i_*) = \mathbb{Z}\,([\mathcal{O}_{X_n}] - [\mathcal{O}_{X'_n}])$ and both sides of (2) are invariant under the involution of $K(X)$ that interchanges $[\mathcal{O}_{X_n}]$ and $[\mathcal{O}_{X'_n}]$.

We also claim that

$$[\mathcal{O}_{X_n}] \cdot [\mathcal{O}_{X_n}] = \begin{cases} [\mathcal{O}_{X_{2n}}] = [\mathcal{O}_{\text{point}}] & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{3}$$

This follows because a variety $\mathbb{P}(E) \subset \mathbb{P}^{2n+1}$ is a translate of X_n under the action of $G = \mathrm{SO}(2n+2)$ if and only if $E \subset \mathbb{C}^{2n+2}$ is a linear subspace of dimension $n+1$ such that $\dim(E \cap \mathbb{C}^{n+1}) \equiv n+1 \pmod 2$. Since the opposite Schubert variety X_n^{op} is a G-translate of X_n, we obtain

$$X_n^{\mathrm{op}} = \begin{cases} \mathbb{P}(\mathrm{Span}\{e_{n+1}, e_{n+3}, \dots, e_{2n+2}\}) & \text{if } n \text{ is even,} \\ \mathbb{P}(\mathrm{Span}\{e_{n+2}, \dots, e_{2n+2}\}) & \text{if } n \text{ is odd,} \end{cases}$$

so $X_n \cap X_n^{\mathrm{op}} = \{\mathbb{C}\, e_{n+1}\}$ is a single point when n is even, while $X_n \cap X_n^{\mathrm{op}} = \emptyset$ when n is odd. We leave it to the reader to check that properties (a), (b), and (c) for the shapes $\mu_1 = (1)$ and $\nu_1 = (n)$ follow from (2) and (3). □

Remark 4.2. The products $\mathcal{O}_{\nu_i} \cdot \mathcal{O}_{\nu_j}$ in the K-theory rings of the Cayley plane E_6/P_6 and the Freudenthal variety E_7/P_7 could in theory be computed as an application of the K-theoretic Chevalley formula from [22]. However, this would require calculations in the much larger rings $K(E_n/B)$ where B denotes a Borel subgroup. It is not clear if this is possible with ordinary computers even for the Cayley plane.

4.2. **Duality.** The sheaf Euler characteristic map $\chi_X : K(X) \to \mathbb{Z}$ is the $\mathbb{Z}$-linear map defined by $\chi_X(\mathcal{O}_\lambda) = 1$ for all straight shapes λ. Since non-empty Richardson varieties are rational [28] and have rational singularities [2], we deduce from Corollary 2.8 that we have

$$\chi_X(\mathcal{O}_\lambda \cdot \mathcal{O}_\mu) = \begin{cases} 1 & \text{if } \lambda \subset \mu^\vee; \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

for all pairs of straight shapes λ and μ (see [7, §2] for more details).

A subset $\theta \subset \Lambda_X$ is called a *rook strip* if no two boxes of θ are comparable by the partial order $\leq$. Equivalently, θ is a skew shape such that no two boxes of θ belong to the same row or column when θ is identified with a subset of $\mathbb{N}^2$. Define the *dual Grothendieck class* of X_λ by

$$\mathcal{O}^*_\lambda \;=\; \sum_{\nu/\lambda^\vee \text{ rook strip}} (-1)^{|\nu/\lambda^\vee|}\, \mathcal{O}_\nu \;\in K(X) \tag{5}$$

where the sum is over all straight shapes ν containing $\lambda^\vee$ such that $\nu/\lambda^\vee$ is a rook strip. The identity (4) implies that

$$\chi_X(\mathcal{O}_\lambda \cdot \mathcal{O}^*_\mu) = \delta_{\lambda,\mu}\,. \tag{6}$$

We remark that the identities (4), (5), (6) are valid on any cominuscule homogeneous space. However, the following result fails for the the Lagrangian Grassmannian $\mathrm{LG}(n,2n)$ by [7, Example 5.7]. Set $c^\nu_{\lambda,\mu} = c^\nu_{\lambda,\mu}(\Lambda_X)$.

Theorem 4.3. *Let X be a minuscule variety and let λ, μ, and ν be straight shapes. Then we have $\mathcal{O}^*_\lambda = (1 - \mathcal{O}_{(1)}) \cdot \mathcal{O}_{\lambda^\vee}$ and $c^\nu_{\lambda,\mu} = c^{\mu^\vee}_{\lambda,\nu^\vee}$.*

Proof. The identity $\mathcal{O}^*_\lambda = (1-\mathcal{O}_{(1)})\cdot\mathcal{O}_{\lambda^\vee}$ follows from Theorem 4.1. We obtain

$$(-1)^{|\nu|-|\lambda|-|\mu|}\, c^\nu_{\lambda,\mu} = \chi_X(\mathcal{O}_\lambda\cdot\mathcal{O}_\mu\cdot\mathcal{O}^*_\nu) = \chi_X((1-\mathcal{O}_{(1)})\cdot\mathcal{O}_\lambda\cdot\mathcal{O}_\mu\cdot\mathcal{O}_{\nu^\vee})\,,$$

which implies that $c^\nu_{\lambda,\mu} = c^{\mu^\vee}_{\lambda,\nu^\vee}$. □

Theorem 4.3 implies that the constants $c(\lambda,\mu,\nu) = c^{\nu^\vee}_{\lambda,\mu}$ are invariant under arbitrary permutations of λ, μ, and ν. This was known for Grassmannians of type A [4, Cor. 1] and for maximal orthogonal Grassmannians [7, Cor. 4.6].

4.3. **Unique rectification targets.** We will say that a subset of Λ_X is an *anti-straight shape* if it is an upper order ideal. Any anti-straight shape has the form $w_X.\lambda = \Lambda_X/\lambda^\vee$ for some straight shape λ (see §2). An *anti-rectification* of an increasing tableau T is any tableau of anti-straight shape that can be obtained by applying a sequence of reverse slides to T.

We require the following involution on the set of all increasing tableaux. If T is an increasing tableau of shape ν/λ, then we let $w_X.T$ be the increasing tableau of shape $w_X.(\nu/\lambda) = \lambda^\vee/\nu^\vee$ defined by $(w_X.T)(\alpha) = -T(w_X.\alpha)$. This involution commutes with jeu de taquin slides in the sense that $w_X.\,\mathrm{jdt}_C(T) = \widehat{\mathrm{jdt}}_{w_X.C}(w_X.T)$.

Theorem 4.4. *Let T be an increasing tableau of straight shape. The following are equivalent.*
(a) *T is a unique rectification target.*
(b) *T has exactly one anti-rectification.*
(c) *The jeu de taquin class $[T]$ contains exactly one tableau of straight shape.*
(d) *The jeu de taquin class $[T]$ contains exactly one tableau of anti-straight shape.*
(e) *All tableaux of $[T]$ can be obtained by applying a sequence of reverse slides to T.*

Proof. It follows from Lemma 3.13 that (a) ⇒ (c), and we clearly have (d) ⇒ (b) and (c) ⇒ (e) ⇒ (a). We will prove the implications (b) ⇒ (a) ⇒ (d).

Assume that T is a unique factorization target and let $T' \in [T]$ be any tableau of anti-straight shape. Define the straight shapes $\lambda = \mathrm{sh}(T)$ and $\mu = w_X.\,\mathrm{sh}(T')$. Then Corollary 3.19 and Theorem 4.3 imply that $c^\mu_{\lambda,\emptyset} = c^{\Lambda_X}_{\lambda,\mu^\vee} > 0$, so we must have $\mu = \lambda$. Furthermore, since $c^\lambda_{\lambda,\emptyset} = 1$ we deduce that T' is the only tableau of anti-straight shape in $[T]$. This proves the implication (a) ⇒ (d).

Assume that (b) holds and let T' be the unique anti-rectification of T. Set $\lambda = \mathrm{sh}(T)$ and let U be any unique rectification target of shape $\lambda^\vee$. Since $c^{\Lambda_X}_{\lambda,\lambda^\vee} = c^\lambda_{\lambda,\emptyset} = 1$, we deduce from Corollary 3.20 that T' has shape $\Lambda_X/\lambda^\vee$. Let S be any rectification of T'. Then $S = \mathrm{jdt}_Z(T')$ for some increasing tableau Z of shape $\lambda^\vee$. To show that T is a unique rectification target, it suffices to prove that $S = T$.

Choose $a \in \mathbb{N}$ such that Z has values in the interval $[-a,a]$. If Z' is any increasing tableau of shape Λ_X/λ, then since $\widehat{\mathrm{jdt}}_{Z'}(T) = T'$ has shape $\Lambda_X/\lambda^\vee$, it follows from Proposition 3.15 that $\mathrm{jdt}_T(Z')$ has shape $\lambda^\vee$. Furthermore, the assignment $Z' \mapsto \mathrm{jdt}_T(Z')$ defines an injective map from the set of increasing tableaux of shape Λ_X/λ with values in $[-a,a]$ to the set of increasing tableaux of shape $\lambda^\vee$ with values in $[-a,a]$. Now since $Z' \mapsto w_X.Z'$ is a bijection between the same two sets, we deduce from the pigeonhole principle that the map $Z' \mapsto \mathrm{jdt}_T(Z')$ is surjective. In particular, we can choose Z' such that $\mathrm{jdt}_T(Z') = Z$. Since $\widehat{\mathrm{jdt}}_{Z'}(T) = T'$, another application of Proposition 3.15 shows that $T = \mathrm{jdt}_Z(T') = S$, as required. □

4.4. **Minimal and maximal increasing tableaux.** For any skew shape ν/λ, define the *minimal increasing tableau* $M_{\nu/\lambda}$ to be the unique increasing tableau that fills the boxes of ν/λ with the smallest possible positive integers. More precisely, $M_{\nu/\lambda}(\alpha)$ is the maximal cardinality of a totally ordered subset of ν/λ with α as its largest element. Define the *maximal increasing tableau* of shape ν/λ by $\widehat{M}_{\nu/\lambda} = w_X.M_{\lambda^\vee/\nu^\vee}$. This tableau fills the boxes of ν/λ with the largest possible *negative* integers.

Example 4.5. Let $X = \mathrm{Gr}(m,n)$ be a Grassmann variety of type A and set $\theta = (9,7,6,6,4)/(5,3,2)$. Then we have

$$M_\theta = \begin{array}{ccccccccc} & & & & & 1 & 2 & 3 & 4 \\ & & & 1 & 2 & 3 & 4 & & \\ & & 1 & 2 & 3 & 4 & & & \\ 1 & 2 & 3 & 4 & 5 & 6 & & & \\ 2 & 3 & 4 & 5 & & & & & \end{array} \quad \text{and} \quad \widehat{M}_\theta = \begin{array}{ccccccccc} & & & & & -4 & -3 & -2 & -1 \\ & & & -5 & -4 & -3 & -1 & & \\ & & -5 & -4 & -3 & -2 & & & \\ -6 & -5 & -4 & -3 & -2 & -1 & & & \\ -4 & -3 & -2 & -1 & & & & & \end{array} .$$

The set of all minimal increasing tableaux is not closed under jeu de taquin slides. However, the rectification of any minimal increasing tableau is a minimal increasing tableau.

Theorem 4.6. *Let $\nu/\lambda \subset \Lambda_X$ be any skew shape. Then $M_{\nu/\lambda}$ has a unique rectification, which is the minimal increasing tableau M_μ of some straight shape μ.*

Proof. By Theorem 3.12 and Lemma 3.13 it is enough to show that the jeu de taquin class $[M_{\nu/\lambda}]$ contains at least one tableau of the form M_μ. This is proved in Theorem 6.16 if X is a Grassmannian of type A, and in Corollary 7.3 if X is a maximal orthogonal Grassmannian. The result is an easy exercise if X is a quadric hypersurface, and it has been verified by computer when X is the Cayley plane or the Freudenthal variety. $\square$

Corollary 4.7. *Let $\lambda \subset \Lambda_X$ be any straight shape. Then the unique anti-rectification of M_λ is $M_{w_X.\lambda}$.*

Proof. Let M_μ be the rectification of $M_{w_X.\lambda}$. Then $c^\lambda_{\mu,\emptyset} = c^{\Lambda_X}_{\mu,\lambda^\vee} > 0$, so we must have $\mu = \lambda$. Theorem 4.4 shows that $M_{w_X.\lambda}$ is the only anti-rectification of M_λ. $\square$

Corollary 4.8. *If λ is a straight shape, then $\widehat{M}_\lambda$ is a unique rectification target.*

Proof. Since the action of w_X on tableaux is compatible with jeu de taquin slides, we deduce from Corollary 4.7 that $w_X.[M_\lambda] = w_X.[M_{w_X.\lambda}] = [w_X.M_{w_X.\lambda}] = [\widehat{M}_\lambda]$. The result therefore follows from Theorem 4.4 and the fact that M_λ is a unique rectification target. $\square$

Given an increasing tableau T of shape ν/λ, define the *greedy rectification* $\mathrm{rect}(T)$ as follows. If $\lambda = \emptyset$ then set $\mathrm{rect}(T) = T$. Otherwise define $\mathrm{rect}(T) = \mathrm{rect}(\mathrm{jdt}_C(T))$, where C is the set of all maximal boxes in λ.

Corollary 4.9. *Let λ, μ, and ν be straight shapes in Λ_X and let T_0 be any increasing tableau of shape μ. Then $c^\nu_{\lambda,\mu}$ is the number of increasing tableaux T of shape ν/λ for which $\mathrm{rect}(T) = T_0$.*

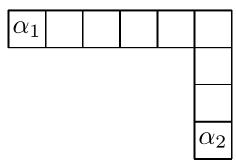
Proof. This follows from corollaries 3.20 and 4.8 because $\mathrm{rect}(T) = \mathrm{jdt}_{\widehat{M}_\lambda}(T)$. $\square$

5. K-Knuth equivalence of hook-closed tableaux

In this section we introduce our main technical tool, the *K-Knuth equivalence relation* on words of integers. This relation governs jeu de taquin equivalence of increasing tableaux for Grassmannians of type A and will be used to show that minimal increasing tableaux for these varieties are unique rectification targets. A weak version of K-Knuth equivalence will be defined in section 7.2, which plays a similar role for maximal orthogonal Grassmannians.

Given two boxes $\alpha_1 = [i_1, j_1]$ and $\alpha_2 = [i_2, j_2]$ in $\mathbb{N}^2$, we will say that α_1 is *north of* α_2 if $i_1 \leq i_2$, and we will say that α_1 is *strictly north of* α_2 if $i_1 < i_2$. Similar terminology will be used for the other directions, where the boxes in $\mathbb{N}^2$ are arranged as in section 2. *North-east of* means north of and east of, and *strictly north-east* means strictly north and strictly east. Notice that α_1 is north-west of α_2 if and only if $\alpha_1 \leq \alpha_2$.

A finite subset $\theta \subset \mathbb{N}^2$ will be called a *hook-closed shape* if, whenever θ contains two boxes $\alpha_1 = [i_1, j_1]$ and $\alpha_2 = [i_2, j_2]$ such that $\alpha_1 \leq \alpha_2$, θ also contains the *north-east hook* spanned by α_1 and α_2. More precisely, θ must contain all boxes $[i_1, c]$ for $j_1 \leq c \leq j_2$ and all boxes $[r, j_2]$ for $i_1 \leq r \leq i_2$.



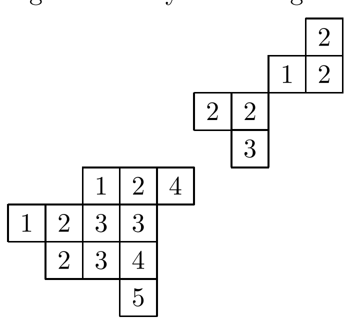
Examples of hook-closed shapes include skew shapes for Grassmannians of type A and maximal orthogonal Grassmannians.

Let θ be a hook-closed shape. A *weakly increasing tableau* T of shape θ is a map $T : \theta \to \mathbb{Z}$ such that all rows of T are weakly increasing from left to right and all columns of T are weakly increasing from top to bottom. The tableau T is *strictly increasing* (or simply *increasing*) if all rows and columns of T are strictly increasing.

Definition 5.1. Let T be a weakly increasing tableaux of hook-closed shape. A *reading word* of T is any word listing the boxes of T in any order for which (i) each box α appears before all boxes north-east of α, (ii) if a box α is equal to the box immediately above α, then α appears before all boxes strictly north of α, and (iii) if a box α is equal to the box immediately to the right of α, then α appears before all boxes strictly east of α.

Notice that some weakly increasing tableaux have no reading words.

Example 5.2. The following is a weakly increasing tableau of hook-closed shape.



This tableau has the following four reading words (and no others):

$$(2,3,1,2,3,1,5,4,3,2,4,2,3,2,1,2,2)\,,\ (2,3,1,2,3,5,1,4,3,2,4,2,3,2,1,2,2)$$
$$(2,3,1,2,3,5,4,1,3,2,4,2,3,2,1,2,2)\,,\ (2,3,1,2,3,5,4,3,1,2,4,2,3,2,1,2,2)\,.$$

Definition 5.3. Define the *K-Knuth equivalence relation* on words of integers, denoted by $\equiv$, as the symmetric transitive closure of the following basic relations. For any words of integers $\mathbf{u}=(u_1,\dots,u_s)$ and $\mathbf{v}=(v_1,\dots,v_t)$, and $a,b,c\in\mathbb{Z}$ we set

$$\begin{aligned}(\mathbf{u},a,a,\mathbf{v}) &\equiv (\mathbf{u},a,\mathbf{v})\,,\\ (\mathbf{u},a,b,a,\mathbf{v}) &\equiv (\mathbf{u},b,a,b,\mathbf{v})\,,\\ (\mathbf{u},a,b,c,\mathbf{v}) &\equiv (\mathbf{u},a,c,b,\mathbf{v}) \quad \text{if } b<a<c\,, \text{ and}\\ (\mathbf{u},a,b,c,\mathbf{v}) &\equiv (\mathbf{u},b,a,c,\mathbf{v}) \quad \text{if } a<c<b\,.\end{aligned}$$

Lemma 5.4. *Let T be a weakly increasing tableau of hook-closed shape. Then all reading words of T are K-Knuth equivalent.*

Proof. Let $\mathbf{u}$ and $\mathbf{v}$ be reading words of T. We prove that $\mathbf{u}\equiv\mathbf{v}$ by induction on the number of boxes in T. Write $\mathbf{u}=(x,\mathbf{u}')$ and $\mathbf{v}=(y,\mathbf{v}')$, where x and y are the first integers in the words. We will identify x and y with the boxes of T that they came from. We may assume that x is strictly north-west of y. Since the shape of T is hook-closed, the tableau $T\smallsetminus\{x,y\}$, obtained by removing the boxes x and y from T, has a south-west corner z, such that z is south-east of x and north-west of y. We must have $x<z<y$, since if one of these inequalities is not strict, then either $\mathbf{v}$ or $\mathbf{u}$ is not a valid reading word of T. Let $\mathbf{w}$ be any reading word of $T\smallsetminus\{x,y,z\}$; for example, such a word can be obtained by removing the first occurrences of x, y, and z from $\mathbf{u}$. Since $\mathbf{u}'$ and $(y,z,\mathbf{w})$ are reading words for $T\smallsetminus\{x\}$, it follows by induction that $\mathbf{u}'\equiv(y,z,\mathbf{w})$, and since $\mathbf{v}'$ and $(x,z,\mathbf{w})$ are reading words for $T\smallsetminus\{y\}$ we similarly obtain $\mathbf{v}'\equiv(x,z,\mathbf{w})$. We deduce that $\mathbf{u}=(x,\mathbf{u}')\equiv(x,y,z,\mathbf{w})\equiv(y,x,z,\mathbf{w})\equiv(y,\mathbf{v}')=\mathbf{v}$, as required. $\square$

We will say that two weakly increasing tableaux T_1 and T_2 are K-Knuth equivalent, written $T_1\equiv T_2$, if a reading word of T_1 is K-Knuth equivalent to a reading word of T_2. This defines an equivalence relation on the set of all weakly increasing tableaux of hook-closed shape that admit reading words.

Lemma 5.5. *Let $[a,b]$ be an integer interval.*

(a) *Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be K-Knuth equivalent words. For $i=1,2$, let $\mathbf{w}_i'$ be the word obtained from $\mathbf{w}_i$ by skipping all integers not contained in the interval $[a,b]$. Then $\mathbf{w}_1'$ and $\mathbf{w}_2'$ are K-Knuth equivalent words.*

(b) *Let T_1 and T_2 be K-Knuth equivalent weakly increasing tableaux of hook-closed shapes. For $i=1,2$, let $T_i|_{[a,b]}$ denote the tableau obtained from T_i by removing all boxes not in $[a,b]$. Then $T_1|_{[a,b]}$ and $T_2|_{[a,b]}$ are K-Knuth equivalent weakly increasing tableaux of hook-closed shapes.*

Proof. This is immediate from the definitions. $\square$

Definition 5.6. Let θ be a hook-closed shape. A *dotted increasing tableau* of shape θ is a map $T_0:\theta\to\mathbb{Z}\cup\{\bullet\}$ such that there exists an integer k with the property that T_0 becomes a strictly increasing tableau (of rational numbers) if all dots in T_0 are replaced with the number $k+\frac{1}{2}$. A *resolution* of a dotted increasing tableau T_0

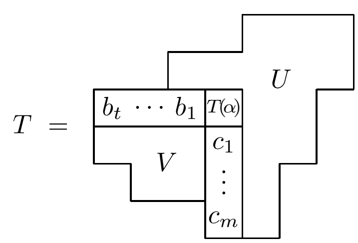
is any tableau obtained by replacing each dot in T_0 with either the maximum of the boxes immediately above and to the left of the dot, or the minimum of the boxes immediately below and to the right of the dot. If there are no boxes immediately above or to the left of a dot, or if there are no boxes immediately below or to the right of a dot, then the box containing the dot may be removed.

Example 5.7. The following dotted increasing tableau is displayed next to one of its resolutions.

$$\begin{array}{cccccccc} & & & & 1 & 3 & \bullet & 8 \\ & & 2 & 3 & 4 & 6 & 9 & \\ & 1 & 3 & 5 & 7 & \bullet & & \\ & 2 & \bullet & 8 & 9 & & & \\ 2 & \bullet & 8 & 9 & & & & \end{array} \qquad \begin{array}{cccccccc} & & & & 1 & 3 & 8 & 8 \\ & & 2 & 3 & 4 & 6 & 9 & \\ & 1 & 3 & 5 & 7 & & & \\ & 2 & 8 & 8 & 9 & & & \\ 2 & 2 & 8 & 9 & & & & \end{array}$$

Lemma 5.8. *If T is a resolution of a dotted increasing tableau, then T has at least one reading word.*

Proof. Let T_0 be a dotted increasing tableau that has T as a resolution. If T_0 contains no dots, then conditions (ii) and (iii) of Definition 5.1 are vacuous and the lemma is clear. Otherwise let α be the south-west most box of T_0 that contains a dot, and let U (respectively U_0) be the tableau of boxes in T (respectively T_0) that are strictly north or strictly east of α. Since U is a resolution of U_0, it follows by induction on the number of dots in T_0 that U admits a reading word $\mathbf{u}$. We also let V be the tableau of boxes in T that are strictly south-west of α, and let $\mathbf{b} = (b_t, b_{t-1}, \dots, b_1)$ be the word listing the boxes to the left of α and $\mathbf{c} = (c_m, c_{m-1}, \dots, c_1)$ the word of boxes below α.



Since T_0 is a dotted increasing tableau, it follows that $b_1 < c_1$. Let $\mathbf{v}$ be a reading word of the strictly increasing tableau V. If $b_1 = T(\alpha)$, then $(\mathbf{v}, \mathbf{b}, \mathbf{c}, T(\alpha), \mathbf{u})$ is a reading word for T, and otherwise $(\mathbf{v}, \mathbf{c}, \mathbf{b}, T(\alpha), \mathbf{u})$ is a reading word. $\square$

The following result will be used to show that K-theoretic jeu de taquin slides preserve K-Knuth equivalence in type A.

Theorem 5.9. *Let T_0 be a dotted increasing tableau of hook-closed shape. Assume that every dot in T_0 is situated to the right of an integer and above another integer. Then all resolutions of T_0 are K-Knuth equivalent.*

Proof. Let T' and T'' be resolutions of T_0. If T_0 contains no dots, then $T' = T''$ and there is nothing to prove. Otherwise let α be the south-west most box of T_0 that contains a dot. By Lemma 5.5(b) and the conditions of the theorem, we may assume that α is surrounded by boxes, except that the box directly south-west of α may be missing. In the following picture the box α is marked with a dot and the box directly north-east of α is labeled y. The picture includes all boxes to the left of (and in the same rows as) α and y, and all boxes below (and in the same columns as) α and y. The numbers s, t, m, n are greater than or equal to one.

$$\begin{array}{cccccc}
a_s & \cdots & \cdots & a_1 & a_0 & y \\
 & b_t & \cdots & b_1 & \bullet & d_0 \\
 & & & & c_1 & d_1 \\
 & & & & \vdots & \vdots \\
 & & & & \vdots & d_n \\
 & & & & c_m &
\end{array}$$

Let U_0 be the tableau of boxes in T_0 that are strictly north or strictly east of α; this includes the row and column of y. It follows by induction on the number of dots in T_0 that all resolutions of U_0 are K-Knuth equivalent. Since both T' and T'' have reading words that terminate in words for resolutions of U_0, we can assume that T' and T'' put the same integers in all boxes except α. Furthermore, if T_0 contains a dot in the same row as y, then we may assume that T' and T'' replace this dot with the maximum of the boxes above and to the left of the dot. Similarly, if T_0 contains a dot in the same column as y but strictly north of y, then we may assume that T' and T'' replace this dot with the minimum of the values below and to the right of the dot. These assumptions imply that $a_0 \leq y < d_0$, and also that T' and T'' have reading words that start with a reading word of the tableau of boxes strictly south-west of α, continue with reading words of the resolutions of the displayed boxes, and terminate in a reading word for the boxes strictly north or strictly east of y. Write $\mathbf{a} = (a_s, \dots, a_0)$, $\mathbf{b} = (b_t, \dots, b_1)$, $\mathbf{c}' = (c_m, \dots, c_2)$, $\mathbf{c} = (\mathbf{c}', c_1)$, $\mathbf{d}' = (d_n, \dots, d_2)$, and $\mathbf{d} = (\mathbf{d}', d_1, d_0)$. There are three cases.

Case 1: Assume that $T'(\alpha) = b_1$ and $T''(\alpha) = c_1$. Then we can choose reading words of T' and T'' such that the word of T'' is obtained from the word of T' if the subword $(\mathbf{b}, \mathbf{c}, b_1)$ is replaced with $(\mathbf{c}, \mathbf{b}, c_1)$, so it is enough to show that $(\mathbf{b}, \mathbf{c}, b_1) \equiv (\mathbf{c}, \mathbf{b}, c_1)$. Notice that $(b_1, \mathbf{c}) \equiv (\mathbf{c}', b_1, c_1)$ and $(b_j, \mathbf{c}, b_{j-1}) \equiv (\mathbf{c}, b_j, b_{j-1})$ for $2 \leq j \leq t$. The result follows because

$$\begin{aligned}
(\mathbf{b}, \mathbf{c}, b_1) &\equiv (b_t, \dots, b_2, \mathbf{c}', b_1, c_1, b_1) \equiv (b_t, \dots, b_2, \mathbf{c}, b_1, c_1) \\
&\equiv (b_t, \dots, b_3, \mathbf{c}, b_2, b_1, c_1) \equiv \cdots \equiv (\mathbf{c}, \mathbf{b}, c_1)\,.
\end{aligned}$$

Case 2: Assume that $T'(\alpha) = b_1$ and $T''(\alpha) = d_0$. (By symmetry this case also takes care of situations with $T'(\alpha) = a_0$ and $T''(\alpha) = c_1$.) We may assume that Case 1 does not apply, so that $d_0 < c_1$. In this case we can choose reading words of T' and T'' such that the word of T'' is obtained from the word of T' if the subword $(b_1, \mathbf{c}, b_1, \mathbf{d})$ is replaced with $(b_1, \mathbf{c}, d_0, \mathbf{d})$. These subwords are K-Knuth equivalent because

$$\begin{aligned}
(b_1, \mathbf{c}, b_1, \mathbf{d}) &\equiv (\mathbf{c}', b_1, c_1, b_1, \mathbf{d}) \equiv (\mathbf{c}, b_1, c_1, \mathbf{d}) \equiv (\mathbf{c}, b_1, \mathbf{d}', c_1, d_1, d_0) \\
&\equiv (\mathbf{c}, b_1, \mathbf{d}', c_1, d_0, d_1) \equiv (\mathbf{c}, b_1, \mathbf{d}', c_1, d_0, d_1, d_1) \equiv (\mathbf{c}, b_1, \mathbf{d}', c_1, d_1, d_0, d_1) \\
&\equiv (\mathbf{c}, b_1, c_1, \mathbf{d}, d_1) \equiv (\mathbf{c}, b_1, c_1, \mathbf{d}', d_0, d_1, d_0) \equiv (\mathbf{c}, b_1, c_1, d_0, \mathbf{d}) \\
&\equiv (\mathbf{c}, c_1, b_1, d_0, \mathbf{d}) \equiv (\mathbf{c}, b_1, d_0, \mathbf{d}) \equiv (b_1, \mathbf{c}, d_0, \mathbf{d})\,.
\end{aligned}$$

Case 3: Assume that $T'(\alpha) = a_0$ and $T''(\alpha) = d_0$. We may assume that Case 1 and Case 2 do not apply, so that $b_1 < a_0 \leq y < d_0 < c_1$. We can choose reading words of T' and T'' such that the word of T'' is obtained from the word of T' if the subword

$(b_1, c_1, a_0, \mathbf{a}, \mathbf{d}, y)$ is replaced with $(b_1, c_1, d_0, \mathbf{a}, \mathbf{d}, y)$. If $y = a_0$ then it follows from Case 1 that $(\mathbf{a}, \mathbf{d}, y) \equiv (\mathbf{d}, \mathbf{a}, y')$, where we set $y' = d_0$. Otherwise Lemma 5.4 implies that $(\mathbf{a}, \mathbf{d}, y) \equiv (\mathbf{d}, \mathbf{a}, y')$, where we set $y' = y$. Notice also that Case 2 applied to subsets of the displayed boxes implies that $(c_1, b_1, a_0, \mathbf{a}) \equiv (c_1, b_1, c_1, \mathbf{a})$ and $(b_1, c_1, b_1, \mathbf{d}) \equiv (b_1, c_1, d_0, \mathbf{d})$. We finally obtain

$$\begin{aligned}(b_1, c_1, a_0, \mathbf{a}, \mathbf{d}, y) &\equiv (c_1, b_1, a_0, \mathbf{a}, \mathbf{d}, y) \equiv (c_1, b_1, c_1, \mathbf{a}, \mathbf{d}, y) \equiv (b_1, c_1, b_1, \mathbf{d}, \mathbf{a}, y') \\ &\equiv (b_1, c_1, d_0, \mathbf{d}, \mathbf{a}, y') \equiv (b_1, c_1, d_0, \mathbf{a}, \mathbf{d}, y)\,.\end{aligned}$$

This completes the proof. □

Remark 5.10. In contrast to the ordinary Knuth relation, it appears to be difficult to determine if two given words are K-Knuth equivalent. The problem is that in order to transform one word into another using the basic relations of Definition 5.3, it may be necessary to go through a sequence of longer words. For example, this happens in Case 2 of the proof of Theorem 5.9. It would be interesting to know if this problem is decidable in general.

6. Grassmannians of type A

6.1. **Tableaux of type A.** In this section we work with the partially ordered set $\Lambda = \mathbb{N}^2$, where the order is given by $[r_1, c_1] \leq [r_2, c_2]$ if and only if $r_1 \leq r_2$ and $c_1 \leq c_2$. A straight shape $\lambda \subset \mathbb{N}^2$ is the same as a *Young diagram* and can be identified with the partition $(\lambda_1 \geq \cdots \geq \lambda_\ell)$ where ℓ is the number of rows in λ and λ_i is the number of boxes in row i. Any tableau defined on a skew shape in $\mathbb{N}^2$ will be called a *tableau of type A*. If $X = \mathrm{Gr}(m, m+k)$ is a Grassmann variety of type A, then we identify Λ_X with the boxes $[r, c] \in \mathbb{N}^2$ for which $1 \leq r \leq m$ and $1 \leq c \leq k$. Every tableau for Λ_X is then a tableau of type A.

Let T and T' be increasing tableaux of type A. Then T and T' are jeu de taquin equivalent for $\mathbb{N}^2$ if and only if T and T' are jeu de taquin equivalent for Λ_X for all Grassmannians $X = \mathrm{Gr}(m, m+k)$ given by sufficiently large integers m and k. Furthermore, an increasing tableau U of straight shape is a unique rectification target for $\mathbb{N}^2$ if and only if U is a unique rectification target for Λ_X for all Grassmannians X satisfying $\mathrm{sh}(U) \subset \Lambda_X$. In this section we study unique rectification targets for $\mathbb{N}^2$.

Lemma 6.1. *Let T be a dotted increasing tableau of type A. Then all resolutions of T are K-Knuth equivalent.*

Proof. Using that the shape of T is a skew Young diagram and Lemma 5.5(b), we may assume that every dot in T is situated to the right of an integer and above another integer. The lemma therefore follows from Theorem 5.9. □

Recall that the *row word* of a tableau T is the reading word obtained by reading the rows of T from left to right, starting with the bottom row.

Theorem 6.2. *Let T and T' be increasing tableaux of type A. Then T and T' are K-Knuth equivalent if and only if T and T' are jeu de taquin equivalent for $\mathbb{N}^2$.*

Proof. For any word $\mathbf{u} = (u_1, \ldots, u_p) \in \mathbb{Z}^p$ we define an increasing tableau $T(\mathbf{u})$ whose shape is the antidiagonal $\{[p+1-j, j] \mid 1 \leq j \leq p\}$ and whose values are given by $T([p+1-j, j]) = u_j$. By examining each of the basic relations of Definition 5.3, it follows easily that, if $\mathbf{u} \equiv \mathbf{u}'$, then $T(\mathbf{u}')$ can be obtained from

$T(\mathbf{u})$ by a sequence of forward and reverse jeu de taquin slides. Furthermore, if $\mathbf{u}$ is the row word of an increasing tableau S, then $T(\mathbf{u})$ can be obtained from S by a sequence of slides. The implication "only if" of the theorem follows from these observations.

For the other implication we must show that K-Knuth equivalence is preserved by arbitrary jeu de taquin slides. Assume that $T' = \mathrm{jdt}_C(T)$ and that T and T' have values in the interval $[a,b]$. Define a sequence of dotted increasing tableaux $T_a, T_{a+1}, \dots, T_{b+1}$ by $T_a = [C \to \bullet] \cup T$ and $T_{i+1} = \mathrm{swap}_{i,\bullet}(T_i)$ for $a \leq i \leq b$. Then T is a resolution of T_a and $T' = T_{b+1}|_{\mathbb{Z}}$ is a resolution of T_{b+1}. Let T'_i be the resolution of T_i obtained by replacing each dot with the minimum of the boxes below and to the right of the dot. Since T'_i is also a resolution of T_{i+1}, it follows from Lemma 6.1 that $T'_i \equiv T'_{i+1}$. We deduce that $T \equiv T'_a \equiv T'_{a+1} \equiv \cdots \equiv T'_{b+1} = T'$. □

Corollary 6.3. *Let T be an increasing tableau of straight shape of type A. The following are equivalent.* (a) *T is a unique rectification target for $\mathbb{N}^2$.* (b) *If T' is any increasing tableau of straight shape such that $T' \equiv T$, then $T' = T$.*

Proof. This follows from Theorem 6.2 and Lemma 3.13. □

6.2. **The Hecke permutation.** Theorem 6.2 implies that the K-Knuth equivalence class of an increasing tableau is an invariant under jeu de taquin slides, in fact the finest such invariant. However, as indicated in Remark 5.10, this invariant may be difficult to work with. We therefore define a coarser invariant which is much easier to understand but still able to identify the jeu de taquin class $[U]$ for many important unique rectification targets U.

Let Σ be the group of bijective maps $w : \mathbb{Z} \to \mathbb{Z}$ such that $w(x) = x$ for all but finitely many integers $x \in \mathbb{Z}$. The elements of Σ will be called *permutations*. For any integer $i \in \mathbb{Z}$ we let $s_i = (i, i+1) \in \Sigma$ be the simple transposition that interchanges i and $i+1$. We will consider Σ as a Coxeter group generated by these simple transpositions. We need the *Hecke product* on Σ, which can be defined as follows. For any permutation $u \in \Sigma$ and simple transposition s_i, we define

$$u \cdot s_i = \begin{cases} us_i & \text{if } \ell(us_i) > \ell(u); \\ u & \text{otherwise.} \end{cases}$$

Given an additional permutation $v \in \Sigma$, we then set

$$u \cdot v = u \cdot s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_\ell}$$

where $v = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is any reduced expression for v, and the simple transpositions are multiplied to u in left to right order. This is independent of the chosen reduced expression and defines an associative monoid product on Σ. The product $u \cdot v$ is called *reduced* if $\ell(u \cdot v) = \ell(u) + \ell(v)$, which is true if and only if $u \cdot v = uv$, i.e. the Hecke product agrees with the usual product in Σ. Proofs of these facts can be found in e.g. [6, §3].

Given a word $\mathbf{a} = (a_1, \dots, a_k)$ of integers, define the permutation $w(\mathbf{a}) = s_{a_1} \cdot s_{a_2} \cdot \ldots \cdot s_{a_k}$. If $\mathbf{a}$ is a reading word of an increasing tableau T of type A, then we also write $w(T) = w(\mathbf{a})$. This permutation is called the *Hecke permutation* of T and was used in [5] for increasing tableaux of straight shape. We will show that this permutation is independent of the chosen reading word and invariant under jeu de taquin slides. This is done by noting that the relations of the Hecke monoid are a weakening of the K-Knuth relations.

Definition 6.4. Define the *Hecke equivalence relation* on words of integers, denoted $\approx$, to be the symmetric transitive closure of the following basic relations. For any words $\mathbf{u}$ and $\mathbf{v}$, and integers a, b we set

$$\begin{aligned}(\mathbf{u}, a, a, \mathbf{v}) &\approx (\mathbf{u}, a, \mathbf{v})\,, \\ (\mathbf{u}, a, b, a, \mathbf{v}) &\approx (\mathbf{u}, b, a, b, \mathbf{v})\,, \\ (\mathbf{u}, a, b, \mathbf{v}) &\approx (\mathbf{u}, b, a, \mathbf{v}) \qquad \text{if } |a-b| \geq 2.\end{aligned}$$

This relation governs the Hecke monoid in the sense that $\mathbf{u} \approx \mathbf{u}'$ if and only if $w(\mathbf{u}) = w(\mathbf{u}')$ for any words of integers $\mathbf{u}$ and $\mathbf{u}'$. On the other hand, notice that the K-Knuth relation $\mathbf{u} \equiv \mathbf{u}'$ implies the Hecke relation $\mathbf{u} \approx \mathbf{u}'$. In fact, the two last basic relations of Definition 5.3 are both special cases of the last relation given in Definition 6.4. We record the following consequence of Theorem 6.2.

Corollary 6.5. *The Hecke permutation $w(T)$ of an increasing tableau of type A is invariant under jeu de taquin slides.*

Theorem 6.6. *Let λ be a Young diagram. Then the minimal increasing tableau M_λ is a unique rectification target for $\mathbb{N}^2$. Furthermore, an increasing tableau T of type A rectifies to M_λ if and only if $w(T) = w(M_\lambda)$.*

Proof. In view of Corollary 6.5 it is enough to show that, if T is any increasing tableau of straight shape such that $w(T) = w(M_\lambda)$, then $T = M_\lambda$.

Set $w = w(T) = w(M_\lambda)$. Using that w is the Hecke permutation defined by the row word of M_λ, we obtain that $w^{-1}(1) = \lambda_1 + 1$. Since w is also the Hecke permutation given by the row word of T, this implies that the first row of T must start with the integers from 1 to λ_1, and if this row has more than λ_1 boxes, then the remaining boxes are greater than or equal to $\lambda_1 + 2$.

Let $\overline{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$ be the Young diagram obtained by removing the first row of λ and let $\overline{M_\lambda}$ be the tableau obtained by removing the first row of M_λ. Let $\overline{T}$ denote the *skew* tableau obtained by removing the first λ_1 boxes from the top row of T, and let T' be a rectification of $\overline{T}$. We also set $u = s_1 s_2 \cdots s_{\lambda_1} \in \Sigma$. Then we have

$$w(\overline{M_\lambda}) \cdot u = w(M_\lambda) = w(T) = w(\overline{T}) \cdot u = w(T') \cdot u\,.$$

Furthermore, since all boxes of $\overline{M_\lambda}$ and T' are greater than or equal to two, it follows that the Hecke products are reduced, hence $w(\overline{M_\lambda}) = w(T')$. By induction on $\ell(\lambda)$ this implies that $T' = \overline{M_\lambda}$. Since all boxes in the top row of $\overline{M_\lambda}$ are strictly smaller than $\lambda_1 + 2$, we deduce that the top row of T contains exactly λ_1 boxes, hence $\overline{T} = T' = \overline{M_\lambda}$. We conclude that $T = M_\lambda$. □

Notice that Theorem 6.6 provides yet another formulation of the Littlewood-Richardson rule for the K-theory of Grassmannians. The structure constant $c^\nu_{\lambda,\mu}$ is equal to the number of increasing tableaux T of shape ν/λ for which $w(T) = w(M_\mu)$.

6.3. The Pieri rule for $\Gamma(\mathbb{N}^2)$.

The following result shows that multiplication with a single-row partition in the combinatorial K-theory ring $\Gamma(\mathbb{N}^2)$ is compatible with Lenart's Pieri rule for the K-theory of Grassmannians [21]. A *horizontal strip* is a skew shape in $\mathbb{N}^2$ that contains at most one box in each column. The following statement is equivalent to [34, Lemma 5.4].

Corollary 6.7. *Let λ be a Young diagram and let p be a positive integer. Then we have*

$$G_{(p)} \cdot G_\lambda = \sum_\nu \binom{r(\nu/\lambda) - 1}{|\nu/\lambda| - p} G_\nu \,,$$

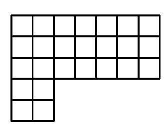
in the ring $\Gamma(\mathbb{N}^2)$, where the sum is over all partitions $\nu \supset \lambda$ such that ν/λ is a horizontal strip, and $r(\nu/\lambda)$ is the number of non-empty rows of ν/λ.

Proof. The coefficient of G_ν in the product $G_{(p)} \cdot G_\lambda$ is equal to the number of increasing tableaux T of shape ν/λ for which $w(T) = s_1 s_2 \cdots s_p$. The condition $w(T) = s_1 s_2 \cdots s_p$ holds if and only if the row word of T is weakly increasing and contains exactly the integers $\{1, 2, \ldots, p\}$. This implies that ν/λ must be a horizontal strip, in which case there are exactly $\binom{r(\nu/\lambda)-1}{|\nu/\lambda|-p}$ choices for T. □

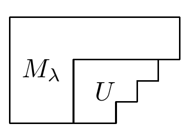
6.4. **Numerical invariants.** We next derive two additional invariants from the K-Knuth class of an increasing tableau. These invariants were also obtained in [34] with different methods. If $\mathbf{u} = (u_1, \ldots, u_\ell)$ is a word of integers, then a *subsequence* of $\mathbf{u}$ is any word of the form $(u_{i_1}, \ldots, u_{i_k})$ for indices $i_1 < \cdots < i_k$. We let $\mathrm{lis}(\mathbf{u})$ denote the length of the longest strictly increasing subsequence of $\mathbf{u}$, and let $\mathrm{lds}(\mathbf{u})$ be the length of the longest strictly decreasing subsequence. It follows by inspection of the basic relations of Definition 5.3 that, if $\mathbf{u}$ and $\mathbf{u}'$ are K-Knuth equivalent words of integers, then $\mathrm{lis}(\mathbf{u}) = \mathrm{lis}(\mathbf{u}')$ and $\mathrm{lds}(\mathbf{u}) = \mathrm{lds}(\mathbf{u}')$. If $\mathbf{u}$ is a reading word of an increasing tableau T of type A, we can therefore write $\mathrm{lis}(T) = \mathrm{lis}(\mathbf{u})$ and $\mathrm{lds}(T) = \mathrm{lds}(\mathbf{u})$ without ambiguity. Notice that if T has straight shape, then $\mathrm{lis}(T)$ is equal to the number of columns in T, and $\mathrm{lds}(T)$ is the number of rows in T. The following result follows from Theorem 6.2.

Corollary 6.8 (Thomas and Yong)**.** *Let T be an increasing tableau of type A. Then $\mathrm{lis}(T)$ and $\mathrm{lds}(T)$ are invariant under jeu de taquin slides.*

We now prove a generalization of Thomas and Yong's result that superstandard tableaux of type A are unique rectification targets. By a *fat hook* we will mean a partition of the form (a^b, c^d) where a, b, c, d are non-negative integers with $a \geq c$. For example:

$(7^3, 2^2) = (7, 7, 7, 2, 2) =$  .

Let $\lambda = (a^b, c^d)$ be a fat hook, let M_λ be the corresponding minimal increasing tableau, and let U be an increasing tableau of straight shape. We will say that U *fits in the corner of* M_λ if U has at most d rows, at most $a - c$ columns, and all integers contained in U are strictly larger than the integers contained in M_λ. In this case we let $M_\lambda \cup U$ denote the increasing tableau obtained by attaching U to the corner of M_λ.

$M_\lambda \cup U =$ 

Theorem 6.9. *Let λ be a fat hook, and let U be any unique rectification target that fits in the corner of M_λ. Then $M_\lambda \cup U$ is a unique rectification target.*

Proof. Let T be a tableau of straight shape that is jeu de taquin equivalent to $M_\lambda \cup U$, and let a be the largest integer contained in M_λ. Then it follows from Theorem 6.6 and Lemma 3.3 that $T|_{[1,a]} = M_\lambda$, after which Corollary 6.8 implies that $T = M_\lambda \cup U'$ for some increasing tableau U' that fits in the corner of M_λ. Since Lemma 3.3 shows that U' is jeu de taquin equivalent to U, the assumption that U is a unique rectification target implies that $U' = U$, as required. □

Corollary 6.10 (Thomas and Yong)**.** *Let λ be a Young diagram. Then the superstandard tableaux S_λ and $\widehat{S}_\lambda$ of shape λ are unique rectification targets.*

Remark 6.11. It is tempting to look for generalizations of Theorem 6.9. However, many natural generalizing statements are ruled out by the fact that the following two tableaux

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array} \quad \equiv \quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2&4\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

are jeu de taquin equivalent, and therefore the first tableau is not a unique rectification target.

6.5. **Stable Grothendieck polynomials.** Our methods can be applied to obtain a simple formula for the product of any stable Grothendieck polynomial with a stable Grothendieck polynomial given by a partition. Let $S_n \subset \Sigma$ be the subgroup of permutations of the integer interval $[1, n]$. For each $w \in S_n$, define a *Grothendieck polynomial* $\mathfrak{G}_w = \mathfrak{G}_w(x_1, \ldots, x_{n-1})$ as follows. If $w = w_0 \in S_n$ is the longest permutation, then we set $\mathfrak{G}_{w_0} = \prod_{i=1}^{n-1} x_i^{n-i}$. Otherwise we set

$$\mathfrak{G}_w = \frac{(1 + x_{i+1})\mathfrak{G}_{ws_i}(x_1, \ldots, x_n) - (1 + x_i)\mathfrak{G}_{ws_i}(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}}$$

for any i such that $w(i) < w(i + 1)$.

Given any permutation $w \in \Sigma$ and $m \in \mathbb{Z}$, the *shifted permutation* $1^m \times w \in \Sigma$ is defined by $(1^m \times w)(i) := w(i - m) + m$ for $i \in \mathbb{Z}$. Notice that for all sufficiently large integers m we have $1^m \times w \in S_{2m}$, hence the polynomial $\mathfrak{G}_{1^m \times w}$ is defined. The *stable Grothendieck polynomial* for w is the power series in infinitely many variables obtained as the limit

$$G_w \; = \; \lim_{m \to \infty} \mathfrak{G}_{1^m \times w} \; \in \; \mathbb{Z}[\![x_1, x_2, \ldots]\!] \, .$$

The polynomials $(-1)^{\ell(w)}\mathfrak{G}_w(-x_1, \ldots, -x_{n-1})$ were introduced by Lascoux and Schützenberger as representatives for the Schubert classes in the K-theory of the flag manifold $\mathrm{F}\ell(\mathbb{C}^n)$ [20], and stabilizations of these polynomials were studied by Fomin and Kirillov in [12]. We have deviated from the original definitions to ensure that the structure constants for products of Grothendieck polynomials are non-negative [2].

If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell > 0)$ is any partition, we let $w_\lambda \in \Sigma$ be the corresponding Grassmannian permutation, defined by $w_\lambda(i) = i + \lambda_{\ell+1-i}$ for $1 \leq i \leq \ell$, $w_\lambda(0) = 0$, and $w_\lambda(i) < w_\lambda(i + 1)$ for $i \neq \ell$. The stable Grothendieck polynomial for λ is defined by $G_\lambda = G_{w_\lambda}$. It was proved in [3, Thm. 6.3] that every stable Grothendieck polynomial G_w can be written as a finite linear combination of the stable polynomials indexed by partitions:

$$G_w = \sum_\lambda a_{w,\lambda} \, G_\lambda \, .$$

By using Theorem 4.1 and [3, Thm. 8.1], we may identify each stable polynomial G_λ with the basis element of the same name in the combinatorial K-theory ring $\Gamma(\mathbb{N}^2)$. In this way we can consider each stable Grothendieck polynomial G_w as an element of $\Gamma(\mathbb{N}^2)$.

A result of Lascoux [19] shows that each coefficient $a_{w,\lambda}$ is non-negative. It was proved in [5] that $a_{w,\lambda}$ is equal to the number of increasing tableaux T of shape λ such that $w(T) = w^{-1}$. This formula has the following generalization.

Corollary 6.12. *For any permutation $w \in \Sigma$ and Young diagram λ we have $G_w \cdot G_\lambda = \sum_\nu c^\nu_{w,\lambda} G_\nu$, where $c^\nu_{w,\lambda}$ is the number of increasing tableaux T of shape ν/λ such that $w(T) = w^{-1}$.*

Proof. Let $\tau = \{T \mid w(T) = w^{-1}\}$ be the set of all increasing tableaux T of type A for which $w(T) = w^{-1}$. It follows from Corollary 6.5 that τ is closed under jeu de taquin slides, and [5, Thm. 1] shows that $G_w = F_\tau(1)$ with the notation of §3.5. Proposition 3.17 therefore shows that $G_w \cdot G_\lambda = F_\tau(G_\lambda)$, as required. □

Remark 6.13. Given a partition λ, there are several ways to choose a permutation w for which $G_w = G_\lambda$, including w_λ, $w(M_\lambda)^{-1}$, and shifts of these permutations. For any such permutation w there exists a unique increasing tableau U of shape λ such that $w(U) = w^{-1}$, and this tableau is a unique rectification target.

Remark 6.14. Corollary 6.12 can also be proved directly from the main result in [5]. The Grassmannian permutation w_λ belongs to S_m where $m = \lambda_1 + \ell(\lambda)$. For $w \in S_n$ we let $w_\lambda \times w \in S_{m+n}$ denote the permutation that acts on $[1,m]$ via w_λ and on $[m+1, m+n]$ via w. We then have $G_\lambda \cdot G_w = G_{w_\lambda \times w}$, so the main result of [5] implies that the coefficient of G_ν in $G_\lambda \cdot G_w$ is the number of increasing tableaux T' of shape ν for which $w(T') = w_\lambda^{-1} \times w^{-1}$. Each such tableaux T' consists of the unique increasing tableau U for which $w(U) = w_\lambda^{-1}$ in its upper-left corner, surrounded by an increasing tableau T of shape ν/λ such that $w(T) = w^{-1}$.

6.6. **Rectification of minimal increasing tableaux.** Given increasing tableaux S and T of type A, we let $S * T$ denote the increasing tableau obtained by attaching the north-east corner of S to the south-west corner of T. We then let $S \cdot T = \mathrm{rect}(S * T)$ denote the greedy rectification of this tableau (see §4.4). We are mainly interested in this construction when S and T have straight shapes. For example, we have

$$\begin{array}{ll}\boxed{1}\boxed{2}\\ \boxed{4}\end{array} * \begin{array}{ll}\boxed{1}\boxed{3}\\ \boxed{3}\end{array} = \begin{array}{llll} & & \boxed{1} & \boxed{3}\\ & & \boxed{3} & \\ \boxed{1} & \boxed{2} & & \\ \boxed{4} & & & \end{array} \quad\text{and}\quad \begin{array}{ll}\boxed{1}\boxed{2}\\ \boxed{4}\end{array} \cdot \begin{array}{ll}\boxed{1}\boxed{3}\\ \boxed{3}\end{array} = \begin{array}{lll}\boxed{1}\boxed{2}\boxed{3}\\ \boxed{2}\\ \boxed{4}\end{array}.$$

It has been proved in [35] that $S \cdot T$ is the only rectification of $S * T$, but we will not use this fact here. On the other hand, the product on increasing tableaux is not associative since

$$\left(\boxed{1} \cdot \begin{array}{ll}\boxed{1}\boxed{4}\\ \boxed{3}\end{array}\right) \cdot \boxed{2} = \begin{array}{lll}\boxed{1}\boxed{2}\boxed{4}\\ \boxed{3}\end{array} \quad\text{whereas}\quad \boxed{1} \cdot \left(\begin{array}{ll}\boxed{1}\boxed{4}\\ \boxed{3}\end{array} \cdot \boxed{2}\right) = \begin{array}{lll}\boxed{1}\boxed{2}\boxed{4}\\ \boxed{3}\boxed{4}\end{array}.$$

However, the product is associative up to jeu de taquin equivalence. More precisely, if S, T, and U are increasing tableaux, then

$$(S \cdot T) \cdot U \equiv S \cdot (T \cdot U)$$

as both sides are jeu de taquin equivalent to $S * T * U$. Notice also that the product of increasing tableaux satisfies the identity

$$S^\dagger \cdot T^\dagger = (T \cdot S)^\dagger \,,$$

where $T^\dagger$ is the conjugate of T, obtained by mirroring the boxes of T in the north-west to south-east diagonal.

Lemma 6.15. *Let M and N be minimal increasing tableaux of straight shapes of type A. Then $M \cdot N$ is a minimal increasing tableau.*

Proof. Since all minimal increasing tableaux of straight shapes are unique rectification targets by Theorem 6.6, it suffices to show that $M * N$ is jeu de taquin equivalent to some minimal increasing tableau of straight shape. We may assume that M and N are both non-empty tableaux. We then proceed by induction on $\mathrm{lds}(M) + \mathrm{lis}(N)$, the number of rows in M plus the number of columns in N.

In the base case $\mathrm{lds}(M) + \mathrm{lis}(N) = 2$, $M = M_{(a)}$ is a minimal increasing tableau with a single row, and $N = M_{(1^b)}$ is a minimal increasing tableau with a single column. In this case we leave it to the reader to check that $M \cdot N = M_{(c,1^{d-1})}$ is a minimal increasing tableau of hook shape, where

$$c = \begin{cases} a & \text{if } a \geq b \\ a+1 & \text{if } a < b \end{cases} \qquad \text{and} \qquad d = \begin{cases} b & \text{if } b \geq a \\ b+1 & \text{if } b < a. \end{cases}$$

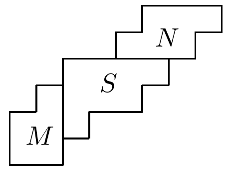
Assume next that $\mathrm{lds}(M) + \mathrm{lis}(N) \geq 3$. By possibly replacing (M, N) with $(N^\dagger, M^\dagger)$, we may assume that N has at least two columns. Let N' be the first column of N and let N'' be the rest of N. Then form the product $T = M \cdot N'$, and let T' be the first column of T and T'' the rest of T. It follows from the induction hypothesis that T is a minimal increasing tableau. Therefore T'' is minimal increasing with smallest element 2, by which we mean that T'' becomes a minimal increasing tableau in the usual sense if all its integers are decreased by one. Since N'' is also minimal increasing with smallest element 2, and since $\mathrm{lds}(T'') \leq \mathrm{lds}(M)$, a second application of the induction hypothesis shows that the tableau $S = T'' \cdot N''$ is minimal increasing with smallest element 2. Notice also that

$$M \cdot N = M \cdot (N' \cdot N'') \equiv (M \cdot N') \cdot N'' = (T' \cdot T'') \cdot N'' \equiv T' \cdot (T'' \cdot N'') = T' \cdot S \,.$$

It follows that $\mathrm{lds}(T') = \mathrm{lds}(T) = \mathrm{lds}(M * N') = \mathrm{lds}(M * N) = \mathrm{lds}(T' \cdot S) \geq \mathrm{lds}(S)$. Since T' is a minimal increasing tableau with a single column, S is minimal increasing with smallest element 2, and T' has at least as many rows as S, we deduce that the product $T' \cdot S$ is obtained by attaching S to the right side of T'. This shows that $T' \cdot S$ is a minimal increasing tableau and completes the proof. □

Theorem 6.16. *Let T be any minimal increasing tableau of type A. Then* $\mathrm{rect}(T)$ *is a minimal increasing tableau.*

Proof. Let $\alpha = [r, c]$ be a box of $\mathrm{sh}(T)$ that is as far north-west as possible, i.e. a box for which $r + c$ is minimal. Let S be the tableau of boxes in T that are south-east of α, let M be the tableau of boxes strictly west of α, and let N be the tableau of boxes strictly north of α.

$T =$ 

The choice of α implies that S, M, and N are minimal increasing tableaux, and S has straight shape. By induction on the number of boxes in T, it follows that $\mathrm{rect}(M)$ and $\mathrm{rect}(N)$ are both minimal increasing tableaux of straight shapes. Lemma 6.15 therefore implies that $(\mathrm{rect}(M) \cdot S) \cdot \mathrm{rect}(N)$ is a minimal increasing tableau. The result follows because T is jeu de taquin equivalent to this product, which is a unique rectification target by Theorem 6.6. $\square$

Remark 6.17. It follows from Lemma 6.15 that the product of increasing tableaux of type A is associative when restricted to the set of all minimal increasing tableaux. More precisely, if S, T, and U are minimal increasing tableaux of type A, then $(S \cdot T) \cdot U = S \cdot (T \cdot U)$, as both sides are minimal increasing tableaux of straight shape, and all such tableaux are unique rectification targets by Theorem 6.6. Let $\mathcal{P}$ denote the set of all Young diagrams. Then the product $\lambda \cdot \mu := \mathrm{sh}(M_\lambda \cdot M_\mu)$ gives $\mathcal{P}$ the structure of associative monoid, with unit equal to the empty Young diagram. Furthermore, the map $\mathcal{P} \to \Sigma$ defined by $\lambda \mapsto w(M_\lambda)$ is a monoid homomorphism, where Σ is equipped with the Hecke product.

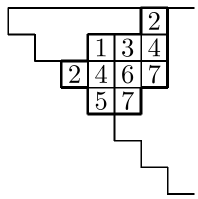
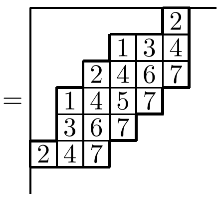
7. Maximal orthogonal Grassmannians

7.1. **Tableaux of type B.** Let $\Delta = \{[r, c] \in \mathbb{N}^2 \mid r \leq c\}$ be the set of boxes in $\mathbb{N}^2$ that are on or above the diagonal. We give Δ the partial order inherited from $\mathbb{N}^2$. A straight shape in Δ is the same as a *shifted Young diagram* and can be identified with the strict partition $(\lambda_1 > \cdots > \lambda_\ell)$ where ℓ is the number of rows in λ and λ_i is the number of boxes in row i. Any tableau defined on a skew shape in Δ will be called a *tableau of type B*. If $X = \mathrm{OG}(n, 2n)$ is a maximal orthogonal Grassmannian, then we identify Λ_X with the boxes $[r, c] \in \Delta$ for which $c \leq n - 1$. Every tableau for Λ_X is then a tableau of type B. Two increasing tableaux T and T' of type B are jeu de taquin equivalent for Δ if and only if they are jeu de taquin equivalent for Λ_X for all $X = \mathrm{OG}(n, 2n)$ with n sufficiently large. An increasing tableau U of straight shape of type B is a unique rectification target for Δ if and only if U is a unique rectification target for Λ_X for all maximal orthogonal Grassmannians X satisfying $\mathrm{sh}(U) \subset \Lambda_X$.

If $\theta \subset \Delta$ is a skew shape, then we let $\theta^2 \subset \mathbb{N}^2$ denote the union of θ with its reflection in the diagonal, i.e. $\theta^2 = \{[r, c] \in \mathbb{N}^2 \mid [r, c] \in \theta \text{ or } [c, r] \in \theta\}$. This is a skew shape in $\mathbb{N}^2$. If T is an increasing tableau of type B with $\mathrm{sh}(T) = \theta$, then we let T^2 denote the tableau of type A of shape θ^2 defined by

$$T^2([r, c]) = \begin{cases} T([r, c]) & \text{if } r \leq c, \\ T([c, r]) & \text{if } r \geq c. \end{cases}$$

We call T^2 the *doubling* of T. For example, the tableau

$T =$  has doubling $T^2 =$  .

If T is an increasing tableau of type B of shape ν/λ, and $C \subset \lambda$ is a subset of the maximal boxes in λ, then it follows immediately from the definitions that

$$\mathrm{jdt}_C(T)^2 = \mathrm{jdt}_{C^2}(T^2)\,. \tag{7}$$

The use of doubled tableaux appears to originate in Worley's thesis [37, §6.3], which gives a slightly different construction such that the number of boxes in a doubled tableau is always twice the number of boxes in the original tableau. A modification of Worley's doubling was used in [9] and is shown to satisfy an identity similar to (7), with the proof requiring slightly more work, see [9, Lemma 3.2]. Equation (7) has the following consequence.

Proposition 7.1. (a) *If S and T are jeu de taquin equivalent increasing tableaux of type B, then S^2 and T^2 are jeu de taquin equivalent tableaux of type A.*

(b) *If U is an increasing tableau of straight shape of type B such that U^2 is a unique rectification target of type A, then U is a unique rectification target of type B.*

Notice that an increasing tableau T of type B is a minimal increasing tableau if and only if T^2 is a minimal increasing tableau of type A. We can therefore use Proposition 7.1 to derive the following two corollaries from Theorem 6.6 and Theorem 6.16.

Corollary 7.2. *Let λ be a shifted Young diagram. Then the minimal increasing tableau M_λ of type B is a unique rectification target. Furthermore, an increasing tableau T of type B rectifies to M_λ if and only if $w(T^2) = w(M_\lambda^2)$.*

Corollary 7.3. *Let T be any minimal increasing tableau of type B. Then* $\mathrm{rect}(T)$ *is a minimal increasing tableau.*

We also recover the result from [9] that row-wise superstandard tableaux of type B are unique rectification targets. Recall that this is not always true for column-wise superstandard tableaux of type B by Example 3.8.

Corollary 7.4 (Clifford, Thomas, Yong)**.** *Let λ be a shifted Young diagram. Then the row-wise superstandard tableau S_λ is a unique rectification target of type B.*

Proof. This follows because S_λ^2 is a unique rectification target by Theorem 6.9. □

We believe that Proposition 7.1(a) is true in both directions. In fact, this follows from Conjecture 7.10 below. However, the following example shows that not all unique rectification targets of type B can be detected by Proposition 7.1(b).

Example 7.5. One can check that the following three increasing tableaux of type A are jeu de taquin equivalent:

$$T_1 = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 2&3&5\\\hline 4&5\\\cline{1-2}\end{array} \quad ; \quad T_2 = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 2&3&5\\\hline 4\\\cline{1-1}\end{array} \quad ; \quad T_3 = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 2&3\\\cline{1-2} 4&5\\\cline{1-2}\end{array}\,.$$

It follows from Theorem 6.6 and Corollary 6.8 that $T_1|_{\{1,2,3,4\}}$ is a unique rectification target of type A, from which we deduce that T_1, T_2, and T_3 are the only tableaux of straight shapes in their jeu de taquin class $[T_1]$.

It follows from this that the increasing tableau of type B defined by

$$U = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 \\ \cline{2-3}\end{array}$$

is a unique rectification target. In fact we have $U^2 = T_1$, so if S is any tableau of straight shape of type B that is jeu de taquin equivalent to U, then $S^2 \in [T_1]$. Since S^2 is a tableau of straight shape that is symmetric across the diagonal, we deduce that $S^2 = T_1$ and $S = U$.

7.2. **Weak K-Knuth equivalence.** Jeu de taquin equivalence of increasing tableaux of type A is governed by K-Knuth equivalence of their reading words. The following relation plays the analogous role for increasing tableaux of type B.

Definition 7.6. Define the *weak K-Knuth equivalence relation* on words of integers, denoted by $\hat{\equiv}$, as the symmetric transitive closure of the following basic relations. For any words of integers $\mathbf{u}$ and $\mathbf{v}$, and integers a, b, c we set

$$\begin{aligned}
(\mathbf{u}, a, a, \mathbf{v}) &\mathrel{\hat{\equiv}} (\mathbf{u}, a, \mathbf{v})\,, \\
(\mathbf{u}, a, b, a, \mathbf{v}) &\mathrel{\hat{\equiv}} (\mathbf{u}, b, a, b, \mathbf{v})\,, \\
(\mathbf{u}, a, b, c, \mathbf{v}) &\mathrel{\hat{\equiv}} (\mathbf{u}, a, c, b, \mathbf{v}) \quad \text{if } b < a < c\,, \\
(\mathbf{u}, a, b, c, \mathbf{v}) &\mathrel{\hat{\equiv}} (\mathbf{u}, b, a, c, \mathbf{v}) \quad \text{if } a < c < b\,, \ \textit{and} \\
(a, b, \mathbf{u}) &\mathrel{\hat{\equiv}} (b, a, \mathbf{u})\,.
\end{aligned}$$

Compared to the K-Knuth relation of Definition 5.3, the weak K-Knuth relation allows the first two integers in any word to be interchanged. Two weakly increasing tableaux T_1 and T_2 of type B are weakly K-Knuth equivalent, written $T_1 \mathrel{\hat{\equiv}} T_2$, if a reading word of T_1 is weakly K-Knuth equivalent to a reading word of T_2. By Lemma 5.4 this defines an equivalence relation on the set of all weakly increasing tableaux of type B that admit reading words.

Lemma 7.7. *Let T be a dotted increasing tableau of type B. Then all resolutions of T are weakly K-Knuth equivalent.*

Proof. Let T' and T'' be resolutions of T. By Theorem 5.9 and Lemma 5.5(b) we may assume that T' and T'' agree except for a single dotted box α which lies on the diagonal and is surrounded by five boxes satisfying $a < b \leq y \leq c < d$:

$$\begin{array}{|c|c|c|}\hline a & b & y \\ \hline \multicolumn{1}{c|}{} & \bullet & c \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & d \\ \cline{3-3}\end{array}$$

We can assume that $T'(\alpha) = b$ and $T''(\alpha) = c$. If T contains a dot in the same row as y, then we may further assume that T' and T'' replace this dot with the maximum of the boxes above and to the left of the dot. And if T contains a box in the same column as y but strictly north of y, then we may assume that T' and T'' replace this dot with the minimum of the boxes below and to the right of the dot. In this situation we must have $b \leq y < c$, and we can choose reading words for T' and T'' which agree except that the word of T' starts with (b, a, b, d, c, y) while the word of T'' starts with (c, a, b, d, c, y). If $y = b$ then set $y' = c$, otherwise set $y' = y$. Then we have

$$\begin{aligned}(b, a, b, d, c, y) \mathrel{\hat{\equiv}} (a, b, b, d, c, y) &\equiv (a, b, d, c, y) \equiv (d, c, a, b, y') \\ &\equiv (d, c, c, a, b, y') \mathrel{\hat{\equiv}} (c, d, c, a, b, y') \equiv (c, a, b, d, c, y)\,,\end{aligned}$$

as required. ☐

Theorem 7.8. *Let T and T' be increasing tableaux of type B. Then the reading words of T and T' are weakly K-Knuth equivalent if and only if T and T' are jeu de taquin equivalent for Δ.*

Proof. This follows from the same argument as proves Theorem 6.2, except that Lemma 6.1 is replaced with Lemma 7.7. ☐

Corollary 7.9. *Let T be an increasing tableau of straight shape of type B. The following are equivalent.* (a) *T is a unique rectification target for Δ.* (b) *If T' is any increasing tableau of straight shape such that $T' \mathrel{\hat{\equiv}} T$, then $T' = T$.*

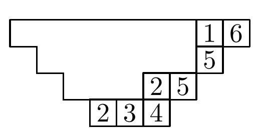
Proof. This follows from from Theorem 7.8 and Lemma 3.13. ☐

Given a word of integers $\mathbf{u} = (u_1, u_2, \dots, u_k)$, we let $\mathbf{u}^\dagger$ denote this word in reverse order, and we let $\mathbf{u}^\dagger\mathbf{u} = (u_k, \dots, u_2, u_1, u_1, u_2, \dots, u_k)$ denote the composition of the two words. It follows from the definitions that, if $\mathbf{u}$ and $\mathbf{v}$ are weakly K-Knuth equivalent words, then $\mathbf{u}^\dagger\mathbf{u}$ and $\mathbf{v}^\dagger\mathbf{v}$ are K-Knuth equivalent. Computer experiments suggest that the converse is also true.

Conjecture 7.10. *Let $\mathbf{u}$ and $\mathbf{v}$ be words of integers. Then we have $\mathbf{u} \mathrel{\hat{\equiv}} \mathbf{v}$ if and only if $\mathbf{u}^\dagger\mathbf{u} \equiv \mathbf{v}^\dagger\mathbf{v}$.*

If this conjecture is true, then the converse of Proposition 7.1(a) holds. Furthermore, if U is any increasing tableau of straight shape of type B, then Conjecture 7.10 implies that U is a unique rectification target for Δ if and only if U^2 is the only tableau in its jeu de taquin class $[U^2]_{\mathbb{N}^2}$ that has straight shape and is *symmetric* across the diagonal.

7.3. **The Pieri rule for $\Gamma(\Delta)$.** Define a *Pieri word of type B* to be a sequence of integers $a = (a_1, a_2, \dots, a_\ell)$ such that each entry a_i is either smaller than or equal to all predecessors a_j with $j < i$, or larger than or equal to all predecessors. We also define a *Pieri tableau of type B* to be any increasing tableau T of type B such that the row word of T is a Pieri word of type B. Such a tableau is called a *KOG-tableau* in [7]. Finally, define the *range* of a tableau to be the set of values contained in its boxes. The following tableau is a Pieri tableau of type B with range $[1, 6] = \{1, 2, 3, 4, 5, 6\}$.



The following result shows that multiplication with single-row shapes in the combinatorial K-theory ring $\Gamma(\Delta)$ is compatible with the Pieri formula for maximal orthogonal Grassmannians proved in [7].

Corollary 7.11. *Let λ be a shifted Young diagram and let p be a positive integer. Then we have*
$$G_{(p)} \cdot G_\lambda = \sum_\nu c^\nu_{p,\lambda}\, G_\nu$$
in the ring $\Gamma(\Delta)$, where the sum is over all shifted Young diagrams $\nu \supset \lambda$, and $c^\nu_{p,\lambda}$ is the number of Pieri tableaux of type B with shape ν/λ and range $[1,p]$.

Proof. An inspection of the basic relations of Definition 7.6 shows that, if $\mathbf{u}$ and $\mathbf{v}$ are weakly K-Knuth equivalent words, then $\mathbf{u}$ is a Pieri word of type B if and only if $\mathbf{v}$ is a Pieri word of type B. Theorem 7.8 therefore implies that the set P of all Pieri tableaux of type B with range $[1,p]$ is closed under jeu de taquin slides. Since $M_{(p)}$ is the only tableau of straight shape in P, we deduce that P is equal to the jeu de taquin class $[M_{(p)}]_\Delta$. The corollary follows from this. □

Remark 7.12. For each permutation $w \in \Sigma$ we can define an element $B_w = \sum_\lambda b_{w,\lambda}\, G_\lambda \in \Gamma(\Delta)$, where $b_{w,\lambda}$ is the number of increasing tableaux T of shape λ such that $w(T^2) = w$. Notice that $B_w \neq 0$ if and only if $w^{-1} = w$. These elements also satisfy the identity $B_w \cdot G_\lambda = \sum_\nu c^\nu_{w,\lambda}\, G_\nu$, where $c^\nu_{w,\lambda}$ is the number of increasing tableaux T of shape ν/λ such that $w(T^2) = w$, which is analogous to Corollary 6.12. It is interesting to ask if the elements B_w have any geometric meaning in the K-theory of a maximal orthogonal Grassmannian.

References

[1] S. Billey, W. Jockusch, and R. Stanley, *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. **2** (1993), no. 4, 345–374. MR 1241505 (94m:05197)

[2] M. Brion, *Positivity in the Grothendieck group of complex flag varieties*, J. Algebra **258** (2002), no. 1, 137–159, Special issue in celebration of Claudio Procesi's 60th birthday. MR 1958901 (2003m:14017)

[3] A. S. Buch, *A Littlewood-Richardson rule for the K-theory of Grassmannians*, Acta Math. **189** (2002), no. 1, 37–78. MR 1946917 (2003j:14062)

[4] ———, *Combinatorial K-theory*, Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser, Basel, 2005, pp. 87–103. MR 2143073 (2007a:14056)

[5] A. S. Buch, A. Kresch, M. Shimozono, H. Tamvakis, and A. Yong, *Stable Grothendieck polynomials and K-theoretic factor sequences*, Math. Ann. **340** (2008), no. 2, 359–382. MR 2368984 (2009c:05250)

[6] A. S. Buch and L. C. Mihalcea, *Curve neighborhoods of Schubert varieties*, arXiv:1303.6013.

[7] A. S. Buch and V. Ravikumar, *Pieri rules for the K-theory of cominuscule Grassmannians*, J. Reine Angew. Math. **668** (2012), 109–132. MR 2948873

[8] P.-E. Chaput, L. Manivel, and N. Perrin, *Quantum cohomology of minuscule homogeneous spaces*, Transform. Groups **13** (2008), no. 1, 47–89. MR 2421317 (2009e:14095)

[9] E. Clifford, H. Thomas, and A. Yong, *K-theoretic Schubert calculus for OG(n,2n+1) and jeu de taquin for shifted increasing tableaux*, arXiv:1002.1664.

[10] C. K. Fan, *A Hecke algebra quotient and some combinatorial applications*, J. Algebraic Combin. **5** (1996), no. 3, 175–189. MR 1394304 (97k:20013)

[11] S. Fomin and C. Greene, *Noncommutative Schur functions and their applications*, Discrete Math. **193** (1998), no. 1-3, 179–200, Selected papers in honor of Adriano Garsia (Taormina, 1994). MR 1661368 (2000c:05149)

[12] S. Fomin and A. N. Kirillov, *Grothendieck polynomials and the Yang-Baxter equation*, Proc. Formal Power Series and Alg. Comb. (1994), 183–190.

[13] W. Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 2, Springer-Verlag, Berlin, 1998. MR 1644323 (99d:14003)

[14] W. Graham, *Equivariant K-theory and Schubert varieties*, preprint, 2002.

[15] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972, Graduate Texts in Mathematics, Vol. 9. MR 0323842 (48 #2197)

[16] ———, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR 1066460 (92h:20002)

[17] A. Knutson, *Schubert patches degenerate to subword complexes*, Transform. Groups **13** (2008), no. 3-4, 715–726. MR 2452612 (2009g:14062)

[18] B. Kostant and S. Kumar, *T-equivariant K-theory of generalized flag varieties*, J. Differential Geom. **32** (1990), no. 2, 549–603. MR 1072919 (92c:19006)

[19] A. Lascoux, *Transition on Grothendieck polynomials*, Physics and combinatorics, 2000 (Nagoya), World Sci. Publ., River Edge, NJ, 2001, pp. 164–179. MR 1872255 (2002k:14082)

[20] A. Lascoux and M.-P. Schützenberger, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 11, 629–633. MR 686357 (84b:14030)

[21] C. Lenart, *Combinatorial aspects of the K-theory of Grassmannians*, Ann. Comb. **4** (2000), no. 1, 67–82. MR 1763950 (2001j:05124)

[22] C. Lenart and A. Postnikov, *Affine Weyl groups in K-theory and representation theory*, Int. Math. Res. Not. IMRN (2007), no. 12, Art. ID rnm038, 65. MR 2344548 (2008j:14105)

[23] D. E. Littlewood and A. R. Richardson, *Group characters and algebra*, Phil. Trans. R. Soc., A **233** (1934), 99–141.

[24] N. Perrin, *Small resolutions of minuscule Schubert varieties*, Compos. Math. **143** (2007), no. 5, 1255–1312. MR 2360316 (2008m:14098)

[25] P. Pragacz, *Algebro-geometric applications of Schur S- and Q-polynomials*, Lecture Notes in Math., Vol. 1478, pp. 130–191. Springer, Berlin, 1991, MR 1180989 (93h:05170)

[26] R. Proctor, *d-Complete posets generalize Young diagrams for the jeu de taquin property*, arXiv:0905.3716.

[27] ———, *Bruhat lattices, plane partition generating functions, and minuscule representations*, European J. Combin. **5** (1984), no. 4, 331–350. MR 782055 (86h:17007)

[28] R. W. Richardson, *Intersections of double cosets in algebraic groups*, Indag. Math. (N.S.) **3** (1992), no. 1, 69–77. MR 1157520 (93b:20081)

[29] M.-P. Schützenberger, *La correspondance de Robinson*, Lecture Notes in Math., Vol. 579, pp. 59–113. Springer, Berlin, 1977, MR 0498826 (58 #16863)

[30] J. R. Stembridge, *Shifted tableaux and the projective representations of symmetric groups*, Adv. Math. **74** (1989), no. 1, 87–134. MR 991411 (90k:20026)

[31] ———, *On the fully commutative elements of Coxeter groups*, J. Algebraic Combin. **5** (1996), no. 4, 353–385. MR 1406459 (97g:20046)

[32] G. P. Thomas, *Baxter algebras and Schur functions*, Ph.D. thesis, University College of Swansea, 1974.

[33] H. Thomas and A. Yong, *A combinatorial rule for (co)minuscule Schubert calculus*, Adv. Math. **222** (2009), no. 2, 596–620. MR 2538022 (2011b:14122)

[34] ———, *A jeu de taquin theory for increasing tableaux, with applications to K-theoretic Schubert calculus*, Algebra Number Theory **3** (2009), no. 2, 121–148. MR 2491941 (2010h:05334)

[35] ———, *The direct sum map on Grassmannians and jeu de taquin for increasing tableaux*, Int. Math. Res. Not. IMRN (2011), no. 12, 2766–2793. MR 2806593 (2012f:14097)

[36] M. Willems, *K-théorie équivariante des tours de Bott. Application à la structure multiplicative de la K-théorie équivariante des variétés de drapeaux*, Duke Math. J. **132** (2006), no. 2, 271–309. MR 2219259 (2007b:19009)

[37] D. Worley, *A theory of shifted Young tableaux*, Ph.D. thesis, Massachusetts Institute of Technology, 1984.

Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA

E-mail address: `asbuch@math.rutgers.edu`

E-mail address: `msamuel@math.rutgers.edu`